

On a combinatorial problem

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N. G. DE BRUIJN and P. ERDÖS: On a combinational problem.

(Communicated at the meeting of November 27, 1948.)

Let there be given n elements $a_1, a_2, ..., a_n$. By $A_1, A_2, ..., A_m$ we shall denote combinations of the a's. We assume that we have given a system of m > 1 combinations $A_1, A_2, ..., A_m$ so that each pair (a_i, a_j) is contained in one and only one A. Then we prove

Theorem 1. We have $m \ge n^1$), with equality occurring only if either the system is of the type $A_1 = (a_1, a_2, ..., a_{n-1})$, $A_2 = (a_1, a_n)$, $A_3 = (a_2, a_n) ... A_n = (a_{n-1}, a_n)$, or if n is of the form n = k(k-1) + 1 and all the A's have k elements, and each a occurs in exactly k of the A's.

Corollary: If the elements a_i are points in the real projective plane the theorem can be stated as follows: Let there be given n points in the plane, not all on a line. Connect any two of these points. Then the number of lines in this system is $\geq n$. In this case equality occurs only if n-1 of the points are on a line.

This corollary can be proved independently of Theorem 1 by aid of the

following theorem of GALLAI (= GRÜNWALD) 2):

Let there be given n points in the plane, not all on a line. Then there exists a line which goes through two and only two of the points.

Remark: The points of inflexion of the cubic show that it is essential that the points should all be real, thus GALLAI's theorem permits no projective and a fortiori no combinatorial formulation. Also the result clearly fails for infinitely many points.

We now give GALLAI's ingenious proof: Assume the theorem false. Then any line through two of the points also goes through a third. Project one of the points, say a_1 to infinity, and connect it with the other points. Thus we get a set of parallel lines each containing two or more points a_i (in the finite part of the plane). Consider the system of lines connecting any two of these points, and assume that the line $(a_i a_j a_k)$ forms the smallest angle with the parallel lines. (This line again contains at least three points). But the line connecting a_j with a_1 (at infinity) contains at least another (finite) point a_r , and clearly (see figure) either the line $(a_i a_r)$

¹⁾ This was also proved by G. SZEKERES but his proof was more complicated.

²⁾ This theorem was first conjectured by SYLVESTER, GALLAI's proof appeared in the Amer. Math. Monthly as a solution to a problem by P. ERDÖS. The corollary to Theorem 1 also appeared as a problem in the Monthly.

See also H. S. M. COXETER, Amer. Math. Monthly 55, 26—28 (1948), where very simple proofs due to KELLY and STEINBERG are given.

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or the line (a_1, a_k) forms a smaller angle with the parallel lines then $(a_1 a_1 a_k)$. This contradiction establishes the result.

Remark: Denote by f(n) the minimum number of lines which go through exactly two points. It is not known whether $\lim_{n \to \infty} f(n) = \infty$. All that we can show is that $f(n) \ge 3$.

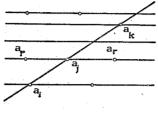


Fig. 1.

Now we prove the corollary as follows: We use induction. Assume that the number of lines determined by n-1 points, not all on a line, is $\geq n-1$. Then we shall prove that n points, not all on a line, determine at least n lines.

Let (a_1, a_2) be a line going through two points only. Consider the points $a_2, a_3, ..., a_n$. If they are all on a line, then (a_1, a_i) , i = 2, 3, ..., n and $(a_2, a_3, ..., a_n)$ clearly determine n lines. If they do not all lie on a line, then they determine at least n-1 lines, and (a_1, a_2) is clearly not one of these lines. Thus together with (a_1, a_2) we again get at least n lines. The same induction argument shows that we get exactly n lines only if n-1 of the points lie on a line, q.e.d.

Proof of theorem 1. For simplicity we shall call the elements $a_1, a_2, ..., a_n$ points and the sets $A_1, A_2, ..., A_m$ lines. Denote by k_i the number of lines passing through the point a_i , and by s_j the number of points on the line A_j . We evidently find (by counting the number of incidences in two ways)

Further if A_i does not pass through a_i , then

(2) follows from the fact that a_i can be connected by a line (i.e. an A) to all the s_j points of A_j , and any two of these lines are different, since otherwise they would have two points in common.

Assume now that k_n is the smallest k_i and that $A_1, A_2, ..., A_{k_n}$ are the lines through a_n . We may suppose that each line contains at least two points, since otherwise it could be omitted. Also $k_n > 1$, for otherwise all the points are on a line. Thus we can find points a_i on A_i , $a_i \neq a_n$, $i = 1, 2, ..., k_n$. Also if $i \neq j$, $i \leq k_n$, $j \leq k_n$ then a_i is not on A_j (for

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otherwise A_i and A_j would have two points in common). Hence by (2) (putting $k_n = \nu$)

 $s_2 \leq k_1$, $s_3 \leq k_2$,..., $s_{\nu} \leq k_{\nu-1}$, $s_1 \leq k_{\nu}$; $s_j \leq k_n$ for $j > \nu$. (3) From (1), (3) and the minimum property of k_n we obtain $m \geq n$, which proves the first part of Theorem 1.

We now determine the cases where m=n. If m=n, then all the inequalities of (3) have to be equalities. Consequently we can renumerate the points so that $s_1=k_1$, $s_2=k_2$, ..., $s_n=k_n$. We may suppose that $k_1 \ge k_2 \ge ... \ge k_n > 1$. There are two cases:

- a) $k_1 > k_2$. Hence by $s_1 = k_1 > k_i$ $(2 \le i \le n)$, (2) shows that all the a_i $(i \ge 2)$ lie on A_1 . Of course a_1 does not lie on A_1 and we have the first case of Theorem 1.
- b) $k_1 = k_2$. If no k_i is less than k_1 then clearly $k_i = s_j$ $(1 \le i, j \le n)$. We shall show that this is the only possibility. If $k_j < k_1$, then we have by (2) that a_j lies on both A_1 and A_2 . Hence k_n is the only k which can be less than k_1 . Now $s_n = k_n$ different lines contain a_n . Any line through a_n contains one further point and all but one contain two further points, since $k_1 = k_2 = \ldots = k_{n-1} > k_n \ge 2$. Thus there are at least two lines which do not contain a_n ; for both of these lines we have by (2) $s_j \le k_n$. This contradicts $s_1 = s_2 = \ldots = s_{n-1} > k_n$.

Apart from case a) we only have the case where $s_i = k_j = k$, $(1 \le i, j \le n)$. It is easily seen that then n = k (k-1)+1, and also that any pair of lines has exactly one intersection point. For if A_i does not intersect A_j ; and if a_i lies on A_i then we infer from (2) that $k_i \ge s_j + 1$ which is not possible since $k_i = s_j = k$. The two dimensional projective finite geometries with $k-1=p^{\alpha}$, p prime, are known to be systems of this type, but F. W. Levi 3) constructed a non-projective example with k=9.

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³⁾ F. W. LEVI, Finite geometrical systems, Calcutta 1942.