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ON A COMPLETE METRIC INVARIANT FOR GROUP AUTOMORPHISMS ON THE TORUS

MASASI KOWADA

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1. Introduction. In ergodic theory, there is an isomorphy problem which asks when two metrical automorphisms on a Lebesgue space are metrically isomorphic. This problem for automorphisms has been discussed by introducing several kinds of metric invariants. Among them, spectral type is known to be a complete metric invariant for ergodic automorphisms with discrete spectrum [2]. Since A.Kolmogorov, by using an invariant, entropy, has shown that spectral type is not necessarily a complete metric invariant for general automorphisms, it has been conjectured that entropy is a complete metric invariant for K-automorphisms. Until now, this conjucture has obtained only a few special classes of K-automorphisms, [1] and [3].

In the present paper, we shall be mainly concerned with multi-dimensional transversal flows for a group automorphism on the finite dimensional torus. Such flows play a role of a complete metric invariant for group automorphisms on the torus.

Let M_n be the n-dimensional torus and et A be a continuous group automorphism on M_n. This mapping A becomes a metrical automorphism on the measure space M_n with which the Haar measure and topological Borel field are associated.

We are concernd with the two types of equivalences, metrical and algebraic ones; we say two group automorphisms A_1 and A_2 are metrically equivalent if there exists a measure preserving, $1-1$ mapping σ such that $\sigma^{-1}A_1\sigma\omega = A_2\omega$ for a. e. $\omega \in M_n$. Group automorphisms A_1 and A_2 are said to be algebraically equivalent if there exists a continuous group automorphism θ on M_n such that $\theta^{-1}A_1\theta = A_2$.

Algebraic equivalence implies metrical one, but the converse implication is false, and so entropy is not complete invariant among algebraic equivalent classes; our invariant, ergodic transversal flow with discrete spectrum may serve to the above two types of equivalence problems.

An m-parameter transversal flow $\{Z_s\}$ for a group automorphism A is a group of measure preserving transformations with parameter $s \in R^m$ which satisfies

1) $Z_{s+t} = Z_s Z_t \pmod 0$

2) the mapping $(s, \omega) \to Z_s\omega$ is measurable

3) $AZ_s = Z_{Ts}A$, where T is an $m \times m$ reqular matrix.

The notion of 1-parameter transversal flow was already introduced by Ya. G. Sinai [5] and he showed a sufficient condition for the existence of it. In general, there does not always exist ergodic 1-parameter transversal flow of translation type and we, therefore, proceed to extend the notion of 1-parameter transversal flow into multi-dimentional one to ensure the existence of ergodic one. Our main results read as follow;

1. *For a group automorphism on M_n, there exists an ergodic transversal flow with discrete spectrum* (Theorem 3. 1).
2. *Group automorphisms A_1 and A_2 are metrically equivalent if and only if there exist ergodic n-dimensional transversal flows $\{Z_s^{(1)}\}$ and $\{Z_s^{(2)}\}$ with common discrete spectrum for A_1 and A_2, respectively, which satisfy the relation* $A_kZ_s^{(k)} = Z_{Ts}^{(k)}A_k(k = 1, 2)$ (Theorem 4. 1).
3. *Group automorphisms A_1 and A_2 are algebraically equivalent if and only if there exist ergodic solutions $\{m, \varphi_k, T\}$ $(k=1, 2)$of the equation $\varphi_k(A_k^* g^\wedge) = T^* \varphi_k(g^\wedge)$ such that $\varphi_1(M_n^\wedge) = \varphi_2(M_n^\wedge)$.* (Theorem 5. 1)

For given two group automorphisms A_1 and A_2 on M_n, we can easily construct ergodic transversal flows with discrete spectrum, $\{Z_s^{(1)}\}$ and $\{Z_s^{(2)}\}$, but it should be noted that it is difficult to construct them in such a way that

$$A_kZ_s^{(k)} = Z_{Ts}^{(k)}A_k, \quad k = 1, 2,$$

that is, the matrix T is common for two flows. On the while, in the case of algebraic equivalence, it is sufficient to seek such pair of transversal flows among the class of transversal flows of translation type, and eigenvalues of A_1 and A_2 give us much informations to construct them.

2. Definitions and notations. In this §, we shall give, mainly, the definition of multi-dimensional transversal flow for metric automorphism on a probabilty space.

