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On a Class of Solvable Higher-Order Difference Equations

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Abstract. Closed form formulas for well-defined solutions of the next difference equation

$$x_n = \frac{x_{n-2}x_{n-k-2}}{x_{n-k}(a_n + b_n x_{n-2}x_{n-k-2})}, \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, and initial values x_{-i}, $i = \overline{1, k+2}$ are real numbers, are given. Long-term behavior of well-defined solutions of the equation when $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant sequences is described in detail by using the formulas. We also describe the domain of undefinable solutions of the equation. Our results explain and considerably improve some recent results in the literature.

1. Introduction and Preliminaries

Studying nonlinear difference equations is an area of a great recent interest (see, e.g. [1]-[45] and the references therein). Since the publication of paper [24], which explains closed form solution to the second-order difference equation in [9], have appeared considerable number of papers on solvable difference equations (see, e.g., [1]-[4], [7], [8], [10], [22], [25], [27], [28], [30]-[45] and the related references therein). Some classical methods for solving difference equations and systems can be found in [19]. Using a method similar to the one in [24], in [44] were found the closed form formulas for well-defined solutions to the following difference equation

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}(a + b x_n x_{n-k})}, \quad n \in \mathbb{N}_0, \tag{1}$$

where $k \in \mathbb{N}$, a, $b \in \mathbb{R}$, and initial values x_{-i}, $i = \overline{0, k}$ are real numbers, and studied behavior of its well-defined solutions. In [42], among others, was noted that the following generalization of equation (1) can be solved similarly

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}(a_n + b_n x_n x_{n-k})}, \quad n \in \mathbb{N}_0, \tag{2}$$

where $k \in \mathbb{N}$, and sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, as well as initial values x_{-i}, $i = \overline{0, k}$ are real numbers.

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A natural problem is to study difference equations related to equation (2). The problem is not so technically easy, since the behavior of solutions to the equations heavily depends on delays and the initial values, and formulas are represented in a complicated way.

We will consider here the following class of difference equations

$$x_n = \frac{x_{n-2}x_{n-k-2}}{x_{n-k}(a_n + b_n x_{n-2}x_{n-k-2})}, \quad n \in \mathbb{N}_0, \tag{3}$$

where $k \in \mathbb{N}$, which is an extension of the equation in [10]. Our results theoretically explain and considerably improve the results in [10].

Assume first that k is even, that is, $k = 2k_1$ for some $k_1 \in \mathbb{N}$. Since every $n \in \mathbb{N}_0$ can be written in the form $n = 2m + i$ for some $m \in \mathbb{N}_0$ and $i \in \{0, 1\}$, we see that for such k, equation (3) can be written as follows

$$x_{2m+i} = \frac{x_{2(m-1)+i}x_{2(m-k_1-1)+i}}{x_{2(m-k_1)+i}(a_{2m+i} + b_{2m+i}x_{2(m-1)+i}x_{2(m-k_1-1)+i})}, \quad m \in \mathbb{N}_0, \tag{4}$$

$i \in \{0, 1\}$, which means that the sequences $(x_{2m+i})_{m\in\mathbb{N}_0}$, $i \in \{0, 1\}$, are respectively solutions to the following two difference equations

$$z_m = \frac{z_{m-1}z_{m-k_1-1}}{z_{m-k_1}(\hat{a}^i_m + \hat{b}^i_m z_{m-1}z_{m-k_1-1})}, \quad m \in \mathbb{N}_0, \tag{5}$$

where $\hat{a}^i_m = a_{2m+i}$, $\hat{b}^i_m = b_{2m+i}$, $i \in \{0, 1\}$. However, two equations in (5) are special cases of equation (2), which implies that the long-term behavior of their solutions essentially follows from the corresponding one of equation (2).

Hence, from now on we will assume that k is an odd positive number, that is, $k = 2t + 1$ for some $t \in \mathbb{N}_0$.

Solution $(x_n)_{n\ge -s}$, $s \in \mathbb{N}$, of the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-s}), \quad n \in \mathbb{N}_0, \tag{6}$$

where $f : \mathbb{R}^s \to \mathbb{R}$ and $s \in \mathbb{N}$, is called eventually periodic with period p, if there is an $n_1 \ge -s$ such that

$$x_{n+p} = x_n, \quad \text{for} \quad n \ge n_1.$$

It is called periodic with period p, if $n_1 = -s$. For some results in this area see, e.g. [6, 12, 16–18, 20, 21, 26, 29] and the references therein.

We now formulate an auxiliary result which will be used frequently throughout the paper. Since the statements in it are well-known we will not prove them.

Lemma 1. *Let $l \in \mathbb{N}_0$, $(a_n)_{n\ge l}$ be a real sequence such that $a_n \ne 0$, $n \ge l$, and*

$$P_n = \prod_{j=l}^{n} a_j, \quad n \ge l.$$

Then the following statements are true.

(a) *If $\limsup_{n\to\infty} |a_n| < 1$, then $P_n \to 0$ as $n \to +\infty$.*

(b) *If $\liminf_{n\to\infty} |a_n| > 1$, then $|P_n| \to +\infty$ as $n \to +\infty$.*

(c) *If $|q| < 1$ and*

$$a_n = 1 + O(q^n)$$

for sufficiently large n, then the sequence $(P_n)_{n\ge l}$ is convergent.

(d) *If*

$$a_n = 1 + \frac{c}{n} + O\Big(\frac{1}{n^{1+\delta}}\Big)$$

for some $\delta > 0$ and sufficiently large n, then:

1) *if $c < 0$, then $P_n \to 0$ as $n \to +\infty$;*
2) *if $c > 0$, then $|P_n| \to +\infty$ as $n \to +\infty$;*
3) *if $c = 0$, then the sequence $(P_n)_{n\ge l}$ is convergent.*

2. Solutions to Equation (3)

If $x_{-i} = 0$ for some $i \in \{1, 2, \ldots, k+2\}$, then from (3) we see that x_{k+2-i} is or not defined or equal to zero. If the later holds then (3) shows that x_{2k+2-i} is not defined. On the other hand, if $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$, and x_n are defined and different from zero for $0 \le n \le n_0 - 1$, then by using again (3) we have that $x_{n_0-2} = 0$ or $x_{n_0-k-2} = 0$, which along with the choice of number n_0 implies $n_0 - 2 < 0$ or $n_0 - k - 2 < 0$ respectively, that is, $x_{-i} = 0$ for some $i \in \{1, 2, \ldots, k+2\}$. This fact along with the previous consideration implies that the solution is not defined.

Hence, from now on in this section, we may assume that $x_{-i} \ne 0$ for every $i \in \{1, 2, \ldots, k+2\}$, which is equivalent to

$$x_n \ne 0 \quad \text{for} \quad n \ge -(k+2).$$

Thus we can use the change of variables

$$y_n = \frac{1}{x_n x_{n-k}}, \quad n \ge -2, \tag{7}$$

and transform equation (3) into the following nonhomogeneous linear second-order difference equation

$$y_n = a_n y_{n-2} + b_n, \quad n \in \mathbb{N}_0. \tag{8}$$

Since for $n \ge -2$ we have $n = 2m + i$, for some $m \ge -1$ and $i \in \{0, 1\}$, equation (8) can be written as

$$y_{2m+i} = a_{2m+i} y_{2(m-1)+i} + b_{2m+i}, \quad m \in \mathbb{N}_0, \tag{9}$$

where $i \in \{0, 1\}$.

