The Annals of Probability
1973, Vol. 1, No. 3, 417-427

ON A CLASS OF SECRETARY PROBLEMS

BY ANTHONY G. MUCCI
University of Maryland

Asymptotic forms for the optimal expected payoffs (minimal costs) for a generalized class of "Secretary" problems are investigated by an analysis of a related family of differential equations. A class of unbounded payoff functions with bounded expected payoffs is determined and methods for generating the expected payoffs are developed.

1. Introduction. The stopping rule problems to be considered in this paper are generalizations of the "secretary problem" for increasing unbounded payoffs. An example of such a problem is found in [2].

Let $\mathscr{A}$ denote the class of functions with domain the positive integers such that $q \in \mathscr{A}$ implies $q(1) = 1$ and $q(k+1) \geqq q(k)$. We will call such functions payoffs. Let a probability $(N!)^{-1}$ be attached to each permutation σ of the first N integers and let $\{X_k\}$, $k = 1, 2, \cdots, N$ be the sequence of independent random variables where X_k is the rank of $\sigma(k)$ among $\sigma(1), \sigma(2), \cdots, \sigma(k)$. Note that $P(X_r = k) = 1/r$ for $k = 1, 2, \cdots, r$. The stopping rule problem consists in determining

$$v_N = \min_t Eq(\sigma(t)) \tag{1.1}$$

where t runs through stopping rules on the sequence $X_1, X_2, \cdots, X_N$. In this paper we investigate $\lim_{N\to\infty} v_N$ for a class of payoffs $q \in \mathscr{A}$ such that

$$\begin{aligned} &q(k) \uparrow \infty \\ &\sum q(k)\lambda^k < \infty \qquad \text{for} \quad 0 \leqq \lambda < 1 \,. \end{aligned} \tag{1.2}$$

Let us set

$$Q(r, k) = E(q(\sigma(r)) \,|\, X_r = k) = \sum_k^{N-(r-k)} q(l) \frac{\binom{l-1}{k-1}\binom{N-l}{r-k}}{\binom{N}{r}} \,.$$

Our objective is to find asymptotic forms and estimates for the values v_N. We have

$$v_N = \min_t Eq(\sigma(t)) = \min_t EQ(t, X_t) \,.$$

We call v_N the optimal payoff. We sometimes call v_N a utility. A recursive technique for generating v_N is the following. Set

$$\begin{aligned} v_N(N) &= Eq(\sigma(N)) = \frac{1}{N} \sum_1^N q(l) \,, \\ v_N(N, k) &= q(k) \,, \qquad\qquad k = 1, 2, \cdots, N \,, \\ v_N(r, k) &= \min\,(Q(r, k), Ev_N(r+1, X_{r+1})) \,, \\ v_N(r) &= Ev_N(r, X_r) = \frac{1}{n} \sum_1^r v_N(r, k) \,. \end{aligned}$$

Received July 21, 1972.
AMS 1970 *subject classifications*. 60G40.
Key words and phrases. Optimal stopping times; pay off.

We then have

$$v_N(r) = E \min(Q(r, X_r), v_N(r+1))$$
$$= \frac{1}{r} \sum_1^N \min(Q(r, k), v_N(r+1)) .$$

Since $v_N(r) = \inf_{t \geqq r} Eq(\sigma(t))$, we see that $v_N(r) \leqq v_N(r+1)$ and $v_N = v_N(1)$. This recursive technique, called a "backwards recursion," defines the optimal stopping rule, t_N, by the prescription that one stop with the observation $X_r = k$ unless $v_N(r, k) < Q(r, k)$, i.e., unless it costs less to continue. It is easily shown that

$$Q(r, k) \leqq Q(r, k+1)$$
$$Q(r+1, k) \leqq Q(r, k) .$$

We can therefore characterize our optimal stopping rule t_N, (which is the collection of pairs (r, k) such that $Q(r, k) \leqq v_N(r+1)$) as a tuple $(r_1, r_2, \cdots, r_N)$ where $r_1 \leqq r_2 \leqq \cdots \leqq r_N$ and where one stops with observation $X_r = k$ provided $r \geqq r_k$.

The procedure outlined above is a rewording of the procedures found in [4], [2] and [6].