DEFINITION 2. 1. Let $(\Omega, \mathfrak{B}, \nu)$ be a probability space. Let $\{Z_s\}$ be a family of metric automorphisms on Ω with parameter s which runs over m-dimensional Euclidean space R^m. Then we call $\{Z_s\}$ is an *m-parameter flow*, if it satisfies the group property,

1) $Z_{s+t} = Z_sZ_t(\text{mod } 0)$, $s, t \in R^m$

and the measurability condition,

2) the mapping, $(s, \omega) \rightarrow Z_s\omega$, is bimeasurable.

DEFINITION 2.2. Denote by V_s the unitary operator on $L^2(\Omega, \nu)$ induced by Z_s;

$$V_s f(\omega) = f(Z_s\omega), \quad f \in L^2(\Omega, \nu).$$

The m-parameter flow $\{Z_s\}$ is said to be *ergodic*, if any invariant function of $\{V_s\}$ is a constant function only If there exists a complete orthonormal system $\{f_n\}$ and a set $\{\xi_n\}$ in R^m such that

$$V_s f_n = e^{i<s,\xi_n>} f_n, \quad s \in R^m,$$

then we say the m-parameter flow $\{Z_s\}$ has *the discrete spectrum* $\{\xi_n\}$, where $< \cdot, \cdot >$ means the inner product in R^m.

DEFINITION 2.3. Let A be a metrical automorphism on Ω. An m-parameter flow $\{Z_s\}$ is *the transversal flow of* A if there exists a regular $m \times m$ matrix T such that

$$AZ_s = Z_{Ts}A, \quad s \in R^m.$$

Although we defined an multi-dimensional transversal flow on a general probability space Ω, in the following through, we consider only the case where Ω is the n-dimensional torus M_n. The general cases are commented shortly in §6.

Let M_n be the n-dimensional torus and ν be the normalized Haar measure of M_n. Then a continuous group automorphism on M_n can be considered as a metrical automorphism on the probability space (M_n, ν). It is well known that a group automorphism A on the torus $M_n = R^n/N^n$ is associated with a unimodular matrix which we also denote by the same letter A, if no confusion occurs. Let $M_n^\wedge$ be the character group of M_n. The elements of M_n and that of $M_n^\wedge$ are denoted by $g, h, \cdots$ and $g^\wedge, h^\wedge, \cdots$, respectively.

We cite the well known theorems of P. Halmos and von Neumann [2] which are concerned with the ordinary 1-parameter ergodic flow with discrete spectrum. These theorems still hold in case of ergodic m-parameter flow. We proceed, without the proof, to the followings

THEOREM A. *Every proper value of an ergodic m-parameter flow is simple, and the set of all proper values forms an additive group. Moreover the family of all eigenfunctions multiplied by suitable constants forms a* *group under the multiplication as function.*

THEOREM B. *Two ergodic m-parameter flows with discrete spectrums are metrically equivalent if and only if they are spectrally equivalent.*

We use the above two theorems in the following.

3. The existence of a transversal flow.

The following is an extension of Ya. G. Sinai's result [5; §7].

LEMMA 3.1. *Let A be a group automorphism on M_n, and suppose that there exist a regular $m \times m$ matrix T and φ, a homomorphic imbedding of $M_n^\wedge$ into R^m such that*

$$\varphi(A^* g^\wedge) = T^*[\varphi(g^\wedge)], \quad g^\wedge \in M_n^\wedge, \tag{3.1}$$

where A^* and T^* denote the transposes of matrices A and T, respectively. Then there exists an m-parameter transversal flow $\{Z_s\}$ of A with discrete spectrum. If, in particular, φ is an isomorphism, then $\{Z_s\}$ is ergodic.

PROOF. We define $\{g_s\}$ by

$$(g_s, g^\wedge) = \exp\{i < s, \varphi(g^\wedge) > \}, \tag{3.2}$$

then $\{g_s\}$ forms an m-parameter subgroup of M_n. Define Z_s by

$$Z_s g = g + g_s, \quad g \in M_n, \tag{3.3}$$

then $\{Z_s\}$ is an m-parameter flow. Moreover we have

$$\begin{aligned}(A g_s, g^\wedge) &= (g_s, A^* g^\wedge) = \exp\{i < s, \varphi(A^* g^\wedge) > \}\\ &= \exp\{i < s, T^* \varphi(g^\wedge) > \} = \exp\{i < Ts, \varphi(g^\wedge) > \}\\ &= (g_{Ts}, g^\wedge)\end{aligned}$$

for any $g^\wedge \in M_n^\wedge$, i.e., $A g_s = g_{Ts}$, which implies that $A Z_s = Z_{Ts} A$.

