ON A CONJECTURE BY WILF ABOUT THE FROBENIUS NUMBER

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Abstract. Given coprime positive integers $a_1 < \cdots < a_d$, the Frobenius number F is the largest integer which is not representable as a non-negative integer combination of the a_i's. Let n denote the number of integers less than F admitting such a representation: Wilf conjectured that $F+1 \leq nd$. We prove that for every fixed value of $\lceil \frac{a_1}{d} \rceil$ the conjecture holds for all values of a_1 which are sufficiently large and are not divisible by a finite set of primes. We also propose a generalization in the context of one-dimensional local rings and a question on the equality $F+1 = nd$.

Introduction

The diophantine Frobenius problem or money-changing problem consists of finding the greatest integer F, called the Frobenius number, that is not representable as a non-negative integer combination of given d coprime positive integers $a_1 < \cdots < a_d$. The problem has stimulated much research over the past decades, due to the applications to several areas of pure and applied mathematics including coding theory, complexity theory, linear algebra, combinatorics and commutative algebra. Since finding general explicit formulas is considerably hard (cf. [5]), the attention is generally focused on solutions in special cases, algorithms or bounds. The monograph [11] explores several viewpoints about the problem.

In 1978 H. S. Wilf proposed a general upper bound for the Frobenius number:

$$F+1 \leq nd \tag{1}$$

where n is the the number of integers less than F that are representable as a non-negative integer combination of $a_1, \dots, a_d$. Although the problem has been considered by several authors (cf. [1], [3], [7], [8], [9], [12], [13], [16]) only special cases have been solved and it remains wide open. The approach we follow in this paper is to study the poset structure of the so-called Apéry set once the value of the ratio $\rho = \lceil \frac{a_1}{d} \rceil$ is fixed. The cases $\rho = 1$ and $\rho = 2$ were solved in [7] and [13] respectively; in this work we focus on the general case. Our main result is the following:

Theorem 1. *For every value of $\rho = \lceil \frac{a_1}{d} \rceil$ Wilf's inequality (1) holds if a_1 is large enough and the prime factors of a_1 are greater than or equal to ρ.*

In the first section of the paper we recall some definitions and preliminary results, whereas the second section is devoted to proving Theorem 1. We conclude the paper with a word about the equality in (1) and a discussion of Wilf's inequality in the context of commutative algebra.

1. Preliminaries

We denote by $\mathbb{N}$ the non-negative integers. Let $\mathcal{S} = \left\{ \sum_{i=1}^{d} \lambda_i a_i : \lambda_i \in \mathbb{N} \right\}$ be the numerical semigroup generated by d relatively prime positive integers $a_1 < \cdots < a_d$. For the rest of the

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paper we assume without loss of generality that $a_1, \dots, a_d$ are the minimal generators of $\mathcal{S}$, i.e. no proper subset generates $\mathcal{S}$; this implies that $d \leq a_1$. The condition $\gcd(a_1, \dots, a_d) = 1$ yields that $\mathbb{N} \setminus \mathcal{S}$ is a finite set and the integer $F = \max \mathbb{Z} \setminus \mathcal{S}$ is called the Frobenius number of $\mathcal{S}$. The Apéry set of $\mathcal{S}$ is defined as

$$\mathrm{Ap}(\mathcal{S}) = \big\{\omega \in \mathcal{S} \,:\, \omega - a_1 \notin \mathcal{S}\big\}$$

and thus it consists of the smallest elements of $\mathcal{S}$ in each residue class modulo a_1. It follows that $0 \in \mathrm{Ap}(\mathcal{S})$, $|\mathrm{Ap}(\mathcal{S})| = a_1$ and that $F + a_1$ is the largest element in $\mathrm{Ap}(\mathcal{S})$. We list the elements of $\mathrm{Ap}(\mathcal{S})$ increasingly setting $\mathrm{Ap}(\mathcal{S}) = \{\omega_0 < \omega_1 < \cdots < \omega_{a_1-1}\}$, notice that we have $\omega_0 = 0, \omega_1 = a_2, \omega_{a_1-1} = F + a_1$. We establish a partial order on $\mathbb{N}$ by setting $n_1 \preceq n_2$ if there exists $s \in \mathcal{S}$ such that $n_1 + s = n_2$ and we consider throughout the paper the poset $(\mathrm{Ap}(\mathcal{S}) \setminus \{0\}, \preceq)$. The minimal elements in $(\mathrm{Ap}(\mathcal{S}) \setminus \{0\}, \preceq)$ are exactly $\{a_2, \dots, a_d\}$ and if $\tau \in \mathcal{S}, \omega \in \mathrm{Ap}(\mathcal{S})$, $\tau \preceq \omega$ then $\tau \in \mathrm{Ap}(\mathcal{S})$. We refer to [12] for details on numerical semigroups.