We now define

$$f_N(1) = \frac{1}{N} \sum_1^N q(l) \tag{1.3}$$
$$f_N\left(\frac{r}{N}\right) = \frac{1}{r} \sum_1^r \min\left(Q(r, k), f_N\left(\frac{r+1}{N}\right)\right) .$$

We extend f_N to all of $[0, 1]$ by linear interpolation. Then f_N is a non-decreasing continuous function on $[0, 1]$ such that

$$f_N\left(\frac{r}{N}\right) = \min_{t \geqq r} Eq(\sigma(t))$$

with

$$v_N = f_N(0) = f_N\left(\frac{1}{N}\right) .$$

For fixed N, the value v_N is achieved for the stopping rule t_N defined as the N-tuple of fractions

$$\left(\frac{r_1}{N}, \frac{r_2}{N}, \cdots, \frac{r_N}{N}\right) \qquad \text{with} \quad r_1 \leqq r_2 \cdots \leqq r_N = N$$

such that one stops on the observation $X_r = k$ if and only if $r \geqq r_k$ where r_k is the minimal r satisfying $Q(r, k) \leqq f_N(r + 1/N)$. In effect, f_N carries all the information necessary to determine v_N and t_N. We will call f_N the N-utility of q. Suppose now that q is bounded with $q_\infty = \lim_{k \to \infty} q(k)$. Set $R_k(\alpha) = \sum_k^\infty q(l)\binom{l-1}{k-1}\alpha^k(1-\alpha)^{l-k}$ for $\alpha \in (0, 1]$. Then we can translate directly from our major results in [9] to obtain

THEOREM 1.1. *There exists a unique* g *on* $[0, 1]$ *such that*

(a) $g' = (1/\alpha) \sum_1^\infty (g - R_k)^+$; $g(1) = q_\infty$,

(b) $f_N \to g$ *uniformly on* $[0, 1]$, *hence* $v_N \to g(0)$,

(c) $r_k/N \to \alpha_k$ *where* α_k *uniquely satisfies* $g(\alpha_k) = R_k(\alpha_k)$ *provided* R_k *is strictly decreasing.*

The boundedness of q plays a major role in establishing these results. We are concerned in the present paper with extending these results in the following way:

THEOREM 1.2. *Let* $q \in \mathscr{A}$, $q(k) \uparrow \infty$, $q(k) \leqq k^A$ *fixed* $A > 0$. *Then*

(a) $v_N \uparrow v < \infty$,

(b) $v = g(0)$ *where* g *is the unique solution on* $[0, 1]$ *of*

$$g' = \frac{1}{\alpha} \sum_1^\infty (g - R_k)^+ ; \qquad g(0) = v .$$

Note that the differential equation in both theorems is the continuous analogue of (1.3) rewritten as

$$f_N\left(\frac{r+1}{N}\right) - f_N\left(\frac{r}{N}\right) = \frac{1}{r} \sum_1^N \left(f_N\left(\frac{r+1}{N}\right) - Q(r, k)\right)^+ .$$

2. The functions g **and** R_k. Let $\mathscr{A}_*$ consist of payoffs q satisfying (1.2). In particular,

$$R_k(\alpha) = \sum_k^\infty q(l)\binom{l-1}{k-1}\alpha^k(1 - \alpha)^{l-k} < \infty \qquad \text{for} \quad \alpha \in (0, 1] .$$

We define

(2.1)
$$q_M(l) = q(l) \quad \text{if} \quad l < M ,$$
$$= q(M) \quad \text{if} \quad l \geqq M ,$$

(2.2)
$$v_{N,M} = \sup_{t \leqq N} Eq_M(\sigma(t)) .$$

We define $f_{N,M} Q_{N,M}(r, k)$, $R_{k,M}$ in an analogous manner. Since q_M is bounded, we can define $g_{(M)}$ with $g_{(M)} = \lim f_{N,M}$ uniformly on $[0, 1]$ and we have $g_{(M)}$ as the unique solution to

(2.3)
$$g'_{(M)} = \frac{1}{\alpha} \sum_1^M (g_{(M)} - R_{k,M})^+ ; \qquad g_{(M)}(1) = q_{(M)} .$$

These are immediate consequences of Theorem 1.1. Let us note some properties of the functions R_k.

(a)
$$R_k' = k \sum_k^\infty (q(l) - q(l+1))\binom{l}{k}\alpha^{k-1}(1 - \alpha)^{l-k}$$
$$= \frac{k}{\alpha}(R_k - R_{k+1}) .$$

(b) R_k is *strictly* decreasing on $(0, 1]$.