The fact that $\{Z_s\}$ has discrete spectrum is deduced from the following;

$$V_s g^\wedge(g) = g^\wedge(Z_s g) = g^\wedge(g + g_s) = \exp\{i < s, \varphi(g^\wedge) > \} \cdot \hat{g}(g);$$

this shows that the spectrum of $\{Z_s\}$ is just the set $\{\varphi(g^\wedge);\ g^\wedge \in M_n^\wedge\}$

Suppose that φ be an isomorphism. Then the relation,

$$(g_s, g^\wedge) = \exp\{i < s, \varphi(g^\wedge) > \} = 1$$

for any g_s, implies $g^\wedge = 0$, therefore, the subgroup $\{g_s\}$ is dense in M_n. Let $f = \sum_{g^\wedge \in M_n^\wedge} C(g^\wedge) g^\wedge$ be the Fourier expansion of $f \in L^2(M_n)$. Suppose f be an invariant function of $\{V_s\}$, then

$$f = V_s f = \sum C(g^\wedge) e^{i<s, \varphi(g^\wedge)>} \cdot \hat{g}$$

$$= \sum C(g^\wedge) g^\wedge.$$

This implies $C(g^\wedge) = 0$, unless $g^\wedge = 0$. Hence f is a constant function, i.e., $\{Z_s\}$ is ergodic.

Thus a triple $\{m, \varphi, T\}$ yields a transversal flow of A. For convenience, we shall agree to say that a triple $\{m, \varphi, T\}$ is *a solution of* (3. 1). If, it satisfies (3. 1). If, in particular, the flow $\{Z_s\}$ defined by (3. 2) and (3. 3) is ergodic, $\{m, \varphi, T\}$ is said to be *an ergodic solution* of (3. 1).

LEMMA 3. 2. *Any group automorphism A has an ergodic solution of* (3. 1).

PROOF. Let $e_k^\wedge = (\overbrace{0, 0, \cdots, 0, 1}^{k}, 0, \cdots, 0) \in M_n^\wedge$, $\varphi(e_k^\wedge) = e_k^\wedge \in R^n$, $(k = 1, \cdots, n)$ and $A = T$. Then (3. 1) has the ergodic solution $\{n, \varphi, A\}$.

Now we obtain the following existence theorem, which is an easy consequence from the previous two lemmas.

THEOREM 3. 1. *For a group automorphism on M_n, there exist ergodic transversal flows with discrete spectrum.*

We shall agree to say that the m-parameter flow $\{Z_s;\ Z_s g = g + g_s\}$ defined in (3. 3) is *of translation type.*

REMARK. There exist, in fact, group automorphisms which have no ergodic 1-parameter transversal flow of translation type, but have ergodic transversal flows with higher dimensional parameter. For example, the group automorphism associated with the unimodular matrix $\begin{pmatrix} 1 & 3 & 4 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \end{pmatrix}$ is such one.

Once eigenvalues of a matrix A are known, it is easy to form a transversal flow of A with discrete spectrum. To clarify this situation, we shall list several examples in the following.

EXAMPLE 1. Suppose that the matrix A has a real eigenvalue λ and let $r=(r_1, r_2, \cdots, r_n)$ be an eigenvector corresponding to λ. Define a homomorphism φ from $M_n^\wedge$ into R^1 by

$$\varphi(e_k^\wedge) = r_k, \quad k = 1, \cdots, n.$$

Then, by Lemma 3.1, the triple $\{1, \varphi, \lambda\}$ is a solution of (3.1) and it determines a transversal flow $\{Z_s\}$ of A. Since φ is a homomorphism, $\{Z_s\}$ is of the following form as was shown in the proof of Lemma 3.1,

$$Z_s g = g + g_s$$

(refer to Ya. G. Sinai [5]).

