Y. FUKUMA
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ON A CONJECTURE OF BELTRAMETTI-SOMMESE FOR POLARIZED 4-FOLDS

Yoshiaki Fukuma

Abstract

Let (X, L) be a polarized manifold of dimension 4. In this paper, we prove that $h^0(K_X + 3L) > 0$ if $K_X + 3L$ is nef, which is a conjecture of Beltrametti-Sommese for polarized 4-folds.

1. Introduction

Let X be a projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then (X, L) is called a *polarized variety*. If X is smooth, then we say that (X, L) is a polarized *manifold*.

The adjoint bundle $K_X + (n-1)L$ of (X, L) plays an important role for investigating (X, L) (for example, see [1, Chapter 7, 9, and 11]), where K_X is the canonical line bundle.

In [1, Conjecture 7.2.7], Beltrametti and Sommese gave the following conjecture.

Conjecture 1 (Beltrametti-Sommese). *Let (X, L) be an n-dimensional polarized manifold with $n \geq 3$. Assume that $K_X + (n-1)L$ is nef. Then $h^0(K_X + (n-1)L) > 0$.*

For this conjecture, the following partial results have been obtained.

(i) In [7, Theorem 2.4], the author proved that this conjecture is true if $n = 3$.

(ii) In [13, 1.2 Theorem], Höring proved that this conjecture is true if $h^0(L) > 0$.

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In this paper, we investigate the conjecture above, and the main purpose of this paper is to prove that the above conjecture is true for $n = 4$.

We will use the customary notation in algebraic geometry.

2. Preliminaries

NOTATION 2.1. Let X be a projective variety of dimension n and let L be a line bundle on X. Then we put

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L) \binom{t+j-1}{j}.$$

DEFINITION 2.1 ([5, Definition 2.1]). Let X be a projective variety of dimension n and let L be a line bundle on X.

(i) For every integer i with $0 \le i \le n$, the *i-th sectional geometric genus $g_i(X, L)$ of (X, L)* is defined by the following.

$$g_i(X, L) = (-1)^i (\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

(ii) For every integer i with $0 \le i \le n$, the *i-th sectional H-arithmetic genus $\chi_i^H(X, L)$ of (X, L)* is defined by the following.

$$\chi_i^H(X, L) = \chi_{n-i}(X, L).$$

Remark 2.1. (i) Since $\chi_{n-i}(X, L) \in \mathbf{Z}$, we see that $\chi_i^H(X, L)$ and $g_i(X, L)$ are integer by definition.

(ii) If $i = \dim X = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$ and $\chi_n^H(X, L) = \chi(\mathcal{O}_X)$.

(iii) If $i = 0$, then $g_0(X, L) = L^n$ and $\chi_0^H(X, L) = L^n$.

(iv) If $i = 1$, then $g_1(X, L) = g(X, L)$, where $g(X, L)$ is the sectional genus of (X, L). If X is smooth, then the sectional genus $g(X, L)$ can be written as

$$g(X, L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1}.$$

(v) If $i = 2$, then we have[1]

$$g_2(X, L) = \frac{1}{12}(K_X + (n-1)L)(K_X + (n-2)L)L^{n-2} + \frac{1}{12}c_2(X)L^{n-2} + \frac{n-3}{24}(2K_X + (n-2)L)L^{n-1} - 1 + h^1(\mathcal{O}_X).$$

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[1] See [6, (2.2.A)].

(vi) If $i = 3$, then we have[2]

$$g_3(X,L) = \frac{(n-2)(n-3)^2}{48}L^n + \frac{(n-3)(3n-8)}{48}K_X L^{n-1} + \frac{n-3}{24}(K_X^2 + c_2(X))L^{n-2} + \frac{1}{24}K_X c_2(X)L^{n-3} + 1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X).$$

(vii) In general for every integer i with $1 \le i \le n$ we get

$$\chi_i^H(X,L) = 1 - h^1(\mathcal{O}_X) + \cdots + (-1)^{i-1}h^{i-1}(\mathcal{O}_X) + (-1)^i g_i(X,L).$$