(c) $R_{k+1} > R_k$ on $[0, 1)$.

(d) $R_k(0^+) = \infty$.

(e) $R_k(1) = q(k)$; $R_k'(1) = k(q(k) - q(k+1))$.
(f) $\lim_{k\to\infty} R_k(\alpha) = \infty$, all $\alpha \in (0, 1]$.
(g) $R_{k,M} \uparrow R_k$ uniformly on $(\alpha, 1]$, each $\alpha > 0$.
(h) $R_{k,M}$ is a polynomial, hence is either constant or is *strictly* decreasing. In fact, $R_{k,M}$ is constant only if for all $l \geqq k$ we have $q(l) = q(m)$. Since $q(l) \uparrow \infty$, we have
(i) For all k, there exists M_k such that $M \geqq M_k$ implies $R_{k,M}$ is a *strictly* decreasing polynomial, and $R_{k,M}$ is a strictly increasing (in M) sequence of polynomials.

All these properties follow easily from (a), which is a straightforward calculation. Property (g) follows from (h), (i), and Dini's theorem:
The following lemma is crucial:

LEMMA 2.1.

$$v_N \leqq v_{N+1}\,.$$

PROOF. A proof can be found in [8], Remarks (2.3D) and (2.3E).
Let us note now that Theorem (1.1) implies $\lim_{N\to\infty} v_{N,M} = g_{(M)}(0) \equiv v_{(M)}$. Further

$$v_{(M)} = g_{(M)}(0) = \lim f_{N,M}(0) \leqq \lim f_{N,M+1}(0) = g_{(M+1)}(0) = v_{(M+1)}\,.$$

We use these inequalities in

PROPOSITION 2.1. *Let*

$$v = \lim \uparrow v_N \qquad \bar{v} = \lim \uparrow v_{(M)}\,.$$

Then $v = \bar{v}$.

PROOF. From the lemma and the inequalities above, both v and $\bar{v}$ are well defined. Using the lemma and the fact that $q \geqq q_M$ we have, for $N \geqq M$, that

$$v_M \leqq v_{N,M} \leqq v_N \leqq v\,.$$

Letting $N \to \infty$, we have $v_{(M)} \leqq v$, so $\bar{v} \leqq v$. On the other hand, $v_M \leqq v_{N,M}$ implies $v_M \leqq v_{(M)}$, so $v_M \leqq \bar{v}$, therefore $v \leqq \bar{v}$. □

REMARK 2.1.

(1) Our objective is to determine $v = \lim \uparrow v_N$. We see that $v = \lim \uparrow v_{(M)}$ where $v_{(M)} = g_{(M)}(0)$. So our approach will be to investigate the functions $g_{(M)}$ satisfying (2.3).
Since $g_{(M)} = \lim f_{N,M} \leqq \lim f_{N,M+1} = g_{(M+1)}$, we can define

$$g = \lim \uparrow g_{(M)}\,.$$

Clearly, $v = \lim \uparrow g_{(M)}(0)$.
(2) Property (i) above, in conjunction with Theorem (1.1) implies that for given k and large enough M there exists a unique $\alpha_{M,k}$ satisfying

$$R_{k,M}(\alpha_{M,k}) = g_{(M)}(\alpha_{M,k})\,.$$

Our next theorem establishes the limiting behavior of the series $\{\alpha_{M,k}\}$ and relates the results to the optimal stopping problem:

THEOREM 2.1.

(1) $\lim_{M\to\infty}\alpha_{M,k}=\alpha_k$ *exists for all* k.

(2) *If* $\alpha_1>0$, *then* $0<\alpha_1<\alpha_2<\cdots$ *with* $\alpha_\infty=\lim\alpha_k$ *satisfying* $\alpha_\infty\leqq 1$.

(3) *If* $\alpha_1>0$, *then the sequence* $\{\alpha_k\}$ *uniquely satisfies* $g(\alpha_k)=R_k(\alpha_k)$. *Further*

$$g<\infty \quad \text{on} \quad [0,\alpha_\infty) \quad \text{and} \quad \lim_{\alpha\to\alpha_\infty} g(\alpha)=\infty\,.$$

(4) $v<\infty$ *if and only if* $\alpha_1>0$, *in which case* g *is the unique solution on* $[0,\alpha_\infty)$ *for the differential equation*:

$$g(0)=v \qquad g'(\alpha)=\frac{1}{\alpha}\sum_1^\infty (g(\alpha)-R_k(\alpha))^+\,.$$

The proofs of all these statements can be found in [8].