Thus, $(y_{2m+i})_{m \ge -1}$, $i \in \{0, 1\}$, are respectively solutions to the next linear first-order equations

$$z_m = a_{2m+i} z_{m-1} + b_{2m+i}, \quad m \in \mathbb{N}_0, \tag{10}$$

for $i \in \{0, 1\}$.

Equations in (10) are solvable. Using the formulas for their solutions it is easy to see that the general solutions to the equations in (9) are

$$y_{2m+i} = y_{i-2} \prod_{j=0}^{m} a_{2j+i} + \sum_{l=0}^{m} b_{2l+i} \prod_{j=l+1}^{m} a_{2j+i}, \quad m \in \mathbb{N}_0, \tag{11}$$

$i \in \{0, 1\}$.

From (7) it follows that

$$x_n = \frac{1}{y_n x_{n-k}} = \frac{y_{n-k}}{y_n} x_{n-2k},$$

for $n \ge k - 2$, and consequently

$$x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{y_{(2j-1)k+i}}{y_{2jk+i}}, \tag{12}$$

for $m \in \mathbb{N}_0$ and $i \in \{k-2, k-1, \ldots, 3k-3\}$.

Hence, if $k = 2t + 1$ we have

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)} \prod_{j=0}^{m} \frac{y_{2(j(2t+1)+s-t-1)+1}}{y_{2(j(2t+1)+s)}}, \tag{13}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \ldots, 3k-3\}$, and

$$x_{2(2t+1)m+2s+1} = x_{2s-2(2t+1)+1} \prod_{j=0}^{m} \frac{y_{2(j(2t+1)+s-t)}}{y_{2(j(2t+1)+s)+1}}, \tag{14}$$

for $m \in \mathbb{N}_0$ and $2s + 1 \in \{k-2, k-1, \ldots, 3k-3\}$.

Employing (11) into (13) and (14) we obtain

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)} \prod_{p=0}^{m} \frac{y_{-1} \prod_{j=0}^{p(2t+1)+s-t-1} a_{2j+1} + \sum_{l=0}^{p(2t+1)+s-t-1} b_{2l+1} \prod_{j=l+1}^{p(2t+1)+s-t-1} a_{2j+1}}{y_{-2} \prod_{j=0}^{p(2t+1)+s} a_{2j} + \sum_{l=0}^{p(2t+1)+s} b_{2l} \prod_{j=l+1}^{p(2t+1)+s} a_{2j}}, \tag{15}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$x_{2(2t+1)m+2s+1} = x_{2s-2(2t+1)+1} \prod_{p=0}^{m} \frac{y_{-2} \prod_{j=0}^{p(2t+1)+s-t} a_{2j} + \sum_{l=0}^{p(2t+1)+s-t} b_{2l} \prod_{j=l+1}^{p(2t+1)+s-t} a_{2j}}{y_{-1} \prod_{j=0}^{p(2t+1)+s} a_{2j+1} + \sum_{l=0}^{p(2t+1)+s} b_{2l+1} \prod_{j=l+1}^{p(2t+1)+s} a_{2j+1}}, \tag{16}$$

for $m \in \mathbb{N}_0$ and $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

3. Case of Constant Coefficients

In this section we study equation (3) for the case when

$$a_n = a, \qquad b_n = b, \qquad n \in \mathbb{N}_0,$$

where a and b are some real constants.

In this case equation (3) becomes

$$x_n = \frac{x_{n-2}x_{n-k-2}}{x_{n-k}(a + bx_{n-2}x_{n-k-2})}, \quad n \in \mathbb{N}_0. \tag{17}$$

From (15) and (16) we have

$$\begin{aligned} x_{2(2t+1)m+2s} &= x_{2s-2(2t+1)} \prod_{p=0}^{m} \frac{y_{-1} \prod_{j=0}^{p(2t+1)+s-t-1} a + \sum_{l=0}^{p(2t+1)+s-t-1} b \prod_{j=l+1}^{p(2t+1)+s-t-1} a}{y_{-2} \prod_{j=0}^{p(2t+1)+s} a + \sum_{l=0}^{p(2t+1)+s} b \prod_{j=l+1}^{p(2t+1)+s} a} \\ &= x_{2s-2(2t+1)} \prod_{p=0}^{m} \frac{y_{-1}a^{p(2t+1)+s-t} + b\sum_{l=0}^{p(2t+1)+s-t-1} a^{p(2t+1)+s-t-1-l}}{y_{-2}a^{p(2t+1)+s+1} + b\sum_{l=0}^{p(2t+1)+s} a^{p(2t+1)+s-l}}, \end{aligned} \tag{18}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$\begin{aligned} x_{2(2t+1)m+2s+1} &= x_{2s-2(2t+1)+1} \prod_{p=0}^{m} \frac{y_{-2} \prod_{j=0}^{p(2t+1)+s-t} a + \sum_{l=0}^{p(2t+1)+s-t} b \prod_{j=l+1}^{p(2t+1)+s-t} a}{y_{-1} \prod_{j=0}^{p(2t+1)+s} a + \sum_{l=0}^{p(2t+1)+s} b \prod_{j=l+1}^{p(2t+1)+s} a} \\ &= x_{2s-2(2t+1)+1} \prod_{p=0}^{m} \frac{y_{-2}a^{p(2t+1)+s-t+1} + b\sum_{l=0}^{p(2t+1)+s-t} a^{p(2t+1)+s-t-l}}{y_{-1}a^{p(2t+1)+s+1} + b\sum_{l=0}^{p(2t+1)+s} a^{p(2t+1)+s-l}}, \end{aligned} \tag{19}$$

for $m \in \mathbb{N}_0$ and $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

Case $a \neq 1$. We have

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)} \prod_{p=0}^{m} \frac{y_{-1}(1-a)a^{p(2t+1)+s-t} + b(1-a^{p(2t+1)+s-t})}{y_{-2}(1-a)a^{p(2t+1)+s+1} + b(1-a^{p(2t+1)+s+1})}, \tag{20}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$x_{2(2t+1)m+2s+1} = x_{2s-2(2t+1)+1} \prod_{p=0}^{m} \frac{y_{-2}(1-a)a^{p(2t+1)+s-t+1} + b(1-a^{p(2t+1)+s-t+1})}{y_{-1}(1-a)a^{p(2t+1)+s+1} + b(1-a^{p(2t+1)+s+1})}, \tag{21}$$

for $m \in \mathbb{N}_0$ and $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

Case $a = 1$. We have

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)} \prod_{p=0}^{m} \frac{y_{-1} + b(p(2t+1) + s - t)}{y_{-2} + b(p(2t+1) + s + 1)}, \tag{22}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$x_{2(2t+1)m+2s+1} = x_{2s-2(2t+1)+1} \prod_{p=0}^{m} \frac{y_{-2} + b(p(2t+1) + s - t + 1)}{y_{-1} + b(p(2t+1) + s + 1)}, \tag{23}$$

for $m \in \mathbb{N}_0$ and $2s + 1 \in \{k-2, k-1, \dots, 3k-3\}$.

