On a Class of Projectively Flat Finsler Metrics with Constant Flag Curvature

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Dec 20, 2006

Introduction

One of important problems in Finsler geometry is to study and characterize Finsler metrics of constant flag curvature. Another problem is to study and characterize projectively flat Finsler metrics on an open domain in R^n. In Riemannian geometry, these two problems are essentially same by Beltrami theorem. However, there are locally projectively flat Finsler metrics which are not of constant flag curvature; and there are Finsler metrics of constant flag curvature which are not locally projectively flat. It's a natural problem to discuss the projectively flat Finsler metrics with constant flag curvature. (α, β)-metrics is a computable class in Finsler metrics. It's interesting to classify the projectively flat (α, β)-metrics with constant flat curvature. In this lecture, I will introduce the recently result by Dr. Zhongmin Shen and me.

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Geodesics:
$\ddot{x} + 2G^i(x,\dot{x}) = 0$, where $G^i = G^i(x,y)$ are given by

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}.$$

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$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

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Flag curvature:

$$\mathbf{K} = \mathbf{K}(P,y) = \frac{\mathbf{g}_y(\mathbf{R}_y(u),u)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,y) - [\mathbf{g}_y(y,u)]^2},$$

where $P = \mathrm{span}\{y,u\} \subset T_xM$.

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For a Riemannian metric $F = \sqrt{g_{ij}(x)y^iy^j}$, $\mathbf{g}_y = \mathbf{g}$,

$$\mathbf{R}_y(u) = R(u,y)y.$$

$$\mathbf{g}(\mathbf{R}_y(u),u) = \mathbf{g}(R(u,y)y,u) = \mathbf{g}(R(y,u)u,y) = \mathbf{g}(\mathbf{R}_u(y),y).$$

$$\mathbf{K}(P,y) = \mathbf{K}(P) \quad \text{(sectional curvature)}.$$

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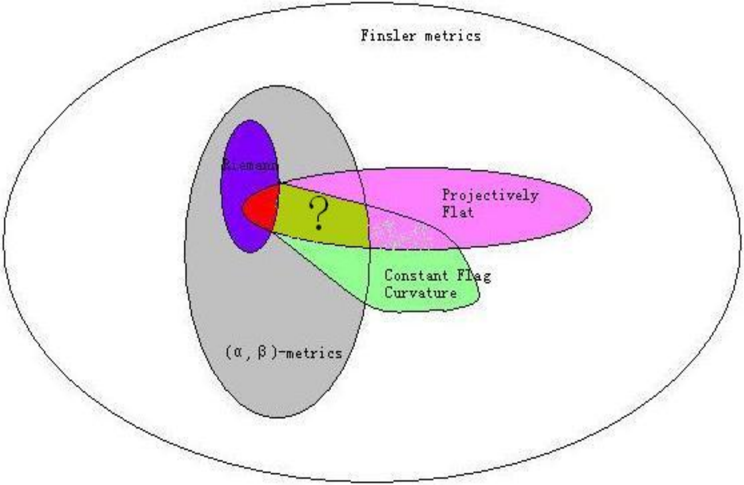
The Relation



Projectively Flat (α, β)-Metrics

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$$G^i = G^i_\alpha + \alpha Q {s^i}_0 + \alpha^{-1}\Theta\Big(-2\alpha Q s_0 + r_{00}\Big)y^i + \Psi\Big(-2\alpha Q s_0 + r_{00}\Big)b^i,$$

where G^i_α is the geodesic coefficient of α and $s_{ij} = b_{i|j} - b_{j|i}$, $r_{ij} = b_{i|j} + b_{j|i}$, $s_{i0} = s_{ij}y^j$, $s_0 = s_{i0}b^i$, $r_{00} = r_{ij}y^iy^j$ and

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'} \\ \Theta &= \frac{\phi - s\phi'}{2\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \cdot \frac{\phi'}{\phi} - s\Psi \\ \Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

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Proposition *A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed.*

Theorem(Shen-Yildirim) *Let $(\alpha + \beta)^2/\alpha$ be a Finsler metric on a manifold M. F is projectively flat if and only if*

(i) $b_{i|j} = \tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\}$,

(ii) *the geodesic coefficients G^i_α of α are in the form:*
$G^i_\alpha = \theta y^i - \tau\alpha^2 b^i$,

where $\tau = \tau(x)$ is a scalar function and $\theta = a_i(x)y^i$ is a 1-form on M.