THEOREM 2.1. *Let (X,L) be a polarized manifold with* $\dim X = n$, *and let i be an integer with $0 \le i \le n-1$. Then*

$$g_i(X,L) = \sum_{j=0}^{n-i-1}(-1)^j\binom{n-i}{j}h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i}(-1)^{n-i-k}h^{n-k}(\mathcal{O}_X).$$

Proof. See [5, Theorem 2.3]. □

DEFINITION 2.2. (i) Let X (resp. Y) be an n-dimensional projective manifold, and let L (resp. H) be an ample line bundle on X (resp. Y). Then (X,L) is called a *simple blowing up of* (Y,H) if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $L = \pi^*(H) - E$, where E is the π-exceptional reduced divisor.

(ii) Let X (resp. M) be an n-dimensional projective manifold, and let L (resp. A) be an ample line bundle on X (resp. M). Then we say that (M,A) is a *reduction of* (X,L) if (X,L) is obtained by a composite of simple blowing ups of (M,A), and (M,A) is not obtained by a simple blowing up of any polarized manifold. The morphism $\phi_1 : X \to M$ is called the *reduction map*.

Remark 2.2. Let (X,L) be a polarized manifold of dimension n and (M,A) a reduction of (X,L).

(i) If (X,L) is not obtained by a simple blowing up of another polarized manifold, then we regard (X,L) as a reduction of itself.

(ii) For any polarized manifold (X,L), there exists a reduction of (X,L). Moreover if $n \ge 3$, then a reduction of (X,L) is unique. (See [4, (11.11), Chapter II].)

(iii) If $\kappa(K_X + (n-2)L) \ge 0$, then we infer that $K_M + (n-2)A$ is nef (see [1, Proposition 7.2.2 and Theorems 7.2.3, 7.2.4, 7.3.2, 7.3.4]).

(iv) $h^0(K_X + tL) = h^0(K_M + tA)$ for every integer t with $1 \le t \le n-1$.

[2] See [6, (2.2.B)].

Definition 2.3 ([9, Definition 3.1 and Definition 3.2]). Let (X, L) be a polarized manifold of dimension n.

(i) Let t be a positive integer. Then set

$$F_0(t) := h^0(K_X + tL),$$
$$F_i(t) := F_{i-1}(t+1) - F_{i-1}(t) \quad \text{for every integer } i \text{ with } 1 \le i \le n.$$

(ii) For every integer i with $0 \le i \le n$, the *ith Hilbert coefficient* $A_i(X, L)$ *of* (X, L) is defined by $A_i(X, L) = F_{n-i}(1)$.

Remark 2.3. (i) If $1 \le i \le n$, then $A_i(X, L)$ can be written as follows (see [9, Proposition 3.2]).

$$\begin{aligned} A_i(X,L) &= (-1)^i \chi_i^H(X,L) + (-1)^{i-1}\chi_{i-1}^H(X,L) \\ &= g_i(X,L) + g_{i-1}(X,L) - h^{i-1}(\mathcal{O}_X). \end{aligned}$$

(ii) By Definition 2.3 and [9, Proposition 3.1 (2)], we have the following:
(ii.1) $A_i(X, L) \in \mathbf{Z}$ for every integer i with $0 \le i \le n$.
(ii.2) $A_0(X, L) = L^n$.
(ii.3) $A_n(X, L) = h^0(K_X + L)$.

(iii) By Remark 2.1 (v) and (vi), and by Remark 2.3 (i), we see that $A_2(X, L)$ and $A_3(X, L)$ are the following.

$$\begin{aligned} A_2(X,L) &= \frac{(3n-2)(n+1)}{24}L^n + \frac{n}{4}K_X L^{n-1} + \frac{1}{12}(K_X^2 + c_2(X))L^{n-2}, \\ A_3(X,L) &= \frac{(n-2)(n^2-1)}{48}L^n + \frac{n(3n-5)}{48}K_X L^{n-1} + \frac{n-1}{24}K_X^2 L^{n-2} \\ &\quad + \frac{1}{24}c_2(X)(K_X + (n-1)L)L^{n-3}. \end{aligned}$$

