Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 34 (3—4), (1979), 347—350.

ON A COMMON GENERALIZATION OF BORSUK'S AND RADON'S THEOREM

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1. The well-known theorem of RADON [3] says that if $A\subset R^d$ and $|A|\geqq d+2$, then there exist $B, C\subset A$, $B\cap C=\emptyset$ such that $\operatorname{conv} B\cap\operatorname{conv} C$ is not empty. It is clear that for each finite set $A=\{a_1, \dots, a_n\}$ in R^d with $n\geqq d+2$ one can find a linear map $f\colon R^{d+1}\to R^d$ and a set $A'=\{a_1', \dots, a_n'\}\subset R^{d+1}$ such that $f(a_i')=a_i$ $i=1, 2, \dots, n$ and $\operatorname{int}\operatorname{conv} A'$ is not empty and $\operatorname{vert}\operatorname{conv} A'=A'$. In view of this fact, Radon's theorem can be stated in the following way.

RADON'S THEOREM. *Let $P\subset R^{d+1}$ be a convex polytope with non-empty interior. Put $A=\operatorname{vert} P$. If $f\colon R^{d+1}\to R^d$ is a linear map, then there exist two disjoint sets $B, C\subset A$ such that $f(\operatorname{conv} B)\cap f(\operatorname{conv} C)$ is non-empty.*

The surprising fact here is that the word "linear" can be replaced by "continuous", namely, a continuous analogue of Radon's theorem is true;

THEOREM 1. *Let $P\subset R^{d+1}$ be a convex polytope with non-empty interior. Given an $f\colon \partial P\to R^d$ continuous map, there exist two disjoint faces, B and C, of P such that $f(B)\cap f(C)\neq\emptyset$.*

COROLLARY. *Let T be a $(d+1)$-dimensional simplex. Denote its d-faces by $L_1, L_2, \dots, L_{d+2}$. If $f\colon \partial T\to R^d$ is a continuous map, then $\bigcap_{i=1}^{d+2} f(L_i)$ is non-empty.*

If f is a linear map, then this statement is an easy consequence (in fact, equivalent) of Helly's theorem (see [3]). The interesting fact here is that in this particular case a continuous version of Helly's theorem holds true.

Let us now introduce some notions. Given a convex compact set $C\subset R^{d+1}$ with non-empty interior and a vector $a\in R^{d+1}$, $a\neq 0$, we write

$$C(a)=\{x\in C\colon \langle a, x\rangle=\max_{t\in C}\langle a, t\rangle\}.$$

Two points, x and y, of C are said to be opposite if for some $a\in R^{d+1}$, $x\in C(a)$ and $y\in C(-a)$. If C happens to be a polytope, then $C(a)$ is a proper face of C. In this case we say that the two faces $C(a)$ and $C(-a)$ are opposite.

THEOREM 2. *Given a polytope $P\subset R^{d+1}$ with non-empty interior and a continuous map $f\colon \partial P\to R^d$, there exist two opposite faces, B and C, of P such that $f(B)\cap f(C)$ is non-empty.*

It is evident that opposite faces of P are disjoint. Thus Theorem 2 implies Theorem 1.

Speaking about points instead of faces Theorem 2 can be formulated as follows.

THEOREM 2'. *Given a polytope* $P\subset R^{d+1}$ *with non-empty interior and a continuous map* $f\colon \partial P\to R^d$, *there exist two opposite points, x and y, of P with* $f(x)=f(y)$.

We shall prove this Theorem 2' which yields a generalization of Borsuk's theorem [1]. In order to state Borsuk's theorem put $S^d=\{x\in R^{d+1}\colon \|x\|=1\}$.

BORSUK'S THEOREM. *If* $f\colon S^d\to R^d$ *is a continuous map, then there is a point* $x\in S^d$ *with* $f(x)=f(-x)$.

THEOREM 3. *Let* $C\subset R^{d+1}$ *be a convex compact set with nonempty interior. If* $f\colon \partial C\to R^d$ *is a continuous map, then there exist two opposite points, x and y, of C with* $f(x)=f(y)$.

Again, Theorem 3 implies Theorem 2'. However, we shall get Theorem 3 from Theorem 2' by a simple continuity argument.