3. A subclass of unbounded payoffs. Our objective in this section is to determine a subclass of unbounded payoffs q for which $\alpha_\infty=1$, or equivalently, for which $g<\infty$ on $[0,1)$.

Let q_M be q truncated at M, i.e., $q_M(l)=q(M)$ for $l\geqq M$. Fix $0<\lambda_1\leqq\lambda_2\leqq\cdots\leqq\lambda_M=1$ and consider the stopping rule

$$S_N=([\lambda_1 N],[\lambda_2 N],\cdots,[\lambda_N N])$$

where

$$\lambda_l=1 \qquad \text{for} \quad l\geqq M\,.$$

This is the stopping rule which prescribes that one stops on observation $X_r=l$ where $[\lambda_k N]\leqq r<[\lambda_{k+1}N]$ provided $l\leqq k$. Since $\lambda_l=1$ for $l\geqq M$, one never stops on the observation $X_r=l$ for $l\geqq M$ unless $r=N$. Thus, we may as well write S_N in the abbreviated form:

$$S_N=([\lambda_1 N],\cdots,[\lambda_{M-1}N],N)\,.$$

For example, if $N=10$, $M=3$ and $\lambda_1 N=5$, $\lambda_2 N=8$, $\lambda_3 N=10$, then our stopping rule $S=(5,8,10)$ prescribes that we let four observations pass, then stop for $5\leqq r\leqq 7$ if $X_r=1$; otherwise we stop for $8\leqq r\leqq 9$ if $X_r\leqq 2$. If neither situation arises, we stop on observation X_{10}.

Set

$$\bar v_N=Eq_M(\sigma(S_N))\,. \tag{3.1}$$

We will obtain an explicit formula for $\lim_{N\to\infty}\bar v_N$.

REMARK 3.1. In the analysis up to and including Remark 3.3 we will be dealing with q_M for some unbounded q. We suppress the M for notational convenience. Set

$$p_1=1\,,$$

$$p_{l,l+1}=P(X_t>l \quad \text{for} \quad [\lambda_l N]\leqq t<[\lambda_{l+1}N])\,,$$

$$p_k=\prod_1^{k-1}p_{l,l+1} \qquad \text{where} \quad k>1\,,$$

$$u_k=\sum_{t=[\lambda_k N]}^{[\lambda_{k+1}N]-1}\sum_{l=1}^k P(X_t=l,\,X_s>k \quad \text{for} \quad [\lambda_k N]\leqq s<t)\cdot Q_N(t,l)\,,$$

$$k\leqq M-1\,.$$

As usual, $Q_N(t, l) = E(q(\sigma(t)) \mid X_t = l)$. Further, we set $u_M = N^{-1} \sum_1^M q(l)$. Then

$$\bar{v}_N = \sum_1^M p_k u_k .$$

Now

$$p_{l,l+1} = \prod_{t=[\lambda_l N]}^{[\lambda_{l+1}N]-1} \left(1 - \frac{l}{t}\right) \to \left(\frac{\lambda_l}{\lambda_{l+1}}\right)^l \qquad \text{as } N \to \infty .$$

Thus

$$\lim_{N\to\infty} p_k = \frac{\prod_1^{k-1} \lambda_l}{\lambda_k^{k-1}}$$

Similar calculations, carried out in detail in [8], give us

$$\lim_{N\to\infty} u_k = \lambda_k^k \int_{\lambda_k}^{\lambda_{k+1}} \frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda)\, d\lambda .$$

This, together with (12.2) yields

PROPOSITION 3.1. *Let q be an increasing payoff truncated at M. Let $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_M = 1$ be fixed and set $S_N = ([\lambda_1 N], [\lambda_2 N], \cdots, [\lambda_M N])$. Then setting $\bar{v}_N = Eq(\sigma(S_N))$, we have*

$$\text{(3.2)} \qquad \bar{v}_{(M)} \equiv \lim_{N\to\infty} \bar{v}_N = \sum_1^{M-1} \left(\prod_1^k \lambda_l\right) \int_{\lambda_k}^{\lambda_{k+1}} \frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda)\, d\lambda + \left(\prod_1^{M-1} \lambda_l\right) \cdot q(M) .$$