Theorem 2.2. *Let (X, L) be a polarized manifold of dimension n and let t be a positive integer. Then for every integer i with $0 \le i \le n$ we have*

$$F_{n-i}(t) = \sum_{j=0}^{i} \binom{t-1}{i-j} A_j(X,L).$$

Proof. See [9, Theorem 3.1]. □

Corollary 2.1 ([9, Corollary 3.1]). *Let (X, L) be a polarized manifold of dimension n, and let t be a positive integer. Then we have*

$$h^0(K_X + tL) = \sum_{j=0}^{n} \binom{t-1}{n-j} A_j(X,L).$$

Remark 2.4. Let (X, L) be a polarized manifold of dimension n. Then by using $A_i(X, L)$, the left hand side of equations (6) and (7) in [13, 4.1 Lemma] can be written as follows.

$$\chi(\mathcal{O}_X) + \frac{1}{2}(K_X + (n-1)L)L^{n-1} = \sum_{i=2}^{n}(-1)^i A_i(X, L), \tag{1}$$

$$L^{n-2}(2(K_X^2 + c_2(X)) + 6nK_XL + (n+1)(3n-2)L^2) = 24A_2(X, L). \tag{2}$$

THEOREM 2.3. *Let X be a projective manifold. Then there exist smooth projective varieties X' and Y, a birational morphism $\mu : X' \to X$ and a fiber space $\phi : X' \to Y$ such that Y is not uniruled and if* $\dim X' > \dim Y$, *then the general fiber of ϕ is rationally connected.*

Proof. See [2], [14] and [11]. □

DEFINITION 2.4. The fiber space $\phi : X' \to Y$ in Theorem 2.3 is called the *MRC-fibration of X*, and Y is called the *base of the MRC-fibration*.

3. Main result

In this section we are going to prove Conjecture 1 for the case of dimension 4.

THEOREM 3.1. *Let (X, L) be a polarized manifold of dimension 4. Assume that $K_X + 3L$ is nef. Then $h^0(K_X + 3L) > 0$ holds.*

Proof. (I) First we consider the case where $q(X) > 0$ (see also the proof of [8, Theorem 3.3]). Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map. Then $1 \leq \dim \alpha(X) \leq 4$. Then by [12, Corollary 10.7, Chapter III, Section 10], a general fiber F_α of α is the following type:

$$F_\alpha = \bigcup_{j=1}^{r} F_j,$$

where F_j is a smooth subvariety for every integer j with $1 \leq j \leq r$, $\dim F_k = \dim F_l$ and $F_k \cap F_l = \emptyset$ for any $k \neq l$. Here we note that if $\kappa(K_X + mL) \geq 0$, then $\kappa(K_{F_j} + mL_{F_j}) \geq 0$ for every integer j with $1 \leq j \leq r$. We also note that $0 \leq \dim F_j \leq 3$ for every j.

(I.1) If $\dim F_\alpha = 0$, then $h^0(K_{F_j} + 3L_{F_j}) > 0$ for every integer j with $1 \leq j \leq r$.

(I.2) Assume that $\dim F_\alpha = 3$ (resp. $1 \leq \dim F_\alpha \leq 2$). Since $K_{F_j} + 3L_{F_j}$ is nef by assumption, we see from [13, 1.5 Theorem] (resp. [8, Theorem 2.8]) that $h^0(K_{F_j} + 3L_{F_j}) > 0$. Hence $h^0(K_{F_\alpha} + 3L_{F_\alpha}) > 0$. So by [3, Lemma 4.1], we have $h^0(K_X + 3L) > 0$ and we get the assertion.

(II) Next we consider the case where $q(X) = 0$.

(II.1) If $\kappa(K_X + 2L) = -\infty$, then $h^0(K_X + tL) = 0$ for $t = 1, 2$. So we get the assertion by [13, 1.2 Theorem].

(II.2) Assume that $\kappa(K_X + 2L) \geq 0$. By taking the reduction of (X, L), if necessary, we may assume that $K_X + 2L$ is nef by Remark 2.2 (iii) and (iv).