4. Long-Term Behavior of Solutions to Equation (17)

Long-term behavior of well-defined solutions to equation (17) will be presented here, in terms of parameters a, b, k and some initial values. Before we formulate our first result we introduce the following notation

$$L_{k,0} = \frac{(1-a)(x_{-1}x_{-k-1})^{-1} - b}{a^{[k/2]+1}((1-a)(x_{-2}x_{-k-2})^{-1} - b)},$$
$$L_{k,1} = \frac{(1-a)(x_{-2}x_{-k-2})^{-1} - b}{a^{[k/2]}((1-a)(x_{-1}x_{-k-1})^{-1} - b)},$$
$$M_{k,0} = \frac{(x_{-1}x_{-k-1})^{-1} - (x_{-2}x_{-k-2})^{-1} - b[k/2] - b}{bk},$$
$$M_{k,1} = \frac{(x_{-2}x_{-k-2})^{-1} - (x_{-1}x_{-k-1})^{-1} - b[k/2]}{bk}.$$

Now we formulate and prove the main results in this section. For the brevity, we will write y_{-2} and y_{-1} instead of $(x_{-2}x_{-k-2})^{-1}$ and $(x_{-1}x_{-k-1})^{-1}$, and will also use the following notation $t = [k/2]$.

4.1. *Case* $a \neq -1, b \neq 0$

Our first result considers the case $a \neq -1$, $b \neq 0$.

Theorem 1. *Assume that* $a \neq -1$, $b \neq 0$, k *is an odd natural number, and* $(x_n)_{n \geq -k-2}$ *is a well-defined solution to equation* (17). *Then the following statements are true.*

(a) *If* $|a| > 1$ *and* $|L_{k,i}| < 1$, *for some* $i \in \{0,1\}$, *then* $x_{2km+2s+i} \to 0$ *as* $m \to +\infty$, *for every* $2s+i \in \{k-2, k-1, \dots, 3k-3\}$.
(b) *If* $|a| > 1$ *and* $|L_{k,i}| > 1$, *for some* $i \in \{0,1\}$, *then* $|x_{2km+2s+i}| \to +\infty$ *as* $m \to +\infty$, *for every* $2s+i \in \{k-2, k-1, \dots, 3k-3\}$.
(c) *If* $|a| > 1$ *and* $y_{-1} = b/(1-a) \neq y_{-2}$, *then* $x_{2km+2s} \to 0$ *as* $m \to +\infty$, *for every* $2s \in \{k-2, k-1, \dots, 3k-3\}$, *and* $|x_{2km+2s+1}| \to +\infty$ *as* $m \to +\infty$, *for every* $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.
(d) *If* $|a| > 1$ *and* $y_{-2} = b/(1-a) \neq y_{-1}$, *then* $x_{2km+2s+1} \to 0$ *as* $m \to +\infty$, *for every* $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$ *and* $|x_{2km+2s}| \to +\infty$ *as* $m \to +\infty$, *for every* $2s \in \{k-2, k-1, \dots, 3k-3\}$.
(e) *If* $|a| > 1$ *and* $L_{k,i} = 1$, *for some* $i \in \{0,1\}$, *then the sequences* $(x_{2km+2s+i})_{m \geq -1}$ *are constant, for every* $2s+i \in \{k-2, k-1, \dots, 3k-3\}$.
(f) *If* $|a| > 1$ *and* $L_{k,i} = -1$, *for some* $i \in \{0,1\}$, *then the sequences* $(x_{4km+2s+i})_{m \in \mathbb{N}_0}$ *and* $(x_{4km+2k+2s+i})_{m \geq -1}$ *are convergent, for every* $2s+i \in \{k-2, k-1, \dots, 3k-3\}$.
(g) *If* $|a| < 1$, *then the sequences* $(x_{2km+j})_{m \geq -1}$, $j \in \{k-2, k-1, \dots, 3k-3\}$, *are convergent.*

(h) *If* $a = 1$ *and* $M_{k,i} < 0$*, for some* $i \in \{0, 1\}$*, then* $x_{2km+2s+i} \to 0$ *as* $m \to +\infty$*, for every* $2s+i \in \{k-2, k-1, \dots, 3k-3\}$.

(i) *If* $a = 1$ *and* $M_{k,i} > 0$*, for some* $i \in \{0, 1\}$*, then* $|x_{2km+2s+i}| \to +\infty$ *as* $m \to +\infty$*, for every* $2s + i \in \{k-2, k-1, \dots, 3k-3\}$.

(j) *If* $a = 1$ *and* $M_{k,i} = 0$*, for some* $i \in \{0, 1\}$*, then the sequence* $(x_{2km+2s+i})_{m\geq -1}$ *is constant, for every* $2s + i \in \{k-2, k-1, \dots, 3k-3\}$.

(k) *If* $y_{-1} = b/(1-a) = y_{-2}$*, then the sequence* $(x_n)_{n\geq -k-2}$ *is* $2k$*-periodic.*

Proof. (a), (b) Let

$$p_m^{t,2s} = \frac{(y_{-1}(1-a)-b)a^{m(2t+1)+s-t}+b}{(y_{-2}(1-a)-b)a^{m(2t+1)+s+1}+b}, \tag{24}$$

for $2s \in \{k-2, k-1, \dots, 3k-3\}$,

$$p_m^{t,2s+1} = \frac{(y_{-2}(1-a)-b)a^{m(2t+1)+s-t+1}+b}{(y_{-1}(1-a)-b)a^{m(2t+1)+s+1}+b}, \tag{25}$$

for $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

By using the condition $|a| > 1$, we have

$$\lim_{m\to+\infty} p_m^{t,2s} = \lim_{m\to+\infty} \frac{(y_{-1}(1-a)-b)+\frac{b}{a^{m(2t+1)+s-t}}}{(y_{-2}(1-a)-b)a^{t+1}+\frac{b}{a^{m(2t+1)+s-t}}} = L_{k,0} \tag{26}$$

for $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$\lim_{m\to+\infty} p_m^{t,2s+1} = \lim_{m\to+\infty} \frac{(y_{-2}(1-a)-b)+\frac{b}{a^{m(2t+1)+s-t+1}}}{(y_{-1}(1-a)-b)a^{t}+\frac{b}{a^{m(2t+1)+s-t+1}}} = L_{k,1}, \tag{27}$$

for $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

From (20), (21), (26), (27), and by using statements (a) and (b) in Lemma 1, these two statements easily follow.

(c) First, note that in this case

$$p_m^{t,2s} = \frac{b}{(y_{-2}(1-a)-b)a^{m(2t+1)+s+1}+b}, \tag{28}$$

for $2s \in \{k-2, k-1, \dots, 3k-3\}$,

$$p_m^{t,2s+1} = \frac{(y_{-2}(1-a)-b)a^{m(2t+1)+s-t+1}+b}{b}, \tag{29}$$

for $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

From (28), (29), and the conditions $|a| > 1$ and $y_{-2} \neq b/(1-a)$, we have

$$\lim_{m\to+\infty} p_m^{t,2s} = \lim_{m\to+\infty} \frac{b}{(y_{-2}(1-a)-b)a^{m(2t+1)+s+1}+b} = 0 \tag{30}$$

for $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$\lim_{m\to+\infty} |p_m^{t,2s+1}| = \lim_{m\to+\infty} \left|\frac{(y_{-2}(1-a)-b)a^{m(2t+1)+s-t+1}+b}{b}\right| = +\infty, \tag{31}$$

for $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

From (20), (21), (30), (31), and by using statements (a) and (b) in Lemma 1, the result easily follows.