Theorem (Shen) *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n ($n\geq 3$), where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta=b_i(x)y^i\neq 0$. Suppose that the following conditions: (a) β is not parallel with respect to α, (b) F is not of Randers type, and (c) $db\neq 0$ everywhere or $b=$ constant on $\mathcal{U}$. Then F is projectively flat on $\mathcal{U}$ if and only if the function $\phi=\phi(s)$ satisfies*

$$\Big\{1+(k_1+k_2s^2)s^2+k_3s^2\Big\}\phi''(s)=(k_1+k_2s^2)\Big\{\phi(s)-s\phi'(s)\Big\},$$

$$b_{i|j}=2\tau\Big\{(1+k_1b^2)a_{ij}+(k_2b^2+k_3)b_ib_j\Big\},$$

$$G^i_\alpha=\xi y^i-\tau\Big(k_1\alpha^2+k_2\beta^2\Big)b^i,$$

where $\tau=\tau(x)$ is a scalar function on $\mathcal{U}$ and k_1,k_2 and k_3 are constants.

Examples: Projectively Flat Finsler Metrics with Constant Flag Curvature

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$\mathbf{K} = -\frac{1}{4}$. Funk metric Θ on the unit ball $B^n \subset R^n$:

$$\Theta = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2} + \frac{\langle x, y\rangle}{1-|x|^2},$$

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$\mathbf{K} = \mathbf{0}$. Berwald's metric

$$B = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}},$$

where $y \in T_x\mathrm{B}^n \approx R^n$.

The Funk metric and Berwald's metric are related and they can be expressed in the form

$$\Theta = \bar{\alpha} + \bar{\beta}, \qquad B = \frac{(\tilde{\alpha} + \tilde{\beta})^2}{\tilde{\alpha}},$$

where

$$\bar{\alpha} := \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2}, \qquad \bar{\beta} := \frac{\langle x, y\rangle}{1 - |x|^2},$$

$$\tilde{\alpha} := \lambda\bar{\alpha}, \qquad \tilde{\beta} := \lambda\bar{\beta}, \qquad \lambda := \frac{1}{1 - |x|^2}.$$

Projectively Flat (α, β)-Metrics with Constant Flag Curvature

Lemma *If*
$\phi(s) = 1 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 + a_7 s^7 + a_8 s^8 + o(s^8)$
satisfies
$\{1 + (k_1 + k_2 s^2) s^2 + k_3 s^2\} \phi''(s) = (k_1 + k_2 s^2)\{\phi(s) - s\phi'(s)\}$,
then

$$a_2 = \frac{k_1}{2}, \quad a_3 = 0, \quad a_5 = 0, \quad a_7 = 0,$$

$$\begin{aligned}
a_4 &= \frac{1}{12}(k_2 - k_1 k_3) - \frac{1}{8} k_1^2, \\
a_6 &= -\frac{11}{120}(k_1 + \frac{4}{11} k_3)(k_2 - k_1 k_3) + \frac{1}{16} k_1^3, \\
a_8 &= \frac{1}{56}(k_2 - k_1 k_3)(\frac{61}{12} k_1^2 + k_3^2) - \frac{5}{224} k_2^2 \\
&\quad + \frac{31}{336} k_1 k_2 k_3 - \frac{47}{672} k_1^2 k_3^2 - \frac{5}{128} k_1^4.
\end{aligned}$$

Lemma (Li-Shen) *Let* $F = \alpha\phi(s)$*,* $s = \beta/\alpha$*, be an* (α, β)*-metric on an open subset* $\mathcal{U} \subset R^n$ $(n \geq 3)$*, where* $\alpha = \sqrt{a_{ij}y^i y^j}$ *and* $\beta = b_i y^i \neq 0$*. Suppose that* F *is not of Randers type and* $db \neq 0$ *everywhere or* $b = constant$ *on* $\mathcal{U}$*. If* F *is projectively flat with constant flag curvature* K*, then* $K = 0$*.*

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- β is parallel with respect to α.