(II.2.1) Assume that $\Omega_X\langle L\rangle$ is generically nef. Since $K_X + 4L$ is ample by assumption, we see from [13, 2.11 Corollary] that

$$(3)\qquad \begin{aligned} c_2(X)(K_X + 4L)L &\geq -\left(\frac{3}{4}K_X(4L) + \frac{3}{8}(4L)^2\right)(K_X + 4L)L \\ &= -3K_X^2L^2 - 18K_XL^3 - 24L^4. \end{aligned}$$

Here we calculate $A_2(X, L) + 2A_3(X, L)$. By Remark 2.3 (iii) we have

$$(4)\qquad \begin{aligned} &A_2(X, L) + 2A_3(X, L) \\ &= \frac{1}{12}(K_X + 3L)(K_X + 8L)L^2 + \frac{1}{24}(2K_X + 2L)L^3 \\ &\quad + \frac{5}{4}L^4 + \frac{7}{6}K_XL^3 + \frac{1}{4}K_X^2L^2 + \frac{1}{12}c_2(X)(K_X + 4L)L \\ &= \frac{1}{3}K_X^2L^2 + \frac{13}{6}K_XL^3 + \frac{10}{3}L^4 + \frac{1}{12}c_2(X)(K_X + 4L)L. \end{aligned}$$

Hence by (3) and (4) we have

$$\begin{aligned} &A_2(X, L) + 2A_3(X, L) \\ &= \frac{1}{3}K_X^2L^2 + \frac{13}{6}K_XL^3 + \frac{10}{3}L^4 + \frac{1}{12}c_2(X)(K_X + 4L)L \\ &\geq \frac{1}{3}K_X^2L^2 + \frac{13}{6}K_XL^3 + \frac{10}{3}L^4 - \frac{1}{12}(3K_X^2L^2 + 18K_XL^3 + 24L^4) \\ &= \frac{1}{12}K_X^2L^2 + \frac{8}{12}K_XL^3 + \frac{4}{3}L^4 \\ &= \frac{1}{12}(K_X + 2L)^2L^2 + \frac{1}{3}(K_X + 3L)L^3. \end{aligned}$$

Since $K_X + 2L$ is nef by assumption, we have $A_2(X, L) + 2A_3(X, L) > 0$. Here we note that $A_4(X, L) = h^0(K_X + L) \geq 0$ by Remark 2.3 (ii.3). Therefore

$$h^0(K_X + 3L) = A_4(X, L) + 2A_3(X, L) + A_2(X, L) > 0.$$

(II.2.2) Assume that $\Omega_X\langle L\rangle$ is not generically nef. Then by [13, 3.1 Theorem] there exist smooth projective varieties X' and Y, a birational morphism $\mu : X' \to X$ and a fiber space $\lambda : X' \to Y$ such that $m := \dim Y < 4$ and a general fiber F_λ of λ is rationally connected and $h^0(D) = 0$ for any Cartier divisor D on F_λ with $D \sim_{\mathbf{Q}} K_{F_\lambda} + j(\mu^*(L))_{F_\lambda}$ with $j \in [0, 4 - m] \cap \mathbf{Q}$.

(II.2.2.1) The case where $\dim Y = 0$ or 1. Then $h^0(K_X + tL) = h^0(K_{X'} + \mu^*(tL)) = 0$ for $t = 1, 2, 3$. But this is impossible by [13, 1.2 Theorem].

(II.2.2.2) The case where $\dim Y = 2$. Then we have $h^0(K_{F_\lambda} + \mu^*(2L)_{F_\lambda}) = 0$. On the other hand, since $\kappa(K_X + 2L) \geq 0$, we have $\kappa(K_{F_\lambda} + \mu^*(2L)_{F_\lambda}) \geq 0$. Here we note that $\dim F_\lambda = 2$. Hence $h^0(K_{F_\lambda} + \mu^*(2L)_{F_\lambda}) > 0$ by [10, Proposition 1] (see also [8, Theorem 2.8]). But this is a contradiction.