Now we give further definitions in order to introduce an equivalent formulation of Wilf's inequality. For each $k \in \mathbb{N}$ let $I_k = \big[ka_1, ka_1 - 1\big]$ and $n_k = \big|\mathcal{S} \cap \big[0, F\big] \cap I_k\big|$. We write

$$F + 1 = Qa_1 + R$$

with $Q, R \in \mathbb{N}$ and $2 \leq R \leq a_1$. Note that $R \neq 1$ as $F \notin \mathcal{S}$ and that I_Q is the interval containing the Frobenius number. We define the numbers

$$\eta_j = \Big|\big\{k \in \mathbb{N} \,:\, \big|I_k \cap \mathcal{S}\big| = j\big\}\Big| \qquad \text{and} \qquad \epsilon_j = \Big|\big\{k \in \mathbb{N} \,:\, \big|I_k \cap \mathcal{S}\big| = j,\, 0 \leq k \leq Q-1\big\}\Big|$$

for each $j \in \big\{1, 2, \dots, a_1 - 1\big\}$. In other words, η_j counts the intervals I_k containing exactly j elements of $\mathcal{S}$, while ϵ_j only counts such intervals among the first Q. The two definitions differ slightly, and the numbers η_j can be expressed in terms of $\mathrm{Ap}(\mathcal{S})$:

Lemma 2 ([13], Lemma 12, Proposition 13)**.** *The following properties hold:*

(1) $\epsilon_j = \eta_j - 1$ *if* $j = \big|I_Q \cap \mathcal{S}\big|$ *and* $\epsilon_j = \eta_j$ *otherwise;*
(2) $\eta_j = \lfloor \frac{\omega_j}{a_1} \rfloor - \lfloor \frac{\omega_{j-1}}{a_1} \rfloor$ *for all* $1 \leq j \leq a_1 - 1$.

The numbers ϵ_j give rise to an expression for the difference of the two sides of (1):

Proposition 3 ([13], Proposition 10, Remark 11)**.** *We have the equation*

$$nd - (F + 1) = \sum_{j=1}^{a_1-1} \epsilon_j (jd - a_1) + (n_Q d - R) =: \Delta.$$

2. Proof of the main theorem

In this section we prove Theorem 1 using Proposition 3. We need a series of lemmas first.

Lemma 4. *If* $x \in \mathbb{N} \setminus \{0\}$ *is smaller than every prime factor of* a_1 *then* $\lfloor \frac{\omega_x}{a_1} \rfloor \leq x \lfloor \frac{\omega_1}{a_1} \rfloor + x - 1$.

Proof. The assumption on x implies that $i\omega_1 \not\equiv j\omega_1 \pmod{a_1}$ for any $0 \leq i < j \leq x$: in fact $i\omega_1 \equiv j\omega_1 \pmod{a_1}$ yields the contradiction $(j-i)\omega_1 \equiv 0 \pmod{a_1}$ as $\gcd(j - i, a_1) = 1$ and $\omega_1 \not\equiv 0 \pmod{a_1}$. Thus the subset $\{\omega_1, 2\omega_1, \dots, x\omega_1\} \subseteq \mathcal{S}$ covers x different residue classes modulo a_1, hence there are elements of at least x different classes less than or equal to $x\omega_1$ in $\mathcal{S}$. By definition of Apéry set we deduce that $\omega_x \leq x\omega_1$ and $\lfloor \frac{\omega_x}{a_1} \rfloor \leq \lfloor \frac{x\omega_1}{a_1} \rfloor \leq x \lfloor \frac{\omega_1}{a_1} \rfloor + x - 1$. □

Lemma 5. *If* $y \in \mathbb{N}$ *satisfies* $y \geq 2$ *and* $a_1 - d \geq \binom{y}{2} + 1$ *then* $\omega_{a_1-1} \geq \omega_y + \omega_1$. *In particular,* $\lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor \geq \lfloor \frac{\omega_y}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor$ *and* $F > \omega_y$.