PROOF. Only the last term needs to be explained. Clearly, $u_M = 1/N \sum_1^N q(l)$, and since $q(l) = q(M)$ for $l \geqq M$, we have $\lim_{N\to\infty} u_M = q(M)$. □

REMARK 3.2. Recall that $g_{(M)} = \lim f_{N,M}$ uniformly on [0, 1]. Thus

$$\begin{aligned} g_{(M)}(\lambda_1) &= \lim_{N\to\infty} f_{N,M}(\lambda_1) = \lim_{N\to\infty} f_{N,M}\left(\frac{[\lambda_1 N]}{N}\right) \\ &= \lim_{N\to\infty} \{\inf_{N \geqq t \geqq [\lambda_1 N]} Eq(\sigma(t))\} \\ &\leqq \lim_{N\to\infty} \bar{v}_N = \bar{v}_{(M)} . \end{aligned}$$

In order to call attention to the dependence of $\bar{v}_{(M)}$ on λ_1, λ_2 etc., we will sometimes write $\bar{v}_{(M)} = \bar{v}_{(M)}(\lambda_1)$. We have

PROPOSITION 3.2. *For any $\lambda_1 > 0$, $g_{(M)}(\lambda_1) \leqq \bar{v}_{(M)}(\lambda_1)$.*

REMARK 3.3. A series of calculations (carried out in detail in [9]) using the equality $1/\lambda^{k+1} \sum_i^k R_l(\lambda) = 1/k - 1[1/\lambda^k \sum_1^{k-1} R_l(\lambda)]'$, $k \geqq 2$ leads directly to the inequality

$$\text{(3.3)} \qquad \bar{v}_{(M)}(\lambda_1) \leqq \sum_1^M \frac{\left(\prod_1^k \lambda_l\right)}{\lambda_k^k} \cdot R_k(\lambda_k) .$$

We can arrive at this inequality by a direct argument. Simply note that

$$\lim_{N\to\infty} p_k = \frac{\prod_1^k \lambda_l}{\lambda_k^k}$$

and that $u_k \leqq Q_N([\lambda_k N], k) \to R_k(\lambda_k)$. Now if, for R_k we used q instead of q_M, then we would strengthen inequality (3.3). Likewise if $\lambda_M \leqq 1$. Thus, for any sequence $0 > \lambda_1 \leqq \lambda_2 < \cdots$ we have $\bar{v}_{(M)}(\lambda_1) \leqq \sum_1^\infty (\prod_1^k \lambda_l/\lambda_k{}^k) R_k(\lambda_k)$.

THEOREM 3.1. *Let $q \in \mathscr{A}_*$. Then for any sequence $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq 1$ we have*

$$g(\lambda_1) \leqq \sum_1^\infty (\textstyle\prod_1^k \lambda_l) \cdot \frac{R_k(\lambda_k)}{\lambda_k{}^k}. \tag{3.4}$$

Let us note again that as in Remark 3.3 this inequality can be achieved directly. The possible advantage in our more detailed approach is that we arrive at an explicit formula for $\bar{v}_{(M)}(\lambda_1)$ which may lead to a stronger inequality.

Our objective in the next section will be to isolate a class of payoffs, q, such that for any $\lambda_1 \in (0, 1)$, we will be able to determine $\lambda_2, \lambda_3 \cdots$ etc., so that the right-hand side in (3.4) is finite.

4. Payoffs that grow like polynomials. We will assume

$$\frac{q(k+1)}{q(k)} = 1 + O\left(\frac{1}{k}\right). \tag{4.1}$$

This assumption merely asserts that $q(K)$ grows at most like a fixed power of k. We will now derive some properties for payoffs satisfying (4.1). Our intention is to show that such payoffs lead to functions g which are finite on [0, 1), which will surely be the case if the summation in Theorem (3.1) is finite.