EXAMPLE 2. Suppose that the matrix A has an eigenvalue $\lambda=\alpha+i\beta$ and let $r=(r_1, r_2, \cdots, r_n)$ be a corresponding eigenvector. We set $u=(\mathrm{Re}\, r_1, \cdots, \mathrm{Re}\, r_n)$ and $v=(\mathrm{Im}\, r_1, \cdots, \mathrm{Im}\, r_n)$. Then we have the relations, $Au=\alpha u-\beta v$, and $Av=\beta u+\alpha v$. Let $\varphi(g^\wedge)=<u, g^\wedge>$ and $\psi(g^\wedge)=<v, g^\wedge>$. Then $\tau(g^\wedge)=(\varphi(g^\wedge), \psi(g^\wedge))$ is a homomorphism from $M_n^\wedge$ into R^2. Denote by T the matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. Then we obtain

$$\tau(A^* g^\wedge) = T^* \tau(g^\wedge), \quad g^\wedge \in M_n^\wedge.$$

Therefore the triple $\{2, \tau, T\}$ is a solution of (3.1) so that it determines a 2-parameter transversal flow $\{Z_{(s,t)}\,;\ (s, t) \in R^2\}$ of A.

EXAMPLE 3. Consider an ergodic solution $\{m, \varphi, T_1\}$ of (3.1). Set $T_2 = S T_1 S^{-1}$, where S is an $m \times m$ regular real matrix. Let $\psi(g^\wedge)=S\varphi(g^\wedge)$, $g^\wedge \in M_n^\wedge$. Then $\{m, \psi, T_2\}$ is also an ergodic solution of (3.1). Hence, if all eigenvalues of the matrix A is real and if we can find S such that $SAS^{-1}=\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, then we have an ergodic solution $\left\{n, \psi, \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\right\}$, where $\psi(e_k^\wedge)=Se_k^\wedge$, $k=1, \cdots, n$. In this case, the ergodic transversal flow $\{Z_{(s_1, \cdots, s_n)}\}$ constructed by the solution has the following form;

$$Z_{(s_1, \cdots, s_n)} g = g + g_{(s_1, \cdots, s_n)}$$

where $g_{(s_1, \cdots, s_n)} = g_{s_1}^{(1)} + \cdots + g_{s_n}^{(n)}$ and $g_{s_k}^{(k)}$ is defined by the same way as in Example 1 with respect to λ_k, $k=1, \cdots, n$.

In the proof of Lemma 3.2, we have constructed a n-parameter ergodic transversal flow for group automorphism A on M_n. On the other hand, group automorphism may have another *ergodic* transversal flow with discrete spectrum of lower dimensional parameter than n. To get such transversal flow, we may well construct subgroups $\{g_{s_1}^{(1)}\}, \cdots, \{g_{s_m}^{(m)}\}\,(m \leqq n)$corresponding to eigenvalues of A, $\lambda_1, \cdots, \lambda_k, c_{k+1}, \cdots, c_m$, respectively, where $\lambda_1, \cdots, \lambda_k$ are reals, $c_{k+1}, \cdots, c_m$ are complexes and $k+2(m-k) \leqq n$. Let s_j be 1 or 2 dimensional parameters according to $1 \leqq j \leqq k$ or $k+1 \leqq j \leqq m$. Then $k+2(m-k)$ dimensional flow

$$Z_{(s_1,\cdots,s_m)}g = g + g_{s_1}^{(1)} + \cdots + g_{s_m}^{(m)}$$

satisfies the relation

$$AZ_{(s_1,\cdots,s_m)}g = Z_{T(s_1,\cdots,s_m)}Ag\ ,$$

where

$$T = \begin{pmatrix} \lambda_1 \ddots & & & 0 \\ & \lambda_k & & \\ & & \boxed{T_{k+1}} \ddots & \\ 0 & & & \boxed{T_m} \end{pmatrix} \quad \text{and} \quad T_j = \begin{pmatrix} \mathrm{Re}c_j & \mathrm{Im}c_j \\ -\mathrm{Im}c_j & \mathrm{Re}c_j \end{pmatrix},$$

$j = k+1, \cdots, m$. It is easy to see that there exists some positive integer $k+2(m-k)$ $(\leqq n)$ such that the above transversal flow is ergodic.