(d) First, note that in this case

$$p_m^{t,2s} = \frac{(y_{-1}(1-a)-b)a^{m(2t+1)+s-t}+b}{b}, \tag{32}$$

for $2s \in \{k-2, k-1, \dots, 3k-3\}$,

$$p_m^{t,2s+1} = \frac{b}{(y_{-1}(1-a)-b)a^{m(2t+1)+s+1}+b}, \tag{33}$$

for $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

From (32), (33), and the conditions $|a| > 1$ and $y_{-1} \neq b/(1-a)$, we have

$$\lim_{m\to+\infty} |p_m^{t,2s}| = \lim_{m\to+\infty} \left|\frac{(y_{-1}(1-a)-b)a^{m(2t+1)+s-t}+b}{b}\right| = +\infty \tag{34}$$

for $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$\lim_{m\to+\infty} p_m^{t,2s+1} = \lim_{m\to+\infty} \frac{b}{(y_{-1}(1-a)-b)a^{m(2t+1)+s+1}+b} = 0, \tag{35}$$

for $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

From (20), (21), (34), (35), and by using statements (a) and (b) in Lemma 1, the result easily follows.

(e) In this case, we have that

$$p_m^{t,2s} = \frac{(y_{-1}(1-a)-b)a^{m(2t+1)+s-t}+b}{(y_{-2}(1-a)-b)a^{m(2t+1)+s+1}+b} = 1, \tag{36}$$

for every $m \in \mathbb{N}_0$ if $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$p_m^{t,2s+1} = \frac{(y_{-2}(1-a)-b)a^{m(2t+1)+s-t+1}+b}{(y_{-1}(1-a)-b)a^{m(2t+1)+s+1}+b} = 1, \tag{37}$$

every $m \in \mathbb{N}_0$ if $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

From (20), (21), (36) and (37) the result easily follows.

(f) By using the asymptotic relation

$$(1+x)^{-1} = 1 - x + O(x^2), \tag{38}$$

when x is in a neighborhood of zero, we have that

$$\begin{aligned} p_m^{t,2s} &= \frac{-(y_{-2}(1-a)-b)a^{m(2t+1)+s+1}+b}{(y_{-2}(1-a)-b)a^{m(2t+1)+s+1}+b} \\ &= -\left(1 - \frac{b}{a^{m(2t+1)+s+1}(y_{-2}(1-a)-b)}\right)\left(1 - \frac{b}{a^{m(2t+1)+s+1}(y_{-2}(1-a)-b)} + O\left(\frac{1}{a^{2m(2t+1)}}\right)\right) \\ &= -\left(1 - \frac{2b}{a^{m(2t+1)+s+1}(y_{-2}(1-a)-b)} + O\left(\frac{1}{a^{2m(2t+1)}}\right)\right), \end{aligned} \tag{39}$$

for sufficiently large m if $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$\begin{aligned} p_m^{t,2s+1} &= \frac{-(y_{-1}(1-a)-b)a^{m(2t+1)+s+1}+b}{(y_{-1}(1-a)-b)a^{m(2t+1)+s+1}+b} \\ &= -\left(1 - \frac{b}{a^{m(2t+1)+s+1}(y_{-1}(1-a)-b)}\right)\left(1 - \frac{b}{a^{m(2t+1)+s+1}(y_{-1}(1-a)-b)} + O\left(\frac{1}{a^{2m(2t+1)}}\right)\right) \\ &= -\left(1 - \frac{2b}{a^{m(2t+1)+s+1}(y_{-1}(1-a)-b)} + O\left(\frac{1}{a^{2m(2t+1)}}\right)\right), \end{aligned} \tag{40}$$

for sufficiently large m if $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

From (20), (21), (39), (40), the assumption $|a| > 1$, and by using statement (c) in Lemma 1, the result easily follows.

(g) By using (38), we have

$$p_m^{t,2s} = \frac{1 + (y_{-1}(1-a) - b)a^{m(2t+1)+s-t}/b}{1 + (y_{-2}(1-a) - b)a^{m(2t+1)+s+1}/b}$$
$$= 1 + a^{m(2t+1)+s-t}\frac{((y_{-1}(1-a) - b) - (y_{-2}(1-a) - b)a^{t+1})}{b} + O(a^{2m(2t+1)}), \tag{41}$$

for sufficiently large m, if $2s \in \{k-2, k-1, \dots, 3k-3\}$,

$$p_m^{t,2s+1} = \frac{1 + (y_{-2}(1-a) - b)a^{m(2t+1)+s-t+1}/b}{1 + (y_{-1}(1-a) - b)a^{m(2t+1)+s+1}/b}$$
$$= 1 + a^{m(2t+1)+s+1}((y_{-2}(1-a) - b)a^{-t} - (y_{-1}(1-a) - b)) + O(a^{2m(2t+1)}), \tag{42}$$

for sufficiently large m, if $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$. From (20), (21), (41), (42), the assumption $|a| < 1$, and by using statement (c) in Lemma 1, the result easily follows.

(h)-(j) Let

$$q_m^{t,2s} = \frac{bm(2t+1) + b(s-t) + y_{-1}}{bm(2t+1) + b(s+1) + y_{-2}}, \tag{43}$$

for $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$q_m^{t,2s+1} = \frac{bm(2t+1) + b(s-t+1) + y_{-2}}{bm(2t+1) + b(s+1) + y_{-1}}, \tag{44}$$

for $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

Then by using (38), we have that

$$q_m^{t,2s} = \frac{1 + \frac{b(s-t)+y_{-1}}{bm(2t+1)}}{1 + \frac{b(s+1)+y_{-2}}{bm(2t+1)}} = \left(1 + \frac{b(s-t) + y_{-1}}{bm(2t+1)}\right)\left(1 - \frac{b(s+1) + y_{-2}}{bm(2t+1)} + O\left(\frac{1}{m^2}\right)\right)$$
$$= 1 + \frac{y_{-1} - y_{-2} - bt - b}{bm(2t+1)} + O\left(\frac{1}{m^2}\right) = 1 + \frac{M_{k,0}}{m} + O\left(\frac{1}{m^2}\right), \tag{45}$$

for sufficiently large m if $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$q_m^{t,2s+1} = \frac{1 + \frac{b(s-t+1)+y_{-2}}{bm(2t+1)}}{1 + \frac{b(s+1)+y_{-1}}{bm(2t+1)}} = \left(1 + \frac{b(s-t+1) + y_{-2}}{bm(2t+1)}\right)\left(1 - \frac{b(s+1) + y_{-1}}{bm(2t+1)} + O\left(\frac{1}{m^2}\right)\right)$$
$$= \left(1 + \frac{y_{-2} - y_{-1} - bt}{bm(2t+1)} + O\left(\frac{1}{m^2}\right)\right) = 1 + \frac{M_{k,1}}{m} + O\left(\frac{1}{m^2}\right), \tag{46}$$

for sufficiently large m if $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

From (22), (23), (45), (46), and by using statement (d) in Lemma 1, these three statements easily follow.