Further, our Theorem 3 contains Borsuk's theorem (put simply $C=\operatorname{conv} S^d$). On the other hand, Theorem 2' is proved using Borsuk's theorem.

2. We need a simple proposition.

PROPOSITION. *If P is a polytope in* R^d *and* $x, y, x_n\in P$ $n=1, 2, \dots$ *and* $\lim x_n=x$, *then there is an* $\varepsilon>0$ *and N such that* $x_n+\varepsilon\cdot(y-x)\in P$ *for* $n>N$.

PROOF. This proposition is true for any cone C (instead of P) whose vertex is x (with arbitrary $\varepsilon>0$ and n), so it is true for $C\cap B(x,\delta)$ where $B(x,\delta)$ is the ball with center x and radius δ. But $P\cap B(x,\delta)=C\cap B(x,\delta)$ for a sufficiently small $\delta>0$ where

$$C=\{z\in R^d\colon\ z=x+\lambda(w-x),\ \lambda>0,\ w\in P\}$$

is a cone with vertex x.

PROOF OF THEOREM 2'. Put $Q=P-P$. Q is a polytope with non-empty interior. It is centrally symmetric with respect to the origin. For $x\in Q$ write

$$h(x)=\max\{z\colon\ x=z-w,\ z, w\in P\}$$

where max is meant in the lexicographic ordering of R^{d+1}. Clearly $h\colon Q\to P$ is well-defined. An easy computation shows that the vector w corresponding to $z=h(x)$ equals $h(-x)$.

We claim that h is continuous. Indeed, let $x, x_n\in Q$, $x=\lim x_n$ and $x_n=z_n-w_n$ where $z_n=h(x_n)$. We can choose a subsequence n_i so that z_{n_i} and, consequently w_{n_i} converge. Put $z=\lim z_{n_i}$ and $w=\lim w_{n_i}$; clearly $x=z-w$. We claim that $z=h(x)$. If not, then $z<h(x)$ in the lexicographic ordering. By the Proposition, for a sufficiently small positive ε and large i

$$z'=z_{n_i}+\varepsilon\big(h(x)-z\big)\in P \quad\text{and}\quad w'=w_{n_i}+\varepsilon\big(h(-x)-w\big)\in P.$$

Now $z'-w'=x_{n_i}$ and $z'>z_{n_i}$ contradicting $z_{n_i}=h(x_{n_i})$. This means that $z=h(x)$. Thus, every convergent subsequence of z_n tends to $h(x)$. Now by compactness $\lim z_n=h(x)$, i.e., h is continuous.

Next we claim that $x \in Q(a)$ implies $h(x) \in P(a)$ and $h(-x) \in P(-a)$. Indeed, if $x \in Q(a)$ then $\max_{t \in Q} \langle a, t \rangle = \langle a, x \rangle$. Of course, $x = h(x) - h(-x)$ and $h(x), h(-x) \in P$. Whence

$$\langle a, h(x) \rangle + \langle -a, h(-x) \rangle = \langle a, x \rangle = \max_{t \in Q} \langle a, t \rangle =$$

$$= \max_{u, v \in P} \langle a, u - v \rangle = \max_{u \in P} \langle a, u \rangle + \max_{v \in P} \langle -a, v \rangle$$

and so $h(x) \in P(a)$ and $h(-x) \in P(-a)$. This further implies that for $x \in \partial Q$ $h(x)$ and $h(-x)$ belong to ∂P.

Now we define a map $g: \partial Q \to R^d$ in the following way: for $x \in \partial Q$ let $g(x) = f(h(x))$. This map is welldefined and continuous. Let us observe now that the conditions of Borsuk's theorem are fulfilled for the map g (instead of S^d we have ∂Q here but this is indifferent). In this case Borsuk's theorem says that there is a point $x \in \partial Q$ with $g(x) = g(-x)$. Now there exists $a \in R^{d+1}$, $a \neq 0$ such that $x \in Q(a)$. Then $z = h(x) \in P(a)$ and $w = h(-x) \in P(-a)$, i.e., z and w are opposite points of P and $f(z) = f(h(x)) = g(x) = g(-x) = f(h(-x)) = f(w)$. And this is what we wanted to prove.