Clearly there must exist $M > 0$ such that M is an integer, and $q(k+1)/q(k) \leqq 1 + 2M/(k+1)$. But then, a straightforward calculation yields $q(l) \leqq (l/k)^{2M} q(k)$ for $l \geqq k$.
Further: $R_k(\lambda) \leqq e^{2M^2} \lambda^{-2M} q(k)$. Let us restate this result as:

PROPOSITION 4.1. *If $q(k+1)/q(k) \leqq 1 + 2M/k$, then $R_k(\lambda) \leqq e^{2M^2}\lambda^{-2M}q(k)$. In particular, if $q(k+1)/p(k) = 1 + O(1/k)$, then there exists a constant C and a positive integer M such that Theorem* 3.1 *can be written*:

$$g(\lambda_1) \leqq \frac{C}{\lambda_1^{2M}} \sum_1^\infty \frac{(\prod_1^k \lambda_l)}{\lambda_k{}^k} q(k) \tag{4.2}$$

for each fixed sequence $0 < \lambda_1 \leqq \cdots \leqq \lambda_M \leqq \cdots \leqq 1$.

REMARK 4.1. We want to choose a sequence $\{\lambda_k\}$ with fixed λ_1 such that $\sum_1^\infty (\prod_1^k \lambda_l)/\lambda_k{}^k q(k) < \infty$. We require further that $\lambda_N \leqq 1$ for all N, hence, that $\prod_1^{N-1} (\lambda_{l+1}/\lambda_l) \leqq 1/\lambda_1$, so that

$$\prod_1^\infty \left(\frac{\lambda_{l+1}}{\lambda_l}\right) \leqq \frac{1}{\lambda_1}. \tag{4.3}$$

Let us set

$$\frac{\lambda_{l+1}}{\lambda_l} = \left[\frac{q(l+m+1)}{q(l+m)}\left(\frac{l+m+1}{l+m}\right)^2\right]^{1/(l+m)} \tag{4.4}$$

with m as yet undetermined. Then

$$\log \prod_1^\infty \left(\frac{\lambda_{l+1}}{\lambda_l}\right) = \sum_1^\infty \log \left\{\frac{q(l+m+1)}{q(l+m)}\left(\frac{l+m+1}{l+m}\right)^2\right\}^{1/(l+m)}$$

$$= \sum_{m+1}^\infty \frac{1}{l} \log \frac{q(l+1)}{q(l)} + 2 \sum_{m+1}^\infty \frac{1}{l} \log\left(1+\frac{1}{l}\right)$$

$$\leqq \sum_{m+1}^\infty \frac{1}{l} \log\left(1+\frac{2M}{l}\right) + 2 \sum_{m+1}^\infty \frac{1}{l} \log\left(1+\frac{1}{l}\right)$$

$$\leqq 2M \sum_{m+1}^\infty \frac{1}{l^2} + 2 \sum_{m+1}^\infty \frac{1}{l^2}$$

$$= 2(M+1) \sum_{m+1}^\infty \frac{1}{l^2} \leqq 2(M+1) \int_m^\infty \frac{ds}{s^2} = \frac{2(M+1)}{m}.$$

So $\prod_1^\infty (\lambda_{l+1}/\lambda_l) \leqq e^{2(M+1)/m}$. Thus, if m_0 is so large that $m \geqq m_0 \Rightarrow 2(M+1)/m \leqq \log 1/\lambda_1$, we will have condition (4.3) satisfied. For any such m, we have

$$\frac{\prod_1^k \lambda_l}{\lambda_k^k} \leqq \frac{q(m+1)}{q(m+k)} \frac{(m+1)^2}{(m+k)^2} \frac{1}{\lambda_1^m}.$$

Thus

$$\frac{\prod_1^k \lambda_l}{\lambda_k^k} q(k) \leqq \frac{(m+1)^2 q(m+1)}{\lambda_1^m} \frac{1}{m(+k)^2}.$$

But then

$$\sum_1^\infty \frac{\prod_1^k \lambda_l}{\lambda_k^k} q(k) \leqq \frac{(m+1)^2 q(m+1)}{\lambda_1^m} \sum_{m+1}^\infty \frac{1}{l^2}$$

$$\leqq \frac{(m+1)^2}{m} \frac{q(m+1)}{\lambda_1^m}.$$

Hence

(4.5) $$g(\lambda_1) \leqq \frac{e^{2m2}}{\lambda_1^{2m}} \cdot \frac{(m+1)^2}{m} \frac{q(m+1)}{\lambda_1^m} < \infty.$$

THEOREM 4.1. *Let* $q(k+1)/q(k) = 1 + O(1/k)$. *Then* $g < \infty$ *on* $[0, 1)$, *hence* $\alpha_\infty = 1$.