Proof. Since the minimal elements in $(\mathrm{Ap}(\mathcal{S})\backslash\{0\}, \preceq)$ are exactly $\{a_2, \dots, a_d\}$ and $|\mathrm{Ap}(\mathcal{S})| = a_1$, there are at least $\binom{y}{2}+1$ non-minimal elements in $(\mathrm{Ap}(\mathcal{S})\backslash\{0\}, \preceq)$. The set $\mathcal{Y} = \{\omega_i+\omega_j \mid 1 \le i \le j \le y-1\}$ contains at most $\binom{y}{2}$ distinct elements, so there exists $\tau \in \mathrm{Ap}(\mathcal{S})\backslash(\{0, a_2, \dots, a_d\}\cup\mathcal{Y})$. In particular τ is not minimal, thus $\tau = \omega_h + \omega_k$ for some $1 \le h \le k$, and $\tau \notin \mathcal{Y}$ yields $k \ge y$. Finally we have $\omega_{a_1-1} \ge \tau = \omega_h + \omega_k \ge \omega_1 + \omega_y$. The inequality with the floor function follows immediately, and from $\omega_1 > a_1$ we obtain $\omega_{a_1-1} = F + a_1 \ge \omega_y + \omega_1 > \omega_y + a_1$ and $F > \omega_y$. $\square$

Lemma 6. *If $y, z \in \mathbb{N}$ satisfy $y \ge 2$, $y \ge z$, $a_1 - d \ge \binom{y}{2} + 1$, and $\lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor = \lfloor \frac{\omega_z}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor$ then $n_Q \ge y - z + 3$.*

Proof. Fix $z \le i \le y$, by Lemma 5 we have that $F + a_1 \ge \omega_1 + \omega_y \ge \omega_1 + \omega_i$ and

$$\left\lfloor \frac{\omega_{a_1-1}}{a_1} \right\rfloor \ge \left\lfloor \frac{\omega_y}{a_1} \right\rfloor + \left\lfloor \frac{\omega_1}{a_1} \right\rfloor \ge \left\lfloor \frac{\omega_i}{a_1} \right\rfloor + \left\lfloor \frac{\omega_1}{a_1} \right\rfloor \ge \left\lfloor \frac{\omega_z}{a_1} \right\rfloor + \left\lfloor \frac{\omega_1}{a_1} \right\rfloor = \left\lfloor \frac{\omega_{a_1-1}}{a_1} \right\rfloor$$

and this leads to $\lfloor \frac{\omega_y}{a_1} \rfloor = \lfloor \frac{\omega_i}{a_1} \rfloor = \lfloor \frac{\omega_z}{a_1} \rfloor$. From $F + 1 = Qa_1 + R$ we obtain $(Q+1)a_1 + (R-1) \ge \omega_1 + \omega_i$. Dividing by a_1 we get $\omega_1 = q_1 a_1 + r_1$ and $\omega_i = q_i a_1 + r_i$ with $0 \le r_i \le a_1 - 1$ and it follows that $Q + 1 = \lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor = \lfloor \frac{\omega_z}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor = \lfloor \frac{\omega_i}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor = q_1 + q_i$. From $\omega_{a_1-1} \ge \omega_i + \omega_1$ we obtain $(Q+1)a_1 + (R-1) \ge (q_i+q_1)a_1 + (r_i+r_1) = (Q+1)a_1 + (r_i+r_1)$, thus $R-1 \ge r_i + r_1$. In particular, $r_1, r_i < R$: we conclude that the elements $Qa_1, \omega_1 + l_1 a_1$ and $\omega_i + l_i a_1$ belong to $\mathcal{S} \cap I_Q \cap \big[0, F\big]$ for suitable $l_1, l_i \in \mathbb{N}$ for each $z \le i \le y$ so that $n_Q \ge 3 + y - z$. $\square$

Lemma 7. *If $y \in \mathbb{N}$ satisfies $y \ge 2$ and $a_1 - d \ge \binom{y}{2} + 1$ then $n_{Q-1} \ge y + 2 - n_Q$.*

Proof. By Lemma 5 we have $a_1 < \omega_y < F$, whence $Q > 0$ and $\big\{\omega_0, \dots, \omega_y\big\} \subseteq \big[0, F\big]$. At most $n_Q - 1$ of these elements belong to $\mathcal{S} \cap I_Q \cap \big[0, F\big]$, because $Qa_1 \notin \mathrm{Ap}(\mathcal{S})$. Hence there are at least $y + 2 - n_Q$ elements ω_i smaller than Qa_1, that is, $\omega_0 < \cdots < \omega_{y-n_Q+1} < Qa_1$: we conclude that $\omega_i + l_i a_1 \in I_{Q-1}$ for suitable $l_i \in \mathbb{N}$ for each $0 \le i \le y - n_Q + 1$ so $n_{Q-1} \ge y + 2 - n_Q$. $\square$

We are now ready to prove the main result.