4. A metrical equivalence theorem for group automorphisms on M_n. In this §, we give a theorem which asserts that a group automorphism on the torus is completely determined by its ergodic transversal flow with discrete spectrum.

LEMMA 4. 1. *Let A be a group automorphism on M_n and $\{Z_s\}$ be an ergodic m-parameter transversal flow of A with discrete spectrum $\Gamma = \{\mu\}$ such that $AZ_s = Z_{T^s}A$. Then there exists a complete orthonormal system $\{f_\mu\}$ in $L^2(M_n)$ with the following properties;*

(1) *f_μ is an eigenfunction of $\{V_s\}$ corresponding to μ*

(2) $f_{\mu+\xi} = f_\mu f_\xi, \quad \mu, \xi \in \Gamma$

(3) $U_A f_\mu = f_{T^*\mu}$,

where $V_s f(g) = f(Z_s g)$ and $U_A f(g) = f(Ag)$.

PROOF. Let $\{h_\mu\,;\ \mu \in \Gamma\}$ be a basis of $L^2(M_n)$ each of which is an eigenfunction of $\{V_s\}$. Since $\{Z_s\}$ is ergodic, the absolute value of h_μ is a constant function, so we may assume $|h_\mu| = 1$. By the relation, $V_s h_\mu h_\xi = e^{i<s,\mu+\xi>} h_\mu h_\xi$ and the simplicity of spectrum of $\{V_s\}$, we get $h_\mu h_\xi = C(\mu, \xi) h_{\mu+\xi}$, where $|C(\mu, \xi)| = 1$. Put $f_\mu = \overline{h_\mu(0)} h_\mu$, where $\overline{h_\mu(0)}$ is the complex conjugate of the value of h_μ at the eidentity 0 of M_n. It is trivial to see (1). The relation (2) is deduced from the following;

$$f_\mu f_\xi = \overline{h_\mu(0)}\,\overline{h_\xi(0)} h_\mu h_\xi = \overline{C(\mu, \xi)}\,\overline{h_{\mu+\xi}(0)} C(\mu, \xi) h_{\mu+\xi} = f_{\mu+\xi} .$$

To obtain (3), note that $U_A V_{Ts} = V_s U_A$ and

$$U_A V_s f_\mu = e^{i<s,\mu>} U_A f_\mu = e^{i<T^{-1}s\ T^*\mu>} U_A f_\mu = V_{T^{-1}s} U_A f_\mu .$$

Hence

$$U_A f_\mu(0) = c f_{T^*\mu}(0) = f_\mu(A(0)) = f_\mu(0) = 1 ,$$

and we get, $c = 1$.

THEOREM 4. 1. *Let A_1 and A_2 be group automorphisms on M_n.*(*) *Then they are metrically equivalent if and only if there exist ergodic transversal flows $\{Z_s^{(1)}\}$ and $\{Z_s^{(2)}\}$ with discrete spectrum of A_1 and A_2, respectively, which satisfy*

(a) *they are spectrally equivalent*
(b) $A_k Z_s^{(k)} = Z_{Ts}^{(k)} A_k$, $k = 1, 2$.

PROOF. Necessity : Let σ be a metrical automorphism on M_n such that $\sigma^{-1} A_1 \sigma = A_2$. By Theorem 3. 1, there exists an ergodic transversal flow $\{Z_s^{(1)}\}$ with discrete spectrum such that $A_1 Z_s^{(1)} = Z_{Ts}^{(1)} A_1$. Let $Z_s^{(2)} = \sigma^{-1} Z_s^{(1)} \sigma$, then we get

$$A_2 Z_s^{(2)} = (\sigma^{-1} A_1 \sigma)(\sigma^{-1} Z_s \sigma) = \sigma^{-1} Z_{Ts}^{(1)} A_1 \sigma = Z_{Ts}^{(2)} A_2 .$$

It is easy to see that $\{Z_s^{(2)}\}$ has the required properties.