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Then $G^i = G^i_\alpha$, which means α is projectively flat. By Beltrami theorem α has constant sectional curvature κ. If $\kappa \neq 0$, it's easy to see that F is a Riemannian metric. This is excluded. Thus $K = 0$.

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- β is not parallel with respect to α.
  By a direct computation we get

$$K\alpha^2\phi^2 = \xi^2 - \xi_{x_m}y^m + \tau^2\alpha^2\Xi - \tau_{x_m}y^m\Xi^2 + 2\tau^2\alpha^2\Gamma,$$

where $\xi = \xi_iy^i$, $\tau = \tau(x)$,

$$\Xi := (1 + (k_1 + k_2s^2)s^2 + k_3s^2)\frac{\phi'}{\phi} - (k_1 + k_2s^2)s,$$

$$\Gamma := (k_1 + k_2s^2)s\Xi - \left\{1 + (k_1 + k_2s^2)s^2 + k_3s^2\right\}\Xi_s.$$

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where $\bar{\alpha} := \sqrt{\sum_{a=2}^n (y^a)^2}$. Then

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And

$$\xi = \frac{s\xi_1}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{\xi}_0, \quad \tau_{x^m} y^m = \frac{s\tau_1}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{\tau}_0,$$

where $\bar{\xi}_0 := \xi_a y^a$, $\bar{\tau}_0 := \tau_{x^a} y^a$. Let

$$\xi_{ij} := \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x^j} + \frac{\partial \xi_j}{\partial x^i}\right).$$

By above identities, we obtain that

$$\frac{Kb^2\phi^2}{b^2-s^2}\bar{\alpha}^2 = \frac{\bar{\alpha}^2}{b^2-s^2}\Big\{s^2(\xi_1^2-\xi_{11})+\tau^2b^2\Xi^2-\tau_1 bs\Xi+2\tau^2b^2\Gamma\Big\} + \frac{1}{\sqrt{b^2-s^2}}\Big\{2s\xi_1\bar{\xi}_0-2s\bar{\xi}_{10}-\bar{\tau}_0 b\Xi\Big\}\bar{\alpha}+\bar{\xi}_0^2-\bar{\xi}_{00}.$$

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$$\frac{1}{b^2-s^2}\Big\{s^2(\xi_1^2-\xi_{11})+\tau^2b^2\Xi^2-\tau_1bs\Xi+2\tau^2b^2\Gamma-Kb^2\phi^2\Big\}\bar{\alpha}^2 + \bar{\xi}_0^2-\bar{\xi}_{00}=0.$$

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$$\frac{1}{b^2-s^2}\Big\{s^2(\xi_1^2-\xi_{11})+\tau^2b^2\Xi^2-\tau_1bs\Xi+2\tau^2b^2\Gamma-Kb^2\phi^2\Big\}\bar{\alpha}^2 + \bar{\xi}_0^2-\bar{\xi}_{00}=0.$$

By Taylor expansion of ϕ

$$\phi=1+a_1s+a_2s^2+a_4s^4+a_6s^6+a_8s^8+o(s^8)$$

we eventually proof $K=0$.

By above identities, we obtain that

$$\frac{Kb^2\phi^2}{b^2-s^2}\bar{\alpha}^2 = \frac{\bar{\alpha}^2}{b^2-s^2}\Big\{s^2(\xi_1^2-\xi_{11})+\tau^2b^2\Xi^2-\tau_1 bs\Xi+2\tau^2b^2\Gamma\Big\} + \frac{1}{\sqrt{b^2-s^2}}\Big\{2s\xi_1\bar{\xi}_0-2s\bar{\xi}_{10}-\bar{\tau}_0 b\Xi\Big\}\bar{\alpha}+\bar{\xi}_0^2-\bar{\xi}_{00}.$$

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we eventually proof $K=0$.
Remark: In this Lemma, we used the equivalent conditions of projectively flat $(\alpha,\beta\ )$-metrics.