Proof of Theorem 1. Let $\rho = \lceil \frac{a_1}{d} \rceil$. Since the cases $\rho = 1, 2$ have been solved, we assume $\rho \ge 3$. Consider the integers

$$y = \frac{3\rho^2 - \rho - 4}{2} \qquad \text{and} \qquad z = \frac{\rho^2 + \rho - 2}{2}$$

notice that $y \ge z \ge \rho + 2$. We are going to make the assumption that $a_1 \ge \frac{\rho}{\rho-2}\binom{y}{2}$, so that the condition appearing in lemmas 5, 6, 7 is satisfied:

$$a_1 - d \ge (\rho-1)d + 1 - d = (\rho-2)d + 1 \ge \frac{\rho-2}{\rho} a_1 + 1 \ge \binom{y}{2} + 1.$$

Our task is to prove (1) by showing that the quantity Δ of Proposition 3 is non-negative; we will actually show that $\Delta > 0$. We are going to break the summation in two parts, at the index $j = \rho$. For the first part we use both properties of Lemma 2 and obtain:

$$\begin{aligned}\sum_{j=1}^{\rho} \epsilon_j (jd - a_1) \;&\ge\; \sum_{j=1}^{\rho} \eta_j (jd - a_1) - (\rho d - a_1) = \sum_{j=1}^{\rho} \left(\left\lfloor \frac{\omega_j}{a_1} \right\rfloor - \left\lfloor \frac{\omega_{j-1}}{a_1} \right\rfloor \right) (jd - a_1) - (\rho d - a_1) \\ &= \left\lfloor \frac{\omega_\rho}{a_1} \right\rfloor (\rho d - a_1) - d \sum_{j=1}^{\rho-1} \left\lfloor \frac{\omega_j}{a_1} \right\rfloor - (\rho d - a_1) \ge -d \sum_{j=1}^{\rho-1} \left\lfloor \frac{\omega_j}{a_1} \right\rfloor\end{aligned}$$

where we used that $\omega_0 = 0$ and $\omega_\rho > \omega_1 > a_1$. By the assumption on the factors of a_1 we can use Lemma 4 for each $j < \rho$, yielding

$$\sum_{j=1}^{\rho-1} \left\lfloor \frac{\omega_j}{a_1} \right\rfloor \leq \sum_{j=1}^{\rho-1} \left(j \left\lfloor \frac{\omega_1}{a_1} \right\rfloor + j - 1 \right) = \binom{\rho}{2} \left\lfloor \frac{\omega_1}{a_1} \right\rfloor + \binom{\rho-1}{2}$$

and therefore

$$\sum_{j=1}^{\rho} \epsilon_j (jd - a_1) \geq -\frac{d(\rho^2 - \rho)}{2} \left\lfloor \frac{\omega_1}{a_1} \right\rfloor - \frac{d(\rho^2 - 3\rho + 2)}{2}.$$

Moreover, since $\frac{1}{2}d(\rho^2 - \rho) \leq \frac{1}{2}d(\rho^2 - \rho) + \rho d - a_1 = \frac{1}{2}d(\rho^2 + \rho) - a_1 = (z+1)d - a_1$ then

$$\sum_{j=1}^{\rho} \epsilon_j (jd - a_1) \geq -\big((z+1)d - a_1\big) \left\lfloor \frac{\omega_1}{a_1} \right\rfloor - \frac{d(\rho^2 - 3\rho + 2)}{2}.$$