Sufficiency : Denote the common discrete spectrum of $\{Z_s^{(1)}\}$ and $\{Z_s^{(2)}\}$ by $\Gamma = \{\mu_j, j = 1, 2, \cdots\}$. Then Γ forms discrete additive group, and so the character group G of Γ is a compact abelian group. Let $\{f_\mu ;\ \mu \in \Gamma\}$ be a family of eigenfunctions of $\{V_s^{(1)}\}$ which satisfies the conditions of (1)~(3) in Lemma 4. 1. For convenience, we rewrite μ by h_μ when μ is regarded as a function in $L^2(G)$; $h_\mu(\omega) = (\omega, \mu)$, $\omega \in G$. The mapping $f_\mu \rightarrow h_\mu$ can be extended to a unitary operator W_1 from $L^2(M_n)$ to $L^2(G)$.

By Theorem A, we see that W_1 is multiplicative, and therefore there exists a metrical isomorphism σ_1 from G into M_n such that

$$h_\mu(\omega) = W_1 f_\mu(\omega) = f_\mu(\sigma_1 \omega), \quad \omega \in G, \quad j = 1, 2, \cdots$$

(*) Note that A_1 and A_2 are not necessarilly ergodic.

.Let

$$\widetilde{A} = \sigma_1^{-1}A\cdot\sigma_1, \quad \widetilde{Z}_s^{(1)} = \sigma_1^{-1}Z_s^{(1)}\sigma_1, \quad \text{and} \quad (\omega_s, \mu) = \exp\{i<s, \mu>\}.$$

Then we get

$$\widetilde{A}_1\widetilde{Z}_s^{(1)} = \widetilde{Z}_{Ts}^{(1)}\widetilde{A}_1$$

$$\begin{aligned}(\widetilde{Z}_s^{(1)}\omega, \mu) &= h_\mu(\sigma_1^{-1}Z_s^{(1)}\sigma_1\omega) = f_\mu(Z_s^{(1)}\sigma_1\omega) \\ &= e^{i<s,\mu>}f_\mu(\sigma_1\omega) = (\omega_s, \mu)(\omega, \mu) = (\omega_s + \omega, \mu),\end{aligned}$$

namely,

$$Z_s^{(1)}\omega = \omega + \omega_s. \qquad -(4.1)$$

Moreover we have

$$\begin{aligned}(\widetilde{A}_1\omega_s, \mu) &= h_\mu(\sigma_1^{-1}A_1\sigma_1\omega_s) = f_\mu(A_1\sigma_1\omega_s) = f_{T^*\mu}(\sigma_1\omega_s) \\ &= (\omega_s, T^*\mu) = e^{i<s,T^*\mu>} = e^{i<Ts,\mu>} = (\omega_{Ts}, \mu).\end{aligned}$$

Hence we obtain

$$\widetilde{A}_1\omega_s = \omega_{Ts} \qquad -(4.2)$$

We shall show that $\widetilde{A}_1$ is a group automorphism on G. We have

$$\begin{aligned}h_\mu(\widetilde{A}_1(\omega + \omega')) &= f_\mu(\sigma_1\widetilde{A}_1(\omega + \omega')) = f_\mu(A_1\sigma_1(\omega + \omega\ )) \\ &= f_{T^*\mu}(\sigma_1(\omega + \omega')) = h_{T^*\mu}(\omega + \omega\ ) \\ &= h_{T^*\mu}(\omega)h_{T^*\mu}(\omega') = f_{T^*\mu}(\sigma_1\omega)f_{T^*\mu}(\sigma_1\omega') \\ &= f_\mu(A_1\sigma_1\omega)f_\mu(A_1\sigma_1\omega\ ) = h_\mu(\widetilde{A}_1\omega)h_\mu(\widetilde{A}_1\omega\ ) \\ &= h_\mu(\widetilde{A}_1\omega + \widetilde{A}_1\omega'), \quad \text{for any} \quad \mu \in G^\wedge.\end{aligned}$$

By the same way as σ_1, we can define an isomorphism σ_2 from G into M_n, and using σ_2, we define a group automorphism $\widetilde{A}_2$ on G and its transversal flow $\{\widetilde{Z}_s^{(2)}\}$. Then we obtain the similar formulas to (4.1) and (4.2); $\widetilde{Z}_s^{(2)}\omega = \omega + \omega_s$ and $\widetilde{A}_2\omega_s = \omega_{Ts} = \widetilde{A}_1\omega_s$. So we conclude, $\widetilde{A}_1 = \widetilde{A}_2$. Let $\sigma = \sigma_1\sigma_2^{-1}$, then we get $\sigma^{-1}A_1\sigma = A_2$. This completes the proof.