5. A recursive technique for determining v. Let $q \in \mathscr{A}_*$ Define

(5.1) $$H_1(\alpha) = \frac{R_1(\alpha)}{\alpha} - \sum_1^\infty \frac{s(k)}{k} (1-\alpha)^k$$

$$G_1(\alpha) = H_1(\alpha) + \frac{1}{\alpha} (R_2(\alpha) - R_1(\alpha))$$

where $s(k) = \sum_1^k q(l)$.

Since we have assumed that $R_1(\alpha)$ has radius of convergence 1 around the point 1, and since $s(k)/k \leqq q(k)$, H_1 and G_1 are well defined. Let us further define

(5.2) $$H_k(\alpha) = \frac{1}{\alpha^k} \sum_1^{k-1} (R_k - R_l)$$

$$G_k(\alpha) = \frac{1}{\alpha^k} \sum_1^{k-1} (R_{k+1} - R_l)$$

for $k \geqq 2$. Clearly H_k and G_k are nonnegative for $k \geqq 2$. Further since $H_1' = -H_2$ and $H_1(1) = 1$, we see that H_1 and G_1 are also nonnegative. Let us list a number of properties of these functions.

PROPOSITION 5.1.

(a) $H_N \leqq G_N \leqq H_{N+1}$ *for* $N \geqq 2$,
(b) $H_1 \leqq G_1$,
(c) H_N *and* G_N *are* strictly *decreasing for* $N \geqq 1$,
(d) $H_N(0^+) = G_N(0^+) = \infty$ *for* $N \geqq 1$,
(e) $H_N' = -(N-1)H_{N+1}$ *for* $N \geqq 2$,
(f) $G_N' \leqq -(N-1)H_{N+1}$ *for* $N \geqq 2$,
(g) $H_N(1) = (N-1)q(N) - s(s-1)$ *for* $N \geqq 2$,
(h) $G_N(1) = (N-1)q(N+1) - s(N-1)$ *for* $N \geqq 2$,
(i) $H_1(1) = 1$, $G_1(1) = q(2)$.

PROOF. We will prove (d), the other parts being straightforward. Note that (a) and (b) show that it suffices to prove $H_1(0^+) = H_2(0^+) = \infty$ in order to establish (d). Now

$$H_1(\alpha) \geqq \textstyle\sum_1^\infty \left[q(k+1) - \frac{s(k)}{k}\right](1-\alpha)^k$$
$$\geqq \textstyle\sum_1^\infty [q(k+1) - q(k)](1-\alpha)^k .$$

Letting $\alpha \to 0^+$, we have, by monotone convergence: $H_1(0^+) \geqq \lim \uparrow q(N) = \infty$.

REMARK 5.1. Let $g < \infty$ on $[0, 1)$. Then g uniquely satisfies $g' = 1/\alpha \sum_1^\infty (g - R_k)^+$; $g(0) = v$. Now on $[\alpha_M, \alpha_{M+1})$ we have $g' = 1/\alpha \sum_1^M (g - R_k)$. From this

$$\left(\frac{g}{\alpha^M}\right)' = -\frac{1}{\alpha^{M+1}} \textstyle\sum_1^M R_k = \frac{1}{1-M}\left(\frac{1}{\alpha^M}\sum_1^{M-1} R_k\right)'$$

provided $M \geqq 2$.

Integrating between $[\alpha_M, \alpha_{M+1})$, we have

$$\frac{R_{M+1}(\alpha_{M+1})}{\alpha_{M+1}^M} - \frac{R_M(\alpha_M)}{\alpha_M^{\;M}} = \int_{\alpha_M}^{\alpha_{M+1}} \left(\frac{g}{\alpha^M}\right)' = \frac{1}{M-1} \int_{\alpha_M}^{\alpha_{M+1}} \left(\frac{1}{\alpha^M} \textstyle\sum_1^{M-1} R_k(\alpha)\right)'$$
$$= \frac{1}{M-1}\left[\frac{1}{\alpha_{M+1}^M} \textstyle\sum_1^{M-1} R_k(\alpha_{M+1}) - \frac{1}{\alpha_M^{\;M}} \sum_1^{M-1} R_k(\alpha_M)\right].$$