For the second part of the summation in Δ, as $z + 1 \geq \rho + 3$ we can write

$$\sum_{j=\rho+1}^{a_1-1} \epsilon_j (jd - a_1) \geq \sum_{j=z+1}^{a_1-1} \epsilon_j (jd - a_1) \geq \sum_{j=z+1}^{a_1-1} \epsilon_j \big((z+1)d - a_1\big) = \big((z+1)d - a_1\big) \sum_{j=z+1}^{a_1-1} \epsilon_j \geq$$

$$\geq \big((z+1)d - a_1\big) \left(\sum_{j=z+1}^{a_1-1} \eta_j - 1 \right) = \big((z+1)d - a_1\big) \left(\left\lfloor \frac{\omega_{a_1-1}}{a_1} \right\rfloor - \left\lfloor \frac{\omega_z}{a_1} \right\rfloor - 1 \right)$$

where we used again both properties of Lemma 2 in the last inequality. Combining the two parts gives

$$\Delta = \sum_{j=1}^{a_1-1} \epsilon_j (jd - a_1) + (n_Q d - R) \geq$$

$$\left(\left\lfloor \frac{\omega_{a_1-1}}{a_1} \right\rfloor - \left\lfloor \frac{\omega_z}{a_1} \right\rfloor - \left\lfloor \frac{\omega_1}{a_1} \right\rfloor - 1 \right) \big((z+1)d - a_1\big) - \frac{d(\rho^2 - 3\rho + 2)}{2} + (n_Q d - R) =: \Pi.$$

Since $z \leq y$, by Lemma 5 we have the inequality $\lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor \geq \lfloor \frac{\omega_z}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor$. If the equality holds, we can use the bound for n_Q in Lemma 6 obtaining

$$\Delta \geq \Pi \geq -\big((z+1)d - a_1\big) - \frac{d(\rho^2 - 3\rho + 2)}{2} + \big((y - z + 3)d - R\big) =$$

$$= d\left(y - 2z - \frac{\rho^2 - 3\rho + 2}{2} + 2 \right) + (a_1 - R) = d + (a_1 - R) > 0$$

by the definitions of y and z and the fact that $a_1 \geq R$. Suppose now we have the strict inequality $\lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor > \lfloor \frac{\omega_z}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor$. In this case the first piece in Π is non-negative because $(z+1)d - a_1 \geq \rho d - a_1 \geq 0$, thus we can ignore it:

$$\Pi \geq -\frac{d(\rho^2 - 3\rho + 2)}{2} + (n_Q d - R).$$

If $n_Q \geq \frac{1}{2}(\rho^2 - \rho + 4)$, then $\Delta \geq \Pi \geq \rho d - R + d \geq \rho d - a_1 + d > 0$. Suppose finally that $n_Q \leq \frac{1}{2}(\rho^2 - \rho + 2)$. By Lemma 7 we know that $n_{Q-1} \geq y + 2 - n_Q$, i.e. $\mathcal{S} \cap I_{Q-1}$ contains at least $y + 2 - n_Q$ elements, and in this case from $\rho \geq 3$ it follows

$$y + 2 - n_Q \geq \frac{3\rho^2 - \rho - 4}{2} + 2 - \frac{\rho^2 - \rho + 2}{2} = \rho^2 - 1 \geq \frac{\rho^2}{2} + \frac{3\rho}{2} - 1 \geq \frac{\rho^2 + \rho - 2}{2} + \rho \geq z + 3.$$

But in the estimation of the second part of the summation in Δ we only used that $\mathcal{S} \cap I_{Q-1}$ contains at least $z+1$ elements, so we can refine the estimation by adding $\big((y+2-n_Q)-(z+1)\big)d$, obtaining

$$\Delta = \sum_{j=1}^{a_1-1} \epsilon_j(jd - a_1) + (n_Q d - R) \geq \Pi + (y + 2 - n_Q - z - 1)d \geq$$

$$-\frac{d(\rho^2 - 3\rho + 2)}{2} - R + (y - z + 1)d = \frac{d(\rho^2 + \rho - 2)}{2} - R \geq \frac{d(3\rho + 3 - 2)}{2} - R \geq (\rho d - R) + \frac{1}{2} > 0$$

where we used that $\rho \geq 3$ and $\rho d \geq a_1 \geq R$.

We have shown that $\Delta > 0$ in each case and the theorem is thus proved. □

3. Two further problems

We focus now on the equality in Wilf's conjecture. In his original paper [15], Wilf also asked whether the equality in (1) is attained if and only if $d = a_1$ and $a_i = a_1 + (i-1)$ for $i = 2, \dots, a_1$. While this is easily seen not to be the case, we believe that it can only occur in two cases.