Projectively Flat (α,β)-Metrics with Zero Flag Curvature

Lemma(Li-Shen) *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric on an open subset $\mathcal{U}\subset R^n$ $(n\geq 3)$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ and $\beta=b_iy^i\neq 0$. Suppose that F is not of Randers type, β is not parallel with respect to α and $db\neq 0$ everywhere or $b=constant$ on $\mathcal{U}$. If F is projectively flat with $K=0$, then*

$$\phi=\frac{(\sqrt{1+ks^2}+\epsilon s)^2}{\sqrt{1+ks^2}},$$

where $k=\frac{1}{5}(3k_1+2k_3)$ and $\epsilon=\pm\frac{2}{\sqrt{5}}\sqrt{k_1-k_3}$.

Projectively Flat (α, β)-Metrics with Zero Flag Curvature

Lemma(Li-Shen) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U} \subset R^n$ $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i \neq 0$. Suppose that F is not of Randers type, β is not parallel with respect to α and $db \neq 0$ everywhere or $b = constant$ on $\mathcal{U}$. If F is projectively flat with $K = 0$, then*

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where $k = \frac{1}{5}(3k_1 + 2k_3)$ and $\epsilon = \pm\frac{2}{\sqrt{5}}\sqrt{k_1 - k_3}$.

Sketch proof: By assumption that $K = 0$, we have

$$3\Big\{ - a_1^4 s^2 - (1 + 2k_3 s^2 + k_1 s^2 + 2k_2 s^4) a_1^2 + (k_1 + k_2 s^2)^2 s^2 \Big\} \tau^2 \phi^2$$
$$+ 6\Big\{ s a_1^2 - s(k_1 + k_2 s^2) \Big\} D(s) \tau^2 \phi' \phi + 3\tau^2 D(s)^2 \phi'^2 = 0,$$

where $D(s) = 1 + (k_1 + k_2 s^2) s^2 + k_3 s^2$. By discuss this equation we get the result.

Classification

By the above two lemmas we obtain the following.

Theorem(Li-Shen) *Let* $F = \alpha\phi(s)$, $s = \beta/\alpha$, *be an* (α, β)*-metric on an open subset* $\mathcal{U}$ *in the n-dimensional Euclidean space* R^n $(n \geq 3)$, *where* $\alpha = \sqrt{a_{ij}y^iy^j}$ *and* $\beta = b_iy^i \neq 0$. *Suppose that* $db \neq 0$ *everywhere or* $b = constant$ *on* $\mathcal{U}$. *Then* F *is projectively flat with constant flag curvature* K *if and only if one of the following holds*

(i) α *is projectively flat and* β *is parallel with respect to* α;

(ii) $F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$ *is projectively flat with constant flag curvature* $K < 0$, *where* k *and* $\epsilon \neq 0$ *are constants;*

(iii) $F = (\sqrt{\alpha^2 + k\beta^2} + \epsilon\beta)^2/\sqrt{\alpha^2 + k\beta^2}$ *is projectively flat with* $K = 0$, *where* k *and* $\epsilon \neq 0$ *are constants.*

Classification

By the above two lemmas we obtain the following.

Theorem(Li-Shen) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i \neq 0$. Suppose that $db \neq 0$ everywhere or $b = constant$ on $\mathcal{U}$. Then F is projectively flat with constant flag curvature K if and only if one of the following holds*

(i) *α is projectively flat and β is parallel with respect to α;*

(ii) *$F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$ is projectively flat with constant flag curvature $K < 0$, where k and $\epsilon \neq 0$ are constants;*

(iii) *$F = (\sqrt{\alpha^2 + k\beta^2} + \epsilon\beta)^2/\sqrt{\alpha^2 + k\beta^2}$ is projectively flat with $K = 0$, where k and $\epsilon \neq 0$ are constants.*

• It is a trivial fact that if F is trivial and the flag curvature $K = constant$, then it is either Riemannian $(K \neq 0)$ or locally Minkowskian $(K = 0)$. (by S. Numata).