(II.2.2.3) The case where $\dim Y = 3$. In this case $F_\lambda \cong \mathbf{P}^1$. Since $h^0(D) = 0$ for any Cartier divisor D on F_λ with $D \sim_{\mathbf{Q}} K_{F_\lambda} + \mu^*(L)_{F_\lambda}$, we have $\deg \mu^*(L)_{F_\lambda} = 1$. In this case $h^0(K_X + L) = 0$ and $h^4(\mathcal{O}_X) = 0$ hold. Hence by Theorem 2.1 we obtain

$$g_4(X, L) = 0, \tag{5}$$

$$g_3(X, L) = h^3(\mathcal{O}_X), \tag{6}$$

$$g_2(X, L) = h^0(K_X + 2L) + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X). \tag{7}$$

Hence by (6), (7) and Remark 2.3 (i) we have $A_3(X, L) = h^0(K_X + 2L)$. By assumption we have

$$h^1(\mathcal{O}_X) = 0. \tag{8}$$

Since $h^0(K_X + L) = 0$, we see from [13, 1.2 Theorem] that we get $h^0(K_X + 2L) > 0$ or $h^0(K_X + 3L) > 0$. If $h^0(K_X + 3L) > 0$, then we get the assertion. So we may assume that

$$h^0(K_X + 2L) > 0 \quad \text{and} \quad h^0(K_X + 3L) = 0. \tag{9}$$

We note that by Definition 2.3 (i) we get $F_0(t) = h^0(K_X + tL)$, $F_1(t) = F_0(t+1) - F_0(t) = h^0(K_X + (t+1)L) - h^0(K_X + tL)$. Hence

$$\begin{aligned} F_1(2) &= F_0(3) - F_0(2) \\ &= h^0(K_X + 3L) - h^0(K_X + 2L) \\ &< 0. \end{aligned}$$

On the other hand, by Theorem 2.2 we get

$$F_{4-i}(t) = \sum_{j=0}^{i} \binom{t-1}{i-j} A_j(X, L).$$

Therefore

$$0 > F_1(2) = A_2(X, L) + A_3(X, L). \tag{10}$$

Furthermore

$$F_1(1) = h^0(K_X + 2L) - h^0(K_X + L) > 0$$

and by Theorem 2.2 we have

$$F_1(1) = A_3(X, L).$$

So we get

$$A_3(X, L) > 0. \tag{11}$$

By (10) and (11) we have

$$A_2(X, L) < 0. \tag{12}$$

Here we prove the following claim.

CLAIM 3.1. *The dimension of the base of the MRC-fibration*[3] *of X is at least* 3.

Proof. Assume that the dimension of the base of the MRC-fibration is less than or equal to two. Then we have

$$h^j(\mathcal{O}_X) = 0 \quad \text{for every integer } j \geq 3. \tag{13}$$

First we note that since $g_4(X, L) = 0$ and $g_3(X, L) = h^3(\mathcal{O}_X)$ by (5) and (6), we see from Remark 2.3 (i) that

$$A_4(X, L) = g_4(X, L) + g_3(X, L) - h^3(\mathcal{O}_X) = 0. \tag{14}$$

So by (11), (12) and (14) we have

$$\sum_{i=2}^{4} (-1)^i A_i(X, L) < 0.$$

On the other hand by Remark 2.4 (1) we have

$$\sum_{i=2}^{4} (-1)^i A_i(X, L) = \chi(\mathcal{O}_X) + \frac{1}{2}(K_X + 3L)L^3.$$

Since $(K_X + 3L)L^3 > 0$ in this case, we have $\chi(\mathcal{O}_X) < 0$. So we see from (13) that $h^1(\mathcal{O}_X) > 0$ and this contradicts the assumption (8). So we get the assertion of Claim 3.1. □

By Claim 3.1 and the argument of [13, Step 2 in Page 741], we see from Remark 2.4 (2) that[4]

$$A_2(X, L) = \frac{1}{24} L^2 (2(K_X^2 + c_2(X)) + 24K_X L + 50L^2) > 0$$

which contradicts (12). Therefore the assumption (9) is impossible.

Therefore we get the assertion of Theorem 3.1. □

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[3] See Definition 2.4.

[4] Here we note that $n = 4$ in this case.

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Yoshiaki Fukuma
Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
E-mail: fukuma@kochi-u.ac.jp