Question 8. *Is it true that the equality $F + 1 = nd$ holds if and only if either $d = 2$ or $d = a_1$ and there exists $K \in \mathbb{N}$ such that $a_i = Ka_1 + (i-1)$ for $i = 2, \dots, a_1$?*

Note that in the latter case the numerical semigroup has the form

$$\mathcal{S} = \{0, a_1, 2a_1, \dots, (K-1)a_1, Ka_1, Ka_1 + 1, Ka_1 + 2, \dots\}$$

and the equality follows from Proposition 3, while in the former case the equality was already known to Sylvester (cf. [14]). We observe that, in order to show that these are the only two cases, it suffices to prove that either $d = 2$ or $d = a_1$. In fact, if $d = a_1$ and $F + 1 = nd$ then Proposition 3 implies the equation

$$\Delta = \sum_{j=1}^{a_1-1} \epsilon_j(j-1)a_1 + (n_Q a_1 - R) = 0$$

and since $n_Q \geq 1, a_1 \geq R$ we conclude that $\epsilon_j = 0$ for every $j \geq 2$, $n_Q = 1, R = a_1$; it follows that $a_i = (Q+1)a_1 + (i-1)$ for $i = 2, \dots, a_1$. Further evidence in support of a positive answer to Question 8 is given by the proof of Theorem 1, as in all the cases investigated therein (where $\rho \geq 3$, hence $d < a_1$) the strict inequality was actually seen to hold.

Finally, we remark that the conjecture has also an interpretation in commutative algebra in terms of length inequalities. Let $\mathcal{R} = \mathbb{k}[[t^{a_1}, \dots, t^{a_d}]]$ be the local ring of a monomial curve, $\overline{\mathcal{R}} = \mathbb{k}[[t]]$ be the integral closure of $\mathcal{R}$ in its field of fractions $\mathcal{Q} = \mathbb{k}((t))$ and $\mathfrak{C} = (\mathcal{R} :_{\mathcal{Q}} \overline{\mathcal{R}})$ be the conductor of $\mathcal{R}$ in $\overline{\mathcal{R}}$, that is the largest common ideal of $\mathcal{R}$ and $\overline{\mathcal{R}}$. Denoting by $\ell(\cdot)$ the length of an $\mathcal{R}$-module, the values of $\ell(\overline{\mathcal{R}}/\mathcal{R})$ and $\ell(\mathcal{R}/\mathfrak{C})$ are both measures of the singularity of $\mathcal{R}$ and they are related by $\ell(\mathcal{R}/\mathfrak{C}) \leq \ell(\overline{\mathcal{R}}/\mathcal{R})$, with equality holding if and only if $\mathcal{R}$ is Gorenstein. Under the notation of this paper, the embedding dimension of the ring is $\nu(\mathcal{R}) = d$, whereas $\ell(\overline{\mathcal{R}}/\mathcal{R}) = F + 1 - n$ and $\ell(\mathcal{R}/\mathfrak{C}) = n$ (cf [2, Section II.1]). Wilf's inequality is therefore equivalent to

$$\ell(\overline{\mathcal{R}}/\mathcal{R}) \leq (\nu(\mathcal{R}) - 1)\ell(\mathcal{R}/\mathfrak{C}). \tag{2}$$

A similar well-known inequality holds in a much more general context: if $\mathcal{R}$ is a one-dimensional Cohen-Macaulay local ring which is excellent, reduced and has an infinite residue field then

$$\ell(\overline{\mathcal{R}}/\mathcal{R}) \leq t(\mathcal{R})\ell(\mathcal{R}/\mathfrak{C})$$

where $t(\cdot)$ denotes the Cohen-Macaulay type and $\overline{\mathcal{R}}$ is the integral closure of $\mathcal{R}$ in its total ring of fractions (cf. [4], [6], [10]). In particular (2) is satisfied whenever $t(\mathcal{R}) < \nu(\mathcal{R})$, but the two invariants are unrelated in general: J. Backelin exhibited a family of rings with $d = 4$ and arbitrarily large type (cf. [8]). It would be interesting to explore the form (2) of Wilf's inequality in this larger class of rings:

Question 9. *Let $\mathcal{R}$ be a one-dimensional Cohen-Macaulay local ring which is excellent, reduced and has an infinite residue field, is it true that $\ell(\overline{\mathcal{R}}/\mathcal{R}) \leq (\nu(\mathcal{R}) - 1)\ell(\mathcal{R}/\mathfrak{C})$?*

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