- The Finsler metric in (ii) is of Randers type, i.e.,

$$F = \bar{\alpha} + \bar{\beta},$$

where $\bar{\alpha} := \sqrt{\alpha^2 + k\beta^2}$ and $\bar{\beta} := \epsilon\beta$.
Shen proved that a Finsler metric in this form is projectively flat with constant flag curvature if and only if it is locally Minkowskian or it is locally isometric to a generalized Funk metric

$$F = c(\bar{\alpha} + \bar{\beta})$$

on the unit ball $B^n \subset R^n$, where $c > 0$ is a constant, and

$$\bar{\alpha} : \;=\; \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2} \tag{1}$$

$$\bar{\beta} : \;=\; \pm\Big\{\frac{\langle x, y\rangle}{1-|x|^2} + \frac{\langle a, y\rangle}{1+\langle a, x\rangle}\Big\}, \tag{2}$$

where $a \in R^n$ is a constant vector.

- The Finsler metric in (iii) is in the form

$$F = (\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha},$$

where $\tilde{\alpha} := \sqrt{\alpha^2 + k\beta^2}$ and $\tilde{\beta} := \epsilon\beta$. It is proved (by Mo, Shen , Yang and Yildirim) that a non-Minkowkian metric $F = (\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha}$ is projectively flat with $K = 0$ if and only if it is, after scaling on x, locally isometric to a metric

$$F = c(\tilde{\alpha} + \tilde{\beta})^2/\tilde{\alpha}$$

on the unit ball $B^n \subset R^n$, where $c = constant$, $\tilde{\alpha} = \lambda\bar{\alpha}$ and $\tilde{\beta} = \lambda\bar{\beta}$, where $\bar{\alpha}$ and $\bar{\beta}$ are given in (1) and (2), and

$$\lambda := \frac{(1 + \langle a, x\rangle)^2}{1 - |x|^2}.$$

An Open Problem

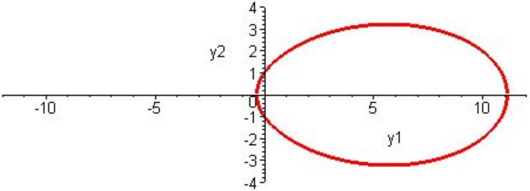
Is there any metric $F = (\alpha + \beta)^2/\alpha$ of constant flag curvature which is not locally projectively flat?

Indicatrix of $F = (\alpha + \beta)^2/\alpha$

Indicatrix of F: Given a Minkowski space (V, F),

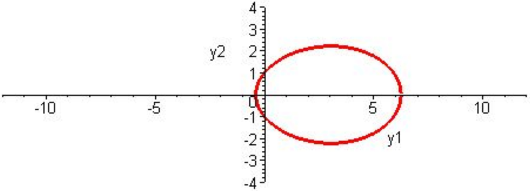
$$S_F := \Big\{ y \in V | F(y) = 1 \Big\}.$$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



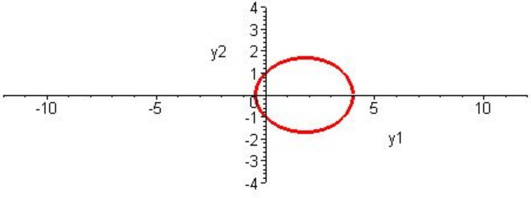
$b = -0.7$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



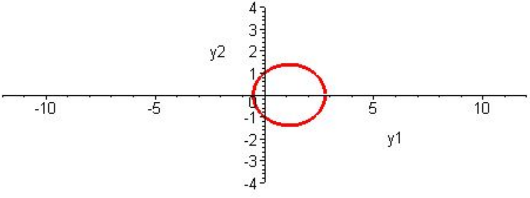
$b = -0.6$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2+(y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



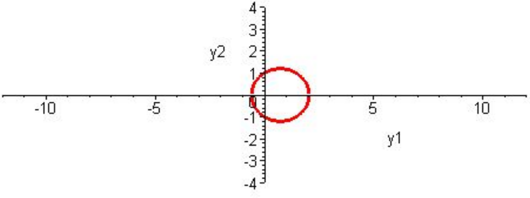
$b = -0.5$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2+(y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