5. An algebraic equivalence for group automorphisms on M_n. As an application of the previous discussions, we shall give a necessary and sufficient condition of algebraic equivalence for group automorphisms on M_n.

THEOREM 5. 1. *Let* A_1 *and* A_2 *be group automorphisms on* M_n. *Then they are algebraically equivalent if and only if there exist ergodic solutions* $\{m, \varphi_k, T\}$ $(k=1, 2)$ *of*

$$\varphi_k(A_k^* g^\wedge) = T^* \varphi_k(g^\wedge) \qquad -(5.1)$$

such that $\varphi_1(M_n^\wedge) = \varphi_2(M_n^\wedge)$.

PROOF. Let $\{Z_s^{(k)}\}$ be ergodic transversal flow of A_k which is defined in the same way as in (3. 3). Since $\{Z_s^{(k)}\}$ is of translation type, all characters $g^\wedge \in M_n^\wedge$ are eigenvectors of $\{Z_s(k)\}$, and moreover the spectrum of it is just the set $\varphi_k(M_n)$; we denote the set $\varphi_1(M_n^\wedge) = \varphi_2(M_n^\wedge)$ by Γ. Let σ_1 and σ_2 be isomorphisms from $G = \Gamma^\wedge$ onto M_n which are defined in the proof of Theorem 4. 1. Since σ_k is character preserving, namely, the function $\hat{g}(\sigma_k \omega)$ in $\omega \in G$ is a character of G, we see that σ_k is a continuous group isomorphism from G onto M_n. Thus we get the continuous group automorphisms $\widetilde{A}_k = \sigma_k^{-1} A_k \sigma_k (k=1, 2)$ such that $\widetilde{A}_1 \omega_s = \widetilde{A}_2 \omega_s$, $s \in R^m$. Since the subgroup $\{\omega_s\} \subset G$ is dense in G, we conclude $A_1 = A_2$, namely, $(\sigma_1 \sigma_2^{-1})^{-1} A_1 (\sigma_1 \sigma_2^{-1}) = A_2$. Clearly $\sigma_1 \sigma_2^{-1}$ is a contnuous group automorphism on M_n.

As illustrated in Examples 1~3, a solution $\{m, \varphi, T\}$ can be constructed from the eigenvalues and eigenvectors of A. Note that it may happen that only one eigenvalue with eigenvector corresponding to it yields us an ergodic solution. We shall state about this case as corollary.

COROLLARY. *Group automorphisms* A_1 *and* A_2 *are algebraically equivalent if the unimodular integral matrices associated with them have eigenvectors* $\mathbf{r}_1 = (r_1^{(1)}, r_2^{(1)}, \cdots, r_n^{(1)})$ *and* $\mathbf{r}_2 = (r_1^{(2)}, r_2^{(2)}, \cdots, r_n^{(2)})$ *corresponding to an eigenvalue* λ *in common such that* $\{r_j^{(1)}\}$ *are integrally independent, and* $\mathbf{r}_2 = B\mathbf{r}_1$, *where* B *is an unimodular integral matrix.*

6. Concluding remarks. As was shown in §2, our invariant can be defined for not only group automorphisms on the torus but also a general class of metrical automorphisms on a probability space. It is still open when it exists for metrical automorphisms; a partial answer to this problem can be given as follows.

1. *If a metrical automorphism* A *on a Lebesgue space has an ergodic m-parameter transversal flow with discrete spectrum,* A *is isomorphic to an*

automorphism (not necessarily group automorphism) on a compact abelian group with which the Haar measure is associated.

2. *Let G be a compact abelian group with the Haar measure and let $G^{\wedge}$ be its character group. Let A be a group automorphism on G and suppose that there exists φ, an isomorphic imbedding of $G^{\wedge}$ into R^m such that*

$$\varphi(A^*g^{\wedge}) = T^*\varphi(g^{\wedge}), \quad g^{\wedge} \in G^{\wedge},$$

where T is an $m \times m$ regular matrix. Then we can construct an ergodic m-parameter transversal flow of A in a similar way as in §3.

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DEPARTMENT OF MATHEMATICS
NAGOYA INST. OF TECHN.
NAGOYA, JAPAN