PROOF OF THE COROLLARY. It is easy to check that if B and C are disjoint faces of the simplex T, then for any $i = 1, 2, \ldots, d+2$ either $B \subset L_i$ or $C \subset L_i$ (or both). This fact proves the Corollary.

PROOF OF THEOREM 3. Without loss of generality we may suppose that $0 \in \text{int}\, C$.

Now let P be a polytope inscribed in C, i.e., vert $P \subset \partial C$ and suppose further that $0 \in \text{int}\, P$. Then a continuous map $f_P: \partial P \to R^d$ can be defined as $f_P(x) = f(\lambda x)$, where λ is the unique positive number with $\lambda x \in \partial C$. By Theorem 2', there are opposite points of P, z_P and w_P with $f_P(z_P) = f_P(w_P)$.

Now choose a sequence of inscribed polytopes $P_1, P_2, \ldots$ with $0 \in \text{int}\, P_n$. Suppose further that vert $P_n \subset$ vert P_{n+1} and $\partial C \cap \bigcup_1^\infty P_n$ is dense in ∂C. Again, for each n there exist opposite (for P_n) points z_n and w_n with $f_n(z_n) = f_n(w_n)$ where $f_n = f_{P_n}$. Since z_n and w_n are opposite points in P_n there is a vector $a_n \in S^d$ such that $z_n \in P_n(a_n)$ and $w_n \in P_n(-a_n)$.

By the compactness of C and S^d we may suppose that z_n, w_n and a_n converge, their limits are $z, w \in \partial C$ and $a \in S^d$ respectively. It is easy to see that z and w are opposite points of C (with normal a) and $f(z) = f(w)$.

3. REMARKS. 1. Theorem 1 can be interpreted in the following way. Let $P \subset R^{d+1}$ be a convex polytope with non-empty interior. Then it is not possible to make a drawing of ∂P in R^d so that disjoint faces of P be disjoint in the drawing.

2. We can give a second proof of Theorem 2 which is more involved than the above one but does not make use of Borsuk's theorem. It relies on a suitably modified version of the main lemma of [2].

3. The following generalization of Theorem 3 holds true.

THEOREM 4. *Let* $C \subset R^{d+1}$ *be a convex compact set with non-empty interior. Let* f *be a point to set map from* ∂C *to the family of all compact convex subsets of a compact set of* R^d. *If* f *is upper semi-continuous (i.e.,* $x_n \to x$, $y_n \in f(x_n)$, *and* $y_n \to y$

implies $y \in f(x)$), *then there exist two opposite points,* z *and* w, *of* C *with* $f(z) \cap f(w) \neq \emptyset$.

This theorem follows from Theorem 3 nearly the same way as Kakutani's fixed-point theorem follows from Brouwer's one.

4. We conclude with a conjecture. There is a generalization of Radon's theorem which is due to H. TVERBERG [5]. In the spirit of our formulation of Radon's theorem this generalization runs as follows:

THEOREM. *Let* $P \subset R^n$ *be a convex polytope with non-empty interior. Here* $n=(r-1)(d+1)$. *Given an* f: $R^n \to R^d$ *linear map there are disjoint proper faces* $B_1, B_2, \dots, B_r$ *of* P, *such that* $\bigcap_{i=1}^{r} f(B_i)$ *is non-empty.*

We think (but can neither prove nor disprove) that in this theorem it is enough to assume that f: $\partial P \to R^d$ is a continuous map.

ACKNOWLEDGEMENT. We are indebted to Prof. M. Bognár for his valuable comments on an earlier version of this paper and to L. Lovász for the question that led us from Theorem 2′ to Theorem 3.

References

[1] P. ALEXANDROFF, H. HOPF, *Topologie*, Springer (Berlin, 1935).
[2] I. BÁRÁNY, *An algorithm to compute fixed points* (unpublished).
[3] B. GRÜNBAUM, *Convex Polytopes*, Interscience (London, 1967).
[4] S. KAKUTANI, A generalization of Brouwer's fixed-point theorem, *Duke Math. J.*, **8** (1941), 457—458.
[5] H. TVERBERG, A generalization of Radon's theorem, *J. London Math. Soc.*, **41** (1966), 123—128.

(Received February 17, 1978)

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