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# On a Conjecture of Chvátal on m -Intersecting Hypergraphs — Source link

Jean-Claude Bermond, Peter Frankl

**Published on:** 01 Nov 1977 - Bulletin of The London Mathematical Society (Oxford University Press)

**Topics:** Conjecture

# ON A CONJECTURE OF CHVÁTAL ON $m$-INTERSECTING HYPERGRAPHS

J. C. BERMOND AND P. FRANKL

Let $X$ be a finite set of cardinality $n$. A hypergraph $(X, \mathscr{F})$ is called $m$-intersecting if for any $F_1, \ldots, F_m \in \mathscr{F}$ the condition $F_1 \cap \ldots \cap F_m \neq \emptyset$ holds.

An $(n, h, m)$-hypergraph is a hypergraph $H = (X, \mathscr{F})$ satisfying

(i) $|X| = n$;

(ii) $H$ is $h$-uniform, that is, $\mathscr{F}$ consists of $h$-subsets of $X$;

(iii) every $m$-intersecting partial hypergraph of $H$ is $(m+1)$-intersecting.

Let $f(n, h, m)$ denote the maximum number of edges in an $(n, h, m)$-hypergraph.

Using this terminology the Erdős–Ko–Rado theorem says that

$$f(n, h, 1) = \binom{n-1}{h-1} \text{ whenever } n \geqslant 2h.$$

Erdős [3] conjectured that

$$f(n, h, 2) = \binom{n-1}{h-1} \text{ whenever } 3 \leqslant h \leqslant 2n/3.$$

In [1] Chvátal made the more general conjecture:

$$f(n, h, m) = \binom{n-1}{h-1} \text{ whenever } m < h \leqslant mn/(m+1).$$

In [2] he proved this conjecture for $h = m+1$.

The aim of this paper is to prove this conjecture in some special cases.

DEFINITION. *A $(v, k, 1)$ $t$-design (called also a Steiner system $S_1(t, k, v)$) $\mathscr{D}$ is a set of*

$$n = \frac{v(v-1) \ldots (v-t+1)}{k(k-1) \ldots (k-t+1)}$$

*different subsets, called blocks, of a set $Y$ of $v$ elements, such that:*

(i)′ $|Y| = v$.

(ii)′ *For any $a \in Y$, there are exactly $h$ blocks of $\mathscr{D}$ containing $a$.*

(iii)′ *For any $t$-tuple of elements $\{a_1, \ldots, a_t\}$ of $Y$, there is exactly one block of $\mathscr{D}$ containing $\{a_1, \ldots, a_t\}$.*

(iv)′ *Every block of $\mathscr{D}$ consists of $k$ elements.*

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Received 24 January, 1977; revised 9 July, 1977.

[BULL. LONDON MATH. SOC., 9 (1977), 310–312]

LEMMA. *Suppose that there exists a $(v, k, 1)$ $t$-design for some $v, k, t$. Let*

$$|X| = n = \frac{v(v-1)\dots(v-t+1)}{k(k-1)\dots(k-t+1)} \quad \text{and} \quad h = \frac{(v-1)\dots(v-t+1)}{(k-1)\dots(k-t+1)}.$$

*Then there exists a family $\mathscr{A}$ of subsets of $X$ such that*

(i) $|\mathscr{A}| = v$.

(ii) *for $A \in \mathscr{A}$, we have $|A| = h$.*

(iii) *For $A_1, \dots, A_t \in \mathscr{A}$, we have $|A_1 \cap \dots \cap A_t| = 1$.*

(iv) *For $x \in X$, there are exactly $k$ members of $\mathscr{A}$ which contain $x$.*

*Proof.* Let $\mathscr{D} = \{X_1, \dots, X_n\}$ be a $(v, k, 1)$ $t$-design on a set $Y = \{a_1, \dots, a_v\}$. Let us define the dual system $\mathscr{A} = \{A_1, \dots, A_v\}$ on the set $X = \{x_1, \dots, x_n\}$ by $x_i \in A_j$ if and only if $a_j \in X_i$. Then it follows from conditions (i)′, (ii)′, (iii)′, (iv)′ that $\mathscr{A}$ satisfies the four conditions of the lemma.

*Remark.* The condition (ii) (resp. (ii)′) is a consequence of the three other conditions.