(k) From (24), (25) and $y_{-1} = b/(1-a) = y_{-2}$, we have that

$$p_m^{t,2s} = 1, \tag{47}$$

for $2s \in \{k-2, k-1, \dots, 3k-3\}$,

$$p_m^{t,2s+1} = 1, \tag{48}$$

for $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$. From (20), (21), (47), (48), the $2k$-periodicity of the sequence $(x_n)_{n \ge -k-2}$ follows. □

4.2. Case $a = -1$, $b \neq 0$

Now we will consider the case $a = -1$, $b \neq 0$, in detail, by using the following formulas

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)} \prod_{p=0}^{m} \frac{(2y_{-1} - b)(-1)^{p(2t+1)+s-t} + b}{(2y_{-2} - b)(-1)^{p(2t+1)+s+1} + b}, \tag{49}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \dots, 3k-3\}$, and

$$x_{2(2t+1)m+2s+1} = x_{2s-2(2t+1)+1} \prod_{p=0}^{m} \frac{(2y_{-2} - b)(-1)^{p(2t+1)+s-t+1} + b}{(2y_{-1} - b)(-1)^{p(2t+1)+s+1} + b}, \tag{50}$$

for $m \in \mathbb{N}_0$ and $2s + 1 \in \{k-2, k-1, \dots, 3k-3\}$, which are obtained from (20) and (21), with $a = -1$.

Let

$$N := \frac{y_{-1}(y_{-1} - b)}{y_{-2}(y_{-2} - b)}.$$

Theorem 2. *Assume that $a = -1$, $b \neq 0$, k is an odd natural number, and $(x_n)_{n \geq -k-2}$ is a well-defined solution to equation* (17)*. Then the following statements are true.*

(a) *If $y_{-1} = b/2 = y_{-2}$, then the sequence $(x_n)_{n \geq -k-2}$ is $2k$-periodic.*

(b) *If $y_{-1} = b/2 \neq y_{-2}$, and $(-\sqrt{2}+1)/2 < y_{-2}/b < (\sqrt{2}+1)/2$, then $|x_{2mk+2s}| \to +\infty$ as $m \to +\infty$, for every $2s \in \{k-2, k-1, \dots, 3k-3\}$.*

(c) *If $y_{-1} = b/2 \neq y_{-2}$, and $y_{-2}/b < (-\sqrt{2}+1)/2$ or $y_{-2}/b > (\sqrt{2}+1)/2$, then $x_{2mk+2s} \to 0$ as $m \to \infty$, for every $2s \in \{k-2, k-1, \dots, 3k-3\}$.*

(d) *If $y_{-1} = b/2$, and $y_{-2} = b(-\sqrt{2}+1)/2$ or $y_{-2} = b(\sqrt{2}+1)/2$, then*

$$x_{2mk+2s} = (-1)^{[\frac{m+1}{2}]}(1 \mp \sqrt{2}(-1)^{s+1})^{\frac{(-1)^{m+1}-1}{2}} x_{2s-2(2t+1)}, \tag{51}$$

for every $2s \in \{k-2, k-1, \dots, 3k-3\}$.

(e) *If $y_{-1} = b/2 \neq y_{-2}$, and $(-\sqrt{2}+1)/2 < y_{-2}/b < (\sqrt{2}+1)/2$, then $x_{2mk+2s+1} \to 0$ as $m \to \infty$, for every $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.*

(f) *If $y_{-1} = b/2 \neq y_{-2}$, and $y_{-2}/b < (-\sqrt{2}+1)/2$ or $y_{-2}/b > (\sqrt{2}+1)/2$, then $|x_{2mk+2s+1}| \to +\infty$ as $m \to \infty$, for every $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.*

(g) *If $y_{-1} = b/2$, and $y_{-2} = b(-\sqrt{2}+1)/2$ or $y_{-2} = b(\sqrt{2}+1)/2$, then*

$$x_{2mk+2s+1} = (-1)^{[\frac{m+1}{2}]}(\pm\sqrt{2}(-1)^{s-t+1} + 1)^{\frac{1+(-1)^m}{2}} x_{2s-2(2t+1)+1}, \tag{52}$$

for every $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

(h) *If $y_{-1} \neq b/2 = y_{-2}$, and $(-\sqrt{2}+1)/2 < y_{-1}/b < (\sqrt{2}+1)/2$, then $x_{2mk+2s} \to 0$ as $m \to \infty$, for every $2s \in \{k-2, k-1, \dots, 3k-3\}$.*

(i) *If $y_{-2} = b/2$, and $y_{-1}/b < (-\sqrt{2}+1)/2$ or $y_{-1}/b > (\sqrt{2}+1)/2$, then $|x_{2mk+2s}| \to +\infty$ as $m \to \infty$, for every $2s \in \{k-2, k-1, \dots, 3k-3\}$.*

(j) *If $y_{-2} = b/2$, and $y_{-1} = b(-\sqrt{2}+1)/2$ or $y_{-1} = b(\sqrt{2}+1)/2$, then*

$$x_{2mk+2s} = (-1)^{[\frac{m+1}{2}]}(1 \mp \sqrt{2}(-1)^{s-t})^{\frac{1+(-1)^m}{2}} x_{2s-2(2t+1)}, \tag{53}$$

for every $2s \in \{k-2, k-1, \dots, 3k-3\}$.

(k) *If $y_{-1} \neq b/2 = y_{-2}$, and $(-\sqrt{2}+1)/2 < y_{-1}/b < (\sqrt{2}+1)/2$, then $|x_{2mk+2s+1}| \to +\infty$ as $m \to \infty$, for every $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.*

(l) *If $y_{-1} \neq b/2 = y_{-2}$, and $y_{-1}/b < (-\sqrt{2}+1)/2$ or $y_{-1}/b > (\sqrt{2}+1)/2$, then $x_{2mk+2s+1} \to 0$ as $m \to \infty$, for every $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.*

(m) *If* $y_{-2} = b/2$, *and* $y_{-1} = b(-\sqrt{2}+1)/2$ *or* $y_{-1} = b(\sqrt{2}+1)/2$, *then*

$$x_{2mk+2s+1} = (-1)^{[\frac{m+1}{2}]}(1 \mp \sqrt{2}(-1)^{s+1})^{\frac{-1+(-1)^{m+1}}{2}} x_{2s-2(2t+1)+1}, \tag{54}$$

for every $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

(n) *If* $y_{-1} \neq b/2 \neq y_{-2}$, *and* $|N| < 1$, *then* $x_{2mk+2s} \to 0$, *as* $m \to \infty$, *for every* $2s \in \{k-2, k-1, \dots, 3k-3\}$, *and* $|x_{2mk+2s+1}| \to +\infty$, *as* $m \to \infty$, *for every* $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

(o) *If* $y_{-1} \neq b/2 \neq y_{-2}$, *and* $|N| > 1$, *then* $|x_{2mk+2s}| \to +\infty$, *as* $m \to \infty$, *for every* $2s \in \{k-2, k-1, \dots, 3k-3\}$, *and* $x_{2mk+2s+1} \to 0$, *as* $m \to \infty$, *for every* $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$.