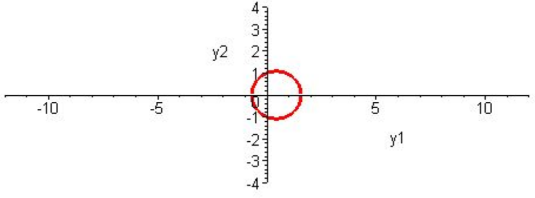
$b = -0.4$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



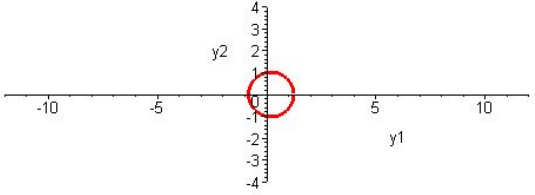
$b = -0.3$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2+(y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



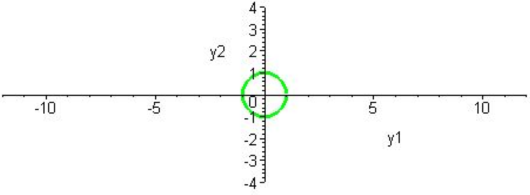
$b = -0.2$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



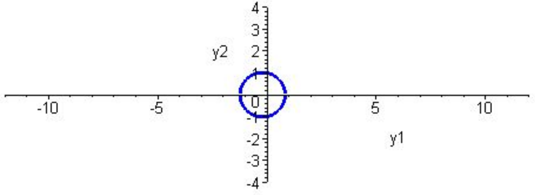
$b = -0.1$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



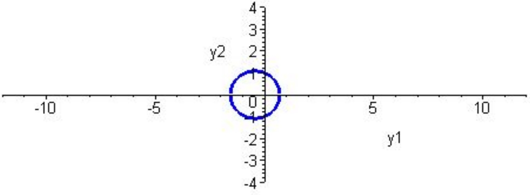
$b = 0$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2+(y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



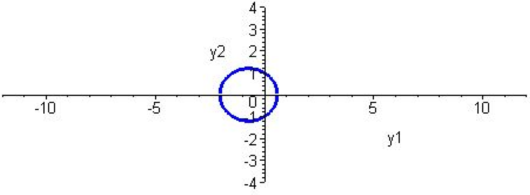
$b = 0.1$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



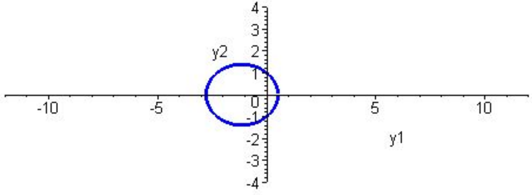
$b = 0.2$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2+(y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



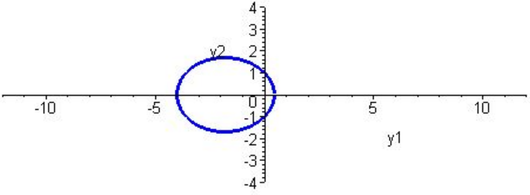
$b = 0.3$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



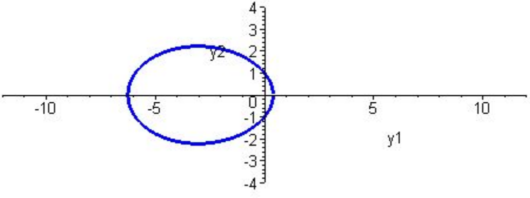
$b = 0.4$

$F = (\alpha + \beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2 + (y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



$b = 0.5$

$F = (\alpha+\beta)^2/\alpha$, $\alpha = \sqrt{(y^1)^2+(y^2)^2}$, $\beta = b(x)y^1$. The indicatrix of F:



$b = 0.6$

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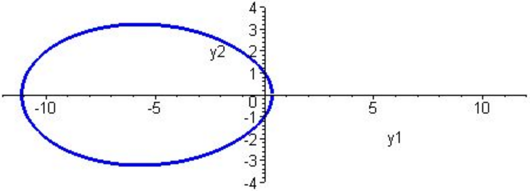

$b = 0.7$

Thank you very much!