THEOREM *If for some $v, k, t$, there exists a $(v, k, 1)$ $t$-design (called also Steiner system $S_1(t, k, v)$), then the conjecture of Chvátal is true for the triple*

$$n = \frac{v(v-1)\dots(v-t+1)}{k(k-1)\dots(k-t+1)}; \quad h = \frac{(v-1)\dots(v-t+1)}{(k-1)\dots(k-t+1)}; \quad m = t.$$

*Proof of the theorem.* Let $(X, \mathscr{F})$ be an $(n, h, m)$-hypergraph. We apply a method of proof due to G. O. H. Katona [6]. Suppose that there exists a $(v, k, 1)$ $t$-design and let $\mathscr{A}$ be the family defined in the lemma. If $P$ is a permutation of the elements of $X$, then the system $P(\mathscr{A}) = \{P(A_1), \dots, P(A_v)\}$ satisfies the conditions (i), (ii), (iii), (iv) as well. ($P(A)$ is the set consisting of the images of the elements of $A$ by the permutation $P$.) We count the number of pairs $(P, F)$, where $P$ is a permutation of the elements of $X$, $F$ an element of $\mathscr{F}$ and $F \in P(\mathscr{A})$. From properties (i) to (iv) it is clear that $P(\mathscr{A})$ is $m$-intersecting and that if $A_1, \dots, A_q$ is an $(m+1)$-intersecting sub-family of $P(\mathscr{A})$ then $A_1 \cap \dots \cap A_q \neq \emptyset$, entailing $q \leqslant k$. Thus the number of pairs $(P, F)$ is at most $kn!$.

On the other hand to any $F \in \mathscr{F}$ and any $A \in \mathscr{A}$, there are exactly $h!\,(n-h)!$ different permutations $P$ such that $F = P(A)$. Hence the number of pairs $(P, F)$ is exactly: $h!\,(n-h)!\,v|\mathscr{F}|$. Thus we have $|\mathscr{F}|h!\,(n-h)!\,v \leqslant kn!$. Thus taking into account that $nk = vh$, we obtain $|\mathscr{F}| \leqslant \binom{n-1}{h-1}$.

COROLLARY. *The conjecture of Chvátal is true for the following triples of integers:*

(a) *$m$ arbitrary, $n = \binom{v}{m}$, $h = \binom{v-1}{m-1}$ for any integer $v > m$.*

(b) $m = 2$, $h = p^\alpha + 1$, $n = h^2 - h + 1$ *or* $n = h^2 - h$ (*p is a prime and* $\alpha$ *an integer*).

$$\left.\begin{array}{lll} \text{(c)} & m = 2,\ h \text{ and } n = h(2h+1)/3 & (h \geq 3) \\ \text{(d)} & m = 2,\ h \text{ and } n = h(3h+1)/4 & (h \geq 4) \\ \text{(e)} & m = 2,\ h \text{ and } n = h(4h+1)/5 & (h \geq 5) \\ \text{(f)} & m = 2,\ h \text{ and } n = h\big(((k-1)h+1)/k\big) & (h \geq h_0(k)) \end{array}\right\} n \text{ is an integer}$$

(g) $m = 3$, $n = \binom{v}{4}$, $h = \binom{v-1}{3}$ *for any integer* $v \equiv 2$ *or* 4 (*mod* 6), $v > 4$.

*Proof.* It follows from the existence of: for (a) trivial $t$-designs; for (b), (c), (d), (e), (f), known $(v, k, 1)$ 2-designs (see [4] or [7]): (b) corresponds to projective and affine planes, (c), (d), (e) to $k = 3, 4, 5$ and (f) follows from the results of Wilson [7] for large $v$. (g) is just the case $k = 4$, $t = 3$ of [3] and follows from [5].

## *References*


1. V. Chvátal, *Problem* 6, in Hypergraph Seminar, *Lecture Notes in Math.* 411 (Springer Verlag 1974), 279–280.
2. V. Chvátal, "An extremal set-intersection theorem", *J. London Math. Soc.*, 9 (1974), 355–359.
3. P. Erdős, *Topics in combinatorial analysis*, Proc. of The Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, 1971), 2–20.
4. H. Hanani, "Balanced incomplete block designs and related designs", *Discrete Maths.*, 11 (1975), 255–369.
5. H. Hanani, "On quadruple systems", *Canadian J. Math.*, 12 (1960), 145–157.
6. G. O. H. Katona, "Extremal problems among subsets of a finite set", in *Combinatorics* (Eds. M. Hall and G. H. Van Lint), Math. Centre Tracts 50 (Amsterdam, 1974), 13–42.
7. R. M. Wilson, "An existence theory for pairwise balanced designs III: Proof of the existence conjectures", *J. Combinatorial Theory*, 18 (1975), 71–79.



CMS, 54 Bd. Raspail, 75006 Paris.
Mathematical Institute,
1053 Budapest, Realtanoda 13–15,
Hungary.