(p) *If* $y_{-1} \neq b/2 \neq y_{-2}$, *and* $N = 1$, *then the sequence* $(x_n)_{n \geq -k-2}$ *is* $4k$*-periodic.*

(q) *If* $y_{-1} \neq b/2 \neq y_{-2}$, *and* $N = -1$, *then the sequence* $(x_n)_{n \geq -k-2}$ *is* $8k$*-periodic.*

Proof. (a) This statement follows directly from (49) and (50), and can be also regarded as a special case of Theorem 1 *(k)*.

(b) From (49) we have that

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)} \prod_{p=0}^{m} \frac{1}{(2y_{-2}/b-1)(-1)^{p(2t+1)+s+1}+1}, \tag{55}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \dots, 3k-3\}$, which implies

$$\begin{aligned} x_{4(2t+1)m+2s} &= x_{2s-2(2t+1)} \prod_{p=0}^{2m} \frac{1}{(2y_{-2}/b-1)(-1)^{p(2t+1)+s+1}+1} \\ &= \frac{x_{2s-2(2t+1)}}{(1-(2y_{-2}/b-1)^2)^m((2y_{-2}/b-1)(-1)^{s+1}+1)}, \end{aligned} \tag{56}$$

and

$$\begin{aligned} x_{4(2t+1)m+4t+2+2s} &= x_{2s-2(2t+1)} \prod_{p=0}^{2m+1} \frac{1}{(2y_{-2}/b-1)(-1)^{p(2t+1)+s+1}+1} \\ &= \frac{x_{2s-2(2t+1)}}{(1-(2y_{-2}/b-1)^2)^{m+1}}. \end{aligned} \tag{57}$$

Now note that conditions, $y_{-2} \neq b/2$ and $(-\sqrt{2}+1)/2 < y_{-2}/b < (\sqrt{2}+1)/2$ are equivalent to $0 < (2y_{-2}/b-1)^2 < 2$, i.e.

$$|(2y_{-2}/b-1)^2 - 1| < 1.$$

From this, (56) and (57), the result easily follows.

(c) Note that the conditions $y_{-2} \neq b/2$, and $y_{-2}/b < (-\sqrt{2}+1)/2$ or $y_{-2}/b > (\sqrt{2}+1)/2$, are equivalent to $(2y_{-2}/b-1)^2 > 2$, that is,

$$(2y_{-2}/b-1)^2 - 1 > 1.$$

From this fact, (56) and (57), the result easily follows.

(d) Note that the conditions $y_{-2} = b(-\sqrt{2}+1)/2$ or $y_{-2} = b(\sqrt{2}+1)/2$, are equivalent to $(2y_{-2}/b-1)^2 = 2$. From this (56) and (57) we have that

$$x_{4(2t+1)m+2s} = \frac{x_{2s-2(2t+1)}}{(-1)^m(\mp\sqrt{2}(-1)^{s+1}+1)}, \tag{58}$$

and

$$x_{4(2t+1)m+4t+2+2s} = \frac{x_{2s-2(2t+1)}}{(-1)^{m+1}}. \tag{59}$$

From (58) and (59), formula (51) easily follows.

(e) From (50) we have that

$$x_{2(2t+1)m+2s+1} = x_{2s-2(2t+1)+1} \prod_{p=0}^{m} \Big((2y_{-2}/b-1)(-1)^{p(2t+1)+s-t+1}+1\Big), \tag{60}$$

for $m \in \mathbb{N}_0$ and $2s+1 \in \{k-2, k-1, \dots, 3k-3\}$, which implies

$$\begin{aligned} x_{4(2t+1)m+2s+1} &= x_{2s-2(2t+1)+1} \prod_{p=0}^{2m} \Big((2y_{-2}/b-1)(-1)^{p(2t+1)+s-t+1}+1\Big) \\ &= x_{2s-2(2t+1)+1}((2y_{-2}/b-1)(-1)^{s-t+1}+1)\Big(1-(2y_{-2}/b-1)^2\Big)^m, \end{aligned} \tag{61}$$

and

$$\begin{aligned} x_{4(2t+1)m+4t+2+2s+1} &= x_{2s-2(2t+1)+1} \prod_{p=0}^{2m+1} \Big((2y_{-2}/b-1)(-1)^{p(2t+1)+s-t+1}+1\Big) \\ &= x_{2s-2(2t+1)+1}\Big(1-(2y_{-2}/b-1)^2\Big)^{m+1}. \end{aligned} \tag{62}$$

According to the proof of (b) we have that $|(2y_{-2}/b-1)^2-1| < 1$. From this, (61) and (62), the result follows.

(f) In this case we have $(2y_{-2}/b-1)^2-1 > 1$. From this, (61) and (62) the result follows.

(g) In this case we have $(2y_{-2}/b-1)^2 = 2$. Using this in (61) and (62) we get

$$x_{4(2t+1)m+2s+1} = x_{2s-2(2t+1)+1}(\mp\sqrt{2}(-1)^{s-t+1}+1)(-1)^m, \tag{63}$$

and

$$x_{4(2t+1)m+4t+2+2s+1} = x_{2s-2(2t+1)+1}(-1)^{m+1}. \tag{64}$$

From (63) and (64), formula (52) easily follows.

(h) From (49) we have that

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)} \prod_{p=0}^{m} \Big((2y_{-1}/b-1)(-1)^{p(2t+1)+s-t}+1\Big), \tag{65}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \dots, 3k-3\}$, which implies

$$\begin{aligned} x_{4(2t+1)m+2s} &= x_{2s-2(2t+1)} \prod_{p=0}^{2m} \Big((2y_{-1}/b-1)(-1)^{p(2t+1)+s-t}+1\Big) \\ &= x_{2s-2(2t+1)}(1-(2y_{-1}/b-1)^2)^m((2y_{-1}/b-1)(-1)^{s-t}+1), \end{aligned} \tag{66}$$

and

$$\begin{aligned} x_{4(2t+1)m+4t+2+2s} &= x_{2s-2(2t+1)} \prod_{p=0}^{2m+1} \Big((2y_{-1}/b-1)(-1)^{p(2t+1)+s-t}+1\Big) \\ &= x_{2s-2(2t+1)}(1-(2y_{-1}/b-1)^2)^{m+1}. \end{aligned} \tag{67}$$

From (66), (67) and since in this case $|(2y_{-1}/b-1)^2-1| < 1$, the result easily follows.

(i) Since in this case $(2y_{-1}/b-1)^2-1 > 1$. From this (66) and (67), the result easily follows.

(j) In this case we have $(2y_{-1}/b-1)^2=2$. From this (66) and (67) we have that

$$x_{4(2t+1)m+2s}=x_{2s-2(2t+1)}(-1)^m(\mp\sqrt{2}(-1)^{s-t}+1), \tag{68}$$

and

$$x_{4(2t+1)m+4t+2+2s}=x_{2s-2(2t+1)}(-1)^{m+1}. \tag{69}$$

From this (68) and (69) formula (53) easily follows.

(k) From (50) we have that

$$x_{2(2t+1)m+2s+1}=x_{2s-2(2t+1)+1}\prod_{p=0}^{m}\frac{b}{(2y_{-1}-b)(-1)^{p(2t+1)+s+1}+b}, \tag{70}$$

for $m\in\mathbb{N}_0$ and $2s+1\in\{k-2,k-1,\dots,3k-3\}$, which implies

$$\begin{aligned}x_{4(2t+1)m+2s+1}&=x_{2s-2(2t+1)+1}\prod_{p=0}^{2m}\frac{b}{(2y_{-1}-b)(-1)^{p(2t+1)+s+1}+b}\\&=\frac{x_{2s-2(2t+1)+1}}{((2y_{-1}/b-1)(-1)^{s+1}+1)\left(1-(2y_{-1}/b-1)^2\right)^m},\end{aligned} \tag{71}$$

and

$$\begin{aligned}x_{4(2t+1)m+4t+2+2s+1}&=x_{2s-2(2t+1)+1}\prod_{p=0}^{2m+1}\frac{b}{(2y_{-1}-b)(-1)^{p(2t+1)+s+1}+b}\\&=\frac{x_{2s-2(2t+1)+1}}{\left(1-(2y_{-1}/b-1)^2\right)^{m+1}}.\end{aligned} \tag{72}$$

Similar to (b) we have that $|(2y_{-1}/b-1)^2-1|<1$. From this, (71) and (72), the result follows.

(l) In this case we have $(2y_{-1}/b-1)^2-1>1$. From this, (71) and (72), the result follows.

(m) In this case we have $(2y_{-1}/b-1)^2=2$. Using this in (71) and (72) we get

$$x_{4(2t+1)m+2s+1}=\frac{x_{2s-2(2t+1)+1}}{(\mp\sqrt{2}(-1)^{s+1}+1)(-1)^m}, \tag{73}$$

and

$$x_{4(2t+1)m+4t+2+2s+1}=\frac{x_{2s-2(2t+1)+1}}{(-1)^{m+1}}. \tag{74}$$

From (73) and (74), formula (54) easily follows.

(n), (o) Note that in this case

$$\begin{aligned}x_{4(2t+1)m+2s}&=x_{2s-2(2t+1)}\prod_{p=0}^{2m}\frac{(2y_{-1}-b)(-1)^{p(2t+1)+s-t}+b}{(2y_{-2}-b)(-1)^{p(2t+1)+s+1}+b}\\&=x_{2s-2(2t+1)}\frac{(2y_{-1}-b)(-1)^{s-t}+b}{(2y_{-2}-b)(-1)^{s+1}+b}N^m,\end{aligned} \tag{75}$$

and

$$x_{4(2t+1)m+4t+2+2s} = x_{2s-2(2t+1)} \prod_{p=0}^{2m+1} \frac{(2y_{-1}-b)(-1)^{p(2t+1)+s-t}+b}{(2y_{-2}-b)(-1)^{p(2t+1)+s+1}+b}$$
$$= x_{2s-2(2t+1)} N^{m+1}, \tag{76}$$

$$x_{4(2t+1)m+2s+1} = x_{2s-2(2t+1)+1} \prod_{p=0}^{2m} \frac{(2y_{-2}-b)(-1)^{p(2t+1)+s-t+1}+b}{(2y_{-1}-b)(-1)^{p(2t+1)+s+1}+b}$$
$$= x_{2s-2(2t+1)+1} \frac{(2y_{-2}-b)(-1)^{s-t+1}+b}{N^m(2y_{-1}-b)(-1)^{s+1}+b}, \tag{77}$$

and

$$x_{4(2t+1)m+4t+2+2s+1} = \frac{x_{2s-2(2t+1)+1}}{N^{m+1}}. \tag{78}$$

Using formulas (75)-(78), and respectively the conditions $|N| < 1$, that is, $|N| > 1$, these two statements easily follow.

(p) Using formulas (75)-(78) and the condition $N = 1$ we have that

$$x_{4(2t+1)m+2s} = x_{2s-2(2t+1)} \frac{(2y_{-1}-b)(-1)^{s-t}+b}{(2y_{-2}-b)(-1)^{s+1}+b},$$
$$x_{4(2t+1)m+4t+2+2s} = x_{2s-2(2t+1)},$$
$$x_{4(2t+1)m+2s+1} = x_{2s-2(2t+1)+1} \frac{(2y_{-2}-b)(-1)^{s-t+1}+b}{(2y_{-1}-b)(-1)^{s+1}+b},$$
$$x_{4(2t+1)m+4t+2+2s+1} = x_{2s-2(2t+1)+1},$$

from which $4k$-periodicity follows.

(q) Using formulas (75)-(78) and the condition $N = -1$ we have that

$$x_{4(2t+1)m+2s} = x_{2s-2(2t+1)}(-1)^m \frac{(2y_{-1}-b)(-1)^{s-t}+b}{(2y_{-2}-b)(-1)^{s+1}+b},$$
$$x_{4(2t+1)m+4t+2+2s} = x_{2s-2(2t+1)}(-1)^{m+1},$$
$$x_{4(2t+1)m+2s+1} = x_{2s-2(2t+1)+1}(-1)^m \frac{(2y_{-2}-b)(-1)^{s-t+1}+b}{(2y_{-1}-b)(-1)^{s+1}+b}$$
$$x_{4(2t+1)m+4t+2+2s+1} = (-1)^{m+1} x_{2s-2(2t+1)+1}.$$

From these relations we see that the subsequences $x_{4(2t+1)m+2s}$, $x_{4(2t+1)m+4t+2+2s}$, $x_{4(2t+1)m+2s+1}$, $x_{4(2t+1)m+4t+2+2s+1}$ are $2k$-periodic, from which the statement follows. □

4.3. *Case $a \neq 0$, $b = 0$*

In this case equation (17) becomes

$$x_n = \frac{x_{n-2}x_{n-k-2}}{a x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{79}$$

and from formulas (20)-(23), for the case $a \neq 1$, we obtain

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)} \left(\frac{y_{-1}}{y_{-2}a^{t+1}} \right)^{m+1}, \tag{80}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \ldots, 3k-3\}$, and

$$x_{2(2t+1)m+2s+1} = x_{2s-2(2t+1)+1}\left(\frac{y_{-2}}{y_{-1}a^t}\right)^{m+1}, \tag{81}$$

for $m \in \mathbb{N}_0$ and $2s+1 \in \{k-2, k-1, \ldots, 3k-3\}$, while for $a = 1$ we have that

$$x_{2(2t+1)m+2s} = x_{2s-2(2t+1)}\left(\frac{y_{-1}}{y_{-2}}\right)^{m+1}, \tag{82}$$

for $m \in \mathbb{N}_0$ and $2s \in \{k-2, k-1, \ldots, 3k-3\}$, and

$$x_{2(2t+1)m+2s+1} = x_{2s-2(2t+1)+1}\left(\frac{y_{-2}}{y_{-1}}\right)^{m+1}, \tag{83}$$

for $m \in \mathbb{N}_0$ and $2s+1 \in \{k-2, k-1, \ldots, 3k-3\}$.

Employing formulas (80)-(83), the following theorem easily follows. We omit the proof for its simplicity.

Theorem 3. *Assume that $a \neq 0$, $b = 0$, k is an odd natural number, and $(x_n)_{n\geq -k-2}$ is a well-defined solution to equation* (17)*. Then the following statements are true.*

(a) *If $|y_{-1}/y_{-2}a^{t+1}| < 1$, then $x_{2(2t+1)m+2s} \to 0$ as $m \to +\infty$, for every $2s \in \{k-2, k-1, \ldots, 3k-3\}$.*
(b) *If $|y_{-1}/y_{-2}a^{t+1}| > 1$, then $|x_{2(2t+1)m+2s}| \to +\infty$ as $m \to +\infty$, for every $2s \in \{k-2, k-1, \ldots, 3k-3\}$.*
(c) *If $y_{-1}/y_{-2}a^{t+1} = 1$, then $(x_{2(2t+1)m+2s})_{m\in\mathbb{N}_0}$ are constant sequences, for every $2s \in \{k-2, k-1, \ldots, 3k-3\}$.*
(d) *If $y_{-1}/y_{-2}a^{t+1} = -1$, then $(x_{2(2t+1)m+2s})_{m\in\mathbb{N}_0}$ is a two-periodic sequence, for every $2s \in \{k-2, k-1, \ldots, 3k-3\}$.*
(e) *If $|y_{-2}/y_{-1}a^{t}| < 1$, then $x_{2(2t+1)m+2s+1} \to 0$ as $m \to +\infty$, for every $2s+1 \in \{k-2, k-1, \ldots, 3k-3\}$.*
(f) *If $|y_{-2}/y_{-1}a^{t}| > 1$, then $|x_{2(2t+1)m+2s+1}| \to +\infty$ as $m \to +\infty$, for every $2s+1 \in \{k-2, k-1, \ldots, 3k-3\}$.*
(g) *If $y_{-2}/y_{-1}a^{t} = 1$, then $(x_{2(2t+1)m+2s+1})_{m\in\mathbb{N}_0}$ is a constant sequence, for every $2s+1 \in \{k-2, k-1, \ldots, 3k-3\}$.*
(h) *If $y_{-2}/y_{-1}a^{t} = -1$, then $(x_{2(2t+1)m+2s+1})_{m\in\mathbb{N}_0}$ is a two-periodic sequence, for every $2s+1 \in \{k-2, k-1, \ldots, 3k-3\}$.*

5. Domain of Undefinable Solutions

As we have seen in Section 2 if $x_{-i} = 0$ for some $i \in \{1, 2, \ldots, k+2\}$, then such solutions are not defined. The set of all initial values for which solutions to equation (3) under some natural conditions are not defined, so called, *domain of undefinable solutions* of equation (3), is described here. Before we formulate the main result in this section we will give definition of the notion ([34]).

Definition 1. *Consider the difference equation*

$$x_n = f(x_{n-1}, \ldots, x_{n-s}, n), \quad n \in \mathbb{N}_0, \tag{84}$$

where $s \in \mathbb{N}$, and $x_{-i} \in \mathbb{R}$, $i = \overline{1,s}$. The string of numbers $x_{-s}, \ldots, x_{-1}, x_0, \ldots, x_{n_0}$ where $n_0 \geq -1$, is called an undefined solution of equation (84) *if*

$$x_j = f(x_{j-1}, \ldots, x_{j-s}, j)$$

for $0 \leq j < n_0 + 1$, and x_{n_0+1} is not defined number, that is, the quantity $f(x_{n_0}, \ldots, x_{n_0-s+1}, n_0+1)$ is not defined.

The set of all initial values $x_{-s}, \ldots, x_{-1}$ which generate undefined solutions to equation (84) *is called domain of undefinable solutions of the equation.*

Now we formulate and prove the main result in this section.

Theorem 4. *Assume that $a_n \neq 0$, $b_n \neq 0$, $n \in \mathbb{N}_0$. Then the domain of undefinable solutions to equation* (3) *is the following set*

$$\mathcal{U} = \bigcup_{m\in\mathbb{N}_0} \bigcup_{i=0}^{1} \Big\{(x_{-(k+2)}, \dots, x_{-1}) \in \mathbb{R}^{k+2} : x_{i-2}x_{i-k-2} = \frac{1}{c_m}, \ \text{when} \ c_m := -\sum_{j=0}^{m} \frac{b_{2j+i}}{a_{2j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{2l+i}} \neq 0\Big\}$$
$$\bigcup \bigcup_{j=1}^{k+2} \Big\{(x_{-(k+2)}, \dots, x_{-1}) \in \mathbb{R}^{k+2} : x_{-j} = 0\Big\}.$$

Proof. The consideration at the beginning of Section 2 shows that the set

$$\bigcup_{j=1}^{k+2} \Big\{(x_{-(k+2)}, \dots, x_{-1}) \in \mathbb{R}^{k+2} : x_{-j} = 0\Big\}$$

belongs to the domain of undefinable solutions to equation (3). Hence, now assume that $x_{-i} \neq 0$, $i \in \{1, \dots, k+2\}$, i.e., that $x_n \neq 0$ for $n \geq -(k+2)$. If such a solution $(x_n)_{n\geq-(k+2)}$ of equation (3) is not defined then clearly $x_{n-2}x_{n-k-2} = -a_n/b_n$ for some $n \in \mathbb{N}_0$. By using the change of variables (7) and the representation of integers $n \geq -2$, in the form $n = 2m + i$, $m \geq -1$, $i \in \{0, 1\}$, equation (3) is transformed into the two equations in (9), which means that a solution x_n of difference equation (3) is not defined when $y_{2(m-1)+i} = -b_{2m+i}/a_{2m+i}$ for some $m \in \mathbb{N}_0$ and $i \in \{0, 1\}$.

Let

$$g_{2m+i}(t) := a_{2m+i}t + b_{2m+i}, \quad m \in \mathbb{N}_0, \ i \in \{0, 1\}.$$

Then $g^{-1}_{2m+i}(t) := (t - b_{2m+i})/a_{2m+i}$, $m \in \mathbb{N}_0$, $i \in \{0, 1\}$, and specially

$$g^{-1}_{2m+i}(0) = -\frac{b_{2m+i}}{a_{2m+i}}, \quad m \in \mathbb{N}_0, \ i \in \{0, 1\}. \tag{85}$$

Note that the equations in (9) can be written in the form $y_{2m+i} = g_{2m+i}(y_{2(m-1)+i})$, $m \geq -1$, $i \in \{0, 1\}$, which implies that

$$y_{2m+i} = g_{2m+i} \circ g_{2(m-1)+i} \circ \cdots \circ g_i(y_{i-2}), \quad m \in \mathbb{N}_0, \ i \in \{0, 1\}. \tag{86}$$

From (85) and (86) it follows that $y_{2(m-1)+i} = -b_{2m+i}/a_{2m+i}$ for some $m \in \mathbb{N}_0$, $i \in \{0, 1\}$, if and only if

$$y_{i-2} = g_i^{-1} \circ \cdots \circ g^{-1}_{2(m-1)+i}(-b_{2m+i}/a_{2m+i}), \quad m \in \mathbb{N}_0, \ i \in \{0, 1\},$$

that is,

$$y_{i-2} = g_i^{-1} \circ \cdots \circ g^{-1}_{2m+i}(0), \quad m \in \mathbb{N}_0, \ i \in \{0, 1\}. \tag{87}$$

It is not difficult to see (similar to getting the formula for general solution to the linear first order difference equation) that (87) implies

$$y_{i-2} = -\sum_{j=0}^{m} \frac{b_{2j+i}}{a_{2j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{2l+i}},$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1\}$. From this and since

$$y_{i-2} = \frac{1}{x_{i-2}x_{i-k-2}}, \quad i \in \{0, 1\},$$

the result easily follows. □

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