Equivalently:

$$H_M(\alpha_M) = G_M(\alpha_{M+1}) . \tag{5.3}$$

This holds for $M = 1$ also. Now $v = g(0) = g(\alpha_1) = R_1(\alpha_1)$, so v can be determined if the recursion (5.3), together with the condition $\alpha_M \uparrow 1$ leads to an explicit determination of α_1. Let us investigate a class of payoffs for which the sequence $\{\alpha_M\}$ can be determined. Set $\beta > 1$ and define

$$q_\beta(k) = 1 \qquad \text{if} \quad k = 1$$
$$= \frac{\beta(\beta+1), \cdots, (\beta+k-2)}{(k-1)!} \qquad \text{if} \quad k \geqq 2 .$$

Note

$$\frac{q_\beta(k+1)}{q_\beta(k)} = 1 + \frac{\beta-1}{k} = 1 + O\left(\frac{1}{k}\right),$$

and

$$\log q_\beta(k) = \sum_1^{k-1} \log\left(1 + \frac{\beta-1}{l}\right)$$
$$\geqq \sum_1^{k-1} \frac{(\beta-1)}{l} \frac{1}{1 + \frac{\beta-1}{l}}$$
$$= (\beta-1) \sum_1^{k-1} \frac{1}{\beta + (l-1)} \to \infty ,$$

hence $q_\beta(k) \uparrow \infty$. These observations establish that $q \in \mathscr{A}_*$ and $g < \infty$ on $[0, 1)$, consequently that $\alpha_k \to 1$. It is a straightforward calculation to show: $R_1(\alpha) = \alpha^{1-\beta}$ $R_k(\alpha) = q(k)\alpha^{1-\beta}$ $k \geqq 2$. Also $H_N(\alpha) = (\beta-1)\beta^{-1}(N-1)q_\beta(N)\alpha^{-\beta-(N-1)}$ $N \geqq 2$. $G_N(\alpha) = (\beta-1)(N-1)(N\beta)^{-1}(N+\beta)q_\beta(N)\alpha^{-\beta-(N-1)}$ $N \geqq 2$. $H_1(\alpha) = (\beta-1)\beta^{-1}\alpha^{-\beta} + 1/\beta$; $G_N(\alpha) = (\beta - 1/\beta)\alpha^{-\beta} = 1/\beta$ i.e., $G_N(\alpha) = (N+\beta)N^{-1}H_N(\alpha)$ $N \geqq 2$. Consequently

$$\alpha_M = \prod_M^\infty \left(1 + \frac{\beta}{l}\right)^{-1/(\beta+l-1)}$$

with

$$v = R_1(\alpha_1) = \prod_1^\infty \left(1 + \frac{\beta}{l}\right)^{\beta-1(\beta+l-1)} .$$

Note in particular that when $\beta = 2$, $q_\beta(k) = k$, and we get

(5.4)
$$\alpha_1 = \prod_1^\infty \left(1 + \frac{2}{l}\right)^{1(l+1)},$$
$$v = \prod_1^\infty \left(1 + \frac{2}{l}\right)^{1(l+1)} .$$

Equations (5.4) were derived in [2] by a different method.

BIBLIOGRAPHY

[1] Breiman, L. (1964). Stopping rule problems. *Applied Combinatorial Mathematics* (E. Beckenbach, ed.). Wiley, New York.

[2] Chow, Y. S., Moriguti, S., Robbins, H., and Samuels, E. (1964). Optimum selection based on relative rank. (The 'Secretary Problem'). *Israel J. Math.* **2** 81–90.

[3] Coddington, E. and Levison, N. (1955). *Theory of Ordinary Differential Equations.* McGraw-Hill, New York. Chapter 1.

[4] DYNKIN, E. B. and YUSHKEVICH, A. A. (1969). *Markov Processes; Theorems and Problems.* Plenum Press, New York. Chapter 3.

[5] GILBERT, J. P. and MOSTELLER, F. (1966). Recognizing the maximum of a sequence. *J. Amer. Statist. Assoc.* **61** 35–73.

[6] GUSEIN-ZADE, S. M. (1966). The problem of choice and the optimal stopping rule for a sequence of independent trials. *Theor. Probability Appl.* **11** 472–476.

[7] LINDLEY, D. V. (1961). Dynamic programming and decision theory. *Appl. Statist.* **10** 39–51.

[8] MUCCI, A. G. (1971a). Differential equations related to optimal selection problems. Unpublished thesis.

[9] MUCCI, A. G. (1971b). Differential equations and optimal choice problems. *Technical Report, University of Maryland, Department of Mathematics, TR 71-54.*

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND