ON A COMBINATORIAL GENERALIZATION OF 27 LINES ASSOCIATED WITH A CUBIC SURFACE

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(Received 26 March 1969)

To Bernhard Hermann Neumann on his 60th birthday

Communicated by G. B. Preston

1.

Given integers $0 < \lambda < k < v$, does there exist a nontrivial graph G with the following properties: G is of order v (i.e. has v vertices), is regular of degree k (i.e. every vertex is adjacent to exactly k other vertices), and every pair of vertices is adjacent to exactly λ others? Two vertices are said to be adjacent if they are connected by an edge. We call a graph with the above properties a symmetric (v, k, λ) graph and refer to the last of the properties as the λ-condition. The complete graph of order v is a trivial example of a symmetric $(v, v-1, v-2)$ graph, but we are of course only interested in non-trivial constructions.

All graphs will be assumed to have no loops or double edges. If the vertices of G are denoted by $x_1, \cdots, x_v$ and S_i, $i = 1, \cdots, v$ denotes the set of vertices which are adjacent to x_i, then the sets S_i form a symmetric block design with parameters (v, k, λ) i.e. $|S_i| = k$, $|S_i \cap S_j| = \lambda$ for $i \neq j$. Thus the parameters v, k, λ must satisfy the Bruck-Ryser-Chowla conditions ([2], p. 107) in order that such a graph should exist. However these conditions are by no means sufficient for the existence of a symmetric (v, k, λ) graph; it follows for instance from a result of Erdös, Rényi and Vera T. Sòs [1] that no non-trivial symmetric $(v, k, 1)$ graph exists, and recently R. C. Bose has proved that for each λ there are at most a finite number of symmetric (v, k, λ) graphs (private communication). From his results it follows among others that for λ prime, $k = \lambda(\lambda+1)$, $v = \lambda^2(\lambda+2)$ is the only possibility. If the condition of regularity is dropped then the star of m triangles with one common vertex is a graph satisfying the λ-condition with $\lambda = 1$; and Erdös has shown recently that all other graphs satisfying the λ-condition for any λ are necessarily regular (private communication).

The insufficiency of the Bruck-Ryser-Chowla condition for the existence of symmetric (v, k, λ) graphs is perhaps not surprising, as these graphs

https://doi.org/10.1017/S144678870000776X Published online by Cambridge University Press

represent block designs of a very special kind. They must satisfy the conditions $x_i \notin S_i$, and $x_i \in S_j \Rightarrow x_j \in S_i$. What is more surprising is that if the last condition is replaced by the "skew" condition

$$x_i \in S_j \Rightarrow x_j \notin S_i,$$

that is, if we ask for a directed graph with similar properties, then it seems likely that such a graph exists for all parameters (v, k, λ) with $2\lambda < k$ for which a block design exists. No counter-example is known to us, of a set of admissible parameters (v, k, λ) for which a skew design definitely does not exist.

2.

The purpose of this paper is to show that there exists a symmetric $(\lambda^2(\lambda+2), \lambda(\lambda+1), \lambda)$ graph for all prime powers λ. We shall prove,

THEOREM 1: *Let λ be a prime power p^n. Then there is a regular graph of order $\lambda^2(\lambda+2)$ and vertex degree $\lambda(\lambda+1)$ with the property that for every pair of vertices there are exactly λ other vertices adjacent to both members of the pair.*

COROLLARY: *For every prime power λ there exists a symmetric block design with parameters $(\lambda^2(\lambda+2), \lambda(\lambda+1), \lambda)$ and satisfying the conditions*

$$x_i \notin S_i, x_i \in S_j \Rightarrow x_j \in S_i.$$

The construction will be slightly different for even and for odd prime powers. Except for $\lambda = 1$ and $\lambda = 2$, the corresponding block designs seem to be new; for instance (45, 12, 3) is listed by Marshall Hall ([2] p. 295) as unknown. This is the more surprising as this design can be obtained quite simply from a well known configuration of 27 lines associated with a general cubic surface. ([1], chapter IV) These 27 lines have a number of interesting combinatorial properties. If we call "points" the points of intersection of the 27 lines then

(i) there are 45 points;
(ii) each line contains exactly 5 points;
(iii) exactly 3 lines meet at each point;
(iv) given a line K and a point x not on K, there is exactly one line containing x which intersects K;
(v) the vertices can be divided into 5 classes containing 9 points each so that no two points in the same class are on a line.

These properties can easily be verified from the incidence relations given in ([1] p. 160). If we connect by an edge every pair of points which are on the same line, a symmetric (45, 12, 3) graph is obtained. Generally if λ is a

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prime powei, we shall show that there exists a graph G with the following properties.

G1: G contains $\lambda^2(\lambda+2)$ vertices

G2: There are λ^3 "lines" in the graph. That is λ^3 subsets, each containing $\lambda+2$ vertices so that every two vertices on a line are adjacent.

G3: Each vertex is contained in eactly λ lines.

G4: Each edge is in exactly one line.

G5: Given a line K and a vertex x not on K there is exactly one edge connecting x with some vertex of K.

G6: The vertices can be divided into $\lambda+2$ classes, each class containg λ^2 vertices, so that no two vertices in the same class are adjacent.

By G2 and G3, each of the λ lines meeting at x contains $\lambda+1$ vertices distinct from x and (by G4) distinct from each other. By G2 and G4 these are exactly the vertices which are adjacent to x. It follows that G is regular with vertex degree $\lambda(\lambda+1)$.

If the vertices x and y are not adjacent then it follows from G5 that to each of the λ lines through x there goes exactly one edge from y, hence there are exactly λ vertices adjacent to both x and y.

If x and y are adjacent then by $G4$ there is exactly one line containing both x and y, and this line contains precisely λ vertices distinct from x and y. By G5 these are exactly the vertices with which both x and y are connected. Hence G has all the required properties. Note that property G6 has not been used in the argument, but all constructions that we have found have this property.

3.

We shall obtain the graph G from a configuration of "points" and "lines" with the following properties

C1: There are $\lambda^2(\lambda+2)$ lines in the configuration.

C2: There are λ^2 points and every point lies on $\lambda+2$ lines.

C3: Every line contains λ points.

C4: Two lines intersect in at most one point.

C5: If the point P is not on the line L then there is exactly one line L' through P which intersects L.

C6: The lines can be divided into $\lambda+2$ families of non intersecting lines, λ^2 lines in each family.

The graph G is obtained as follows:
The vertices of G are the "lines" of the configuration, and two vertices

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are joined by an edge if and only if the corresponding lines intersect. Clearly properties G1 to G6 correspond exactly to properties C1 to C6.

Property C5 follows from C1 to C4 and

C5′: Three mutually intersecting lines meet in a point. Or, there are no triangles in the configuration.

For suppose that C5′ is true, and let L be a given line. There are exactly $\lambda+1$ lines distinct from L, through each of the λ points of L, and exactly $\lambda-1$ points on each of these lines; other than the points on L. These points are distinct by C4 and C5′. Hence there are $(\lambda-1)\lambda(\lambda+1) = \lambda^3-\lambda$ distinct points outside L each of which lies on exactly one line meeting L. But the total number of points outside L is $\lambda^3-\lambda$ by C2 and C3 and so the truth of C5 has been established.

4.

Let us first assume that $\lambda = 2^\alpha$. We want a configuration of λ^3 points and $\lambda^2(\lambda+2)$ lines which has the properties C1 to C6 of §3. For the points of the configuration we take the λ^3 points of the affine 3-space S over $GF(\lambda)$. Elements of S are 3-vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ with $x_i \in GF(\lambda)$. Choose a set of $\lambda+2$ vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{\lambda+2}$ in S with the property that any 3 of them are linearly independent. Such a set exists by Lemma 1 below.

For each $\boldsymbol{a}_i$ we form a family of parallel lines of the form $\boldsymbol{r} = \boldsymbol{r}_0+t\boldsymbol{a}_i$, the parameter t taking all values in $GF(\lambda)$. Each family contains λ^2 lines and each line contains λ points. Since no two lines in a family intersect, every point in S lies on exactly one line in each family. Thus properties C1, C2, C3, C4, C6 are all satisfied.

To show that C5′ is also satisfied, take the lines $\boldsymbol{r} = \boldsymbol{r}_0+t\boldsymbol{a}_i$ and $\boldsymbol{r} = \boldsymbol{r}_0+s\boldsymbol{a}_j$, $j \neq i$ which meet at $\boldsymbol{r}_0$ and suppose that a third line $\boldsymbol{r} = \boldsymbol{r}_1+q\boldsymbol{a}_k$ meets the first one when $t = t_1$, $q = q_1$ and the second one when $s = s_2$, $q = q_2 \neq q_1$. We now have

$$\boldsymbol{r}_0+t_1\boldsymbol{a}_i = \boldsymbol{r}_1+q_1\boldsymbol{a}_k$$
$$\boldsymbol{r}_0+s_2\boldsymbol{a}_j = \boldsymbol{r}_1+q_2\boldsymbol{a}_k.$$

Subtracting leads to a linear relation between $\boldsymbol{a}_i$, $\boldsymbol{a}_j$ and a_k, contrary to our assumption of linear independence.

It remains to show the existence of the vectors $\boldsymbol{a}_i$. We shall prove

LEMMA 1. *If $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_r\}$ is a set of r vectors in affine 3-space S over $GF(\lambda)$ such that any three members of the set are linearly independent, then $r \leqq \lambda+2$ for λ even and $r \leqq \lambda+1$ for λ odd. Both these values are attainable.*

PROOF. We may assume that $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ since this can always be achieved by a suitable non-singular linear

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transformation which will not affect the linear independence property. No vector $\boldsymbol{a}_i$, $3 < i \leqq r$ has any component equal to zero or it would be dependent on two of $\boldsymbol{a}_1, \boldsymbol{a}_2$ and $\boldsymbol{a}_3$. We now choose suitable multiples of these vectors so that the first component of each is 1. Let

$$\boldsymbol{a}_i = (1, \alpha_i, \beta_i), 3 < i \leqq r.$$

If $i \neq j$ then $\alpha_i \neq \alpha_j$ for otherwise we would have $\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_3$ linearly dependent. Thus there are $\lambda - 1$ possible values for α_i and we have $r \leqq \lambda+2$. Similarly the components β_i are all distinct. Also if $\beta_i/\alpha_i = \beta_j/\alpha_j$ we will have $\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_1$ linearly dependent and so all these ratios are distinct. Now suppose that $r = \lambda+2$. Then α_i, β_i and β_i/α_i run through all non-zero elements of $GF(\lambda)$ when $3 < i \leqq r$, and

$$\prod_{i=4}^{\lambda+2} \alpha_i = \prod_{i=4}^{\lambda+2} \beta_i = \prod_{i=4}^{\lambda+2} \frac{\beta_i}{\alpha_i} = 1, \text{ giving } \prod_{\substack{\alpha \in GF(\lambda) \\ \alpha \neq 0}} \alpha = 1.$$

This is clearly so if and only if λ is even. (If λ is odd, the value of the product is -1). We owe this last proof to C.D. Cox.

A suitable set of $\lambda+2$ vectors when $\lambda = 2^\alpha$ is

$$\boldsymbol{a}_i = (1, \rho^i, \rho^{2i}), \ 0 \leqq i \leqq \lambda - 2$$

together with $\boldsymbol{a}_{\lambda-1} = (1, 0, 0)$, $\boldsymbol{a}_\lambda = (0, 1, 0)$, $\boldsymbol{a}_{\lambda+1} = (0, 0, 1)$, where ρ is a primitive root of the multiplicative group of $GF(\lambda)$. If λ is an odd prime power, we take the same set of vectors with $\boldsymbol{a}_\lambda$ omitted. Proof of independence is immediate.

5.

We assume now that $\lambda = p^n$ is an odd prime power. S is again the affine 3-space over $GF(\lambda)$. Points of S are "points" of our configuration, and the following curves will be the "lines".

(i) $x = t \quad y = A \quad z = B$

(ii) $x = A \quad y = t \quad z = B$

(iii) $x = Ct^2 - Bt + A \quad y = -2Ct + B, \quad z = t$

where the parameter t ranges over $GF(\lambda)$ and where A, B, C are arbitrary elements of $GF(\lambda)$. Clearly there are $\lambda^2(\lambda+2)$ such curves, and λ points on each curve. We shall refer to (iii) as the curve A, B, C.

For each value of C there are λ^2 curves of type (iii), and these curves have no point in common. For suppose that the curves A_1, B_1, C and A_2, B_2, C intersect. Then for some t

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$$Ct^2-B_1t+A_1 = Ct^2-B_2t+A_2$$
$$-2Ct+B_1 = -2Ct+B_2$$

which clearly implies $B_1 = B_2$, $A_1 = A_2$.

Similarly no two curves of the form (i) (or of the form (ii)) intersect. Thus we have $\lambda+2$ families of non intersecting curves, λ^2 curves in each family and λ points on each curve. Hence each point of the space lies on exactly $\lambda+2$ curves, one from each family.

LEMMA 2. *Two curves in different families meet in at most one point.*

This is clear if one of the curves is a line of the type (i) or (ii), and we only need to consider two curves of the type (iii). Suppose the curve A_1, B_1, C_1 meets the curve A_2, B_2, C_2 at two different parameter values. Then we have

$$\begin{aligned} -2C_1t+B_1 &= -2C_2t+B_2 \\ -2C_1s+B_1 &= -2C_2s+B_2 \\ C_1(t-s) &= C_2(t-s) \qquad t \neq s \end{aligned}$$

hence $C_1 = C_2$ and the two curves are identical.

So far we have shown that our configuration has properties C1, C2, C3, C4, C6. We now verify C5′ by showing that no triangles exist. We must consider the following possibilities for L_1, L_2, L_3 which form the triangle.

(a) L_1 of type (i), L_2 of type (ii), L_3 of type (iii)
(b) L_1 of type (i), L_2, L_3 of type (iii)
(c) L_1 of type (ii), L_2, L_3 of type (iii)
(d) All three curves of type (iii)

(a) Let L_1 be $x = t_1 \quad y = A_1 \quad z = B_1$
L_2 be $x = A_2 \quad y = t_2 \quad z = B_2$
L_3 be $x = Ct_3^2-Bt_3+A, \quad y = -2Ct_3+B, \quad z = t_3$.

Since L_1 and L_2 meet, we must have $B_1 = B_2$. But then both L_1 and L_2 meet L_3 at the same point, with parameter value $t_3 = B_1 = B_2$, and there is no triangle.

(b) Let L_1 be $x = t_1\, y = A_1\, z = B_1$ and suppose that it meets both A_2, B_2, C_2 and A_3, B_3, C_3 at points with parameter value $t_2 = t_3 = B_1$. Then we have

$$A_1 = -2C_2B_1+B_2 = -2C_3B_1+B_3.$$

If L_2, L_3 meet at the point with parameter value $t = t_2 = t_3 \neq B_1$ then

$$2C_2t+B_2 = 2C_3t+B_3$$

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which with the previous equation gives

$$C_2(B_1-t) = C_3(B_1-t),\ B_1 \neq t.$$

Hence $C_2 = C_3$ and L_2, L_3 are in the same family and so do not meet.

(c) Let L_1 be $x = A_1$, $y = t_1$ $z = B_1$ and suppose that it meets both A_2, B_2, C_2 and A_3, B_3, C_3 in the points with parameter value $t_2 = t_3 = B_1$. We now have

$$A_1 = C_2B_1^2-B_2B_1+A_2 = C_3B_1^2-B_3B_1+A_3$$

and for some $t \neq B_1$

$$\begin{aligned} C_2t^2-B_2t+A_2 &= C_3t^2-B_3t+A_3, \\ -2C_2t+B_2 &= -2C_3t+B_3 \end{aligned}$$

giving

$$\begin{aligned} C_2(B_1+t)-B_2 &= C_3(B_1+t)-B_3 \\ C_2(B_1-t) &= C_3(B_1-t),\ C_2 = C_3 \end{aligned}$$

which is a contradiction as in (b).

(d) Finally suppose that L_1, L_2, L_3 are of type (iii), with L_i having coefficients A_i, B_i, C_i. Suppose that L_i, L_j, $i \neq j$, meet each other at the parameter value $t_{ij} = t_{ji}$ where t_{12}, t_{23}, t_{31} are distinct. Then

$$\begin{aligned} C_{ij}t_{ij}^2-B_it_{ij}+A_i &= C_jt_{ij}^2-B_jt_{ij}+A_j \\ -2C_it_{ij}+B_i &= -2C_jt_{ij}+B_j \end{aligned} \tag{1}$$

giving

$$-B_it_{ij}+2A_i = -B_jt_{ij}+2A_j \tag{2}$$

From (1) we obtain

$$B_1t_{13}(t_{12}-t_{13})+B_2t_{31}(t_{23}-t_{21})+B_3t_{12}(t_{31}-t_{32}) = 0$$

and from (2)

$$B_1(t_{12}-t_{13})+B_2(t_{23}-t_{21})+B_3(t_{31}-t_{32}) = 0.$$

Eliminating B_1

$$\begin{aligned} &B_2(t_{23}-t_{21})(t_{23}-t_{31}) = B_3(t_{23}-t_{21})(t_{23}-t_{31}) \\ &B_2 = B_3, \text{ hence by (1) } C_2 = C_3 \end{aligned}$$

which is again a contradiction, and thus property C5′ of our configuration is established.

In the case of $\lambda = 3$, the construction of this section yields precisely the dual (obtained by interchanging the roles of lines and points) of the configuration of 27 lines associated with a cubic surface.

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Added in proof. Block designs with parameters $(\lambda^2(\lambda+2), \lambda(\lambda+2), \lambda)$, where λ is a prime power, have been constructed previously by K. Takenuchi [4]. These designs do not seem to supply solutions of the corresponding graph problem. Solutions of the graph problem for (45, 12, 3) (and many other parameter systems which do not overlap ours) have been obtained recently by R. C. Bose and S. S. Shrikande, Mimeographed Note No. 600.6, Institute of Statistics, Consolidated University of North Carolina, 1969.

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