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ON A CONJECTURE OF CHVÁTAL ON m -INTERSECTING HYPERGRAPHS

J. C. BERMOND AND P. FRANKL

Let X be a finite set of cardinality n . A hypergraph (X, \mathcal{F}) is called m -intersecting if for any $F_1, \dots, F_m \in \mathcal{F}$ the condition $F_1 \cap \dots \cap F_m \neq \emptyset$ holds.

An (n, h, m) -hypergraph is a hypergraph $H = (X, \mathcal{F})$ satisfying

(i) $|X| = n$;

(ii) H is h -uniform, that is, \mathcal{F} consists of h -subsets of X ;

(iii) every m -intersecting partial hypergraph of H is $(m+1)$ -intersecting.

Let $f(n, h, m)$ denote the maximum number of edges in an (n, h, m) -hypergraph.

Using this terminology the Erdős–Ko–Rado theorem says that

$$f(n, h, 1) = \binom{n-1}{h-1} \text{ whenever } n \geq 2h.$$

Erdős [3] conjectured that

$$f(n, h, 2) = \binom{n-1}{h-1} \text{ whenever } 3 \leq h \leq 2n/3.$$

In [1] Chvátal made the more general conjecture:

$$f(n, h, m) = \binom{n-1}{h-1} \text{ whenever } m < h \leq mn/(m+1).$$

In [2] he proved this conjecture for $h = m+1$.

The aim of this paper is to prove this conjecture in some special cases.

DEFINITION. A $(v, k, 1)$ t -design (called also a Steiner system $S_1(t, k, v)$) \mathcal{D} is a set of

$$n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)}$$

different subsets, called blocks, of a set Y of v elements, such that:

(i)' $|Y| = v$.

(ii)' For any $a \in Y$, there are exactly h blocks of \mathcal{D} containing a .

(iii)' For any t -tuple of elements $\{a_1, \dots, a_t\}$ of Y , there is exactly one block of \mathcal{D} containing $\{a_1, \dots, a_t\}$.

(iv)' Every block of \mathcal{D} consists of k elements.

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LEMMA. Suppose that there exists a $(v, k, 1)$ t -design for some v, k, t . Let

$$|X| = n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)} \quad \text{and} \quad h = \frac{(v-1) \dots (v-t+1)}{(k-1) \dots (k-t+1)}.$$

Then there exists a family \mathcal{A} of subsets of X such that

- (i) $|\mathcal{A}| = v$.
- (ii) for $A \in \mathcal{A}$, we have $|A| = h$.
- (iii) For $A_1, \dots, A_t \in \mathcal{A}$, we have $|A_1 \cap \dots \cap A_t| = 1$.
- (iv) For $x \in X$, there are exactly k members of \mathcal{A} which contain x .

Proof. Let $\mathcal{D} = \{X_1, \dots, X_n\}$ be a $(v, k, 1)$ t -design on a set $Y = \{a_1, \dots, a_v\}$. Let us define the dual system $\mathcal{A} = \{A_1, \dots, A_v\}$ on the set $X = \{x_1, \dots, x_n\}$ by $x_i \in A_j$ if and only if $a_j \in X_i$. Then it follows from conditions (i)', (ii)', (iii)', (iv)' that \mathcal{A} satisfies the four conditions of the lemma.

Remark. The condition (ii) (resp. (ii)') is a consequence of the three other conditions.

THEOREM *If for some v, k, t , there exists a $(v, k, 1)$ t -design (called also Steiner system $S_1(t, k, v)$), then the conjecture of Chvátal is true for the triple*

$$n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)}; \quad h = \frac{(v-1) \dots (v-t+1)}{(k-1) \dots (k-t+1)}; \quad m = t.$$

Proof of the theorem. Let (X, \mathcal{F}) be an (n, h, m) -hypergraph. We apply a method of proof due to G. O. H. Katona [6]. Suppose that there exists a $(v, k, 1)$ t -design and let \mathcal{A} be the family defined in the lemma. If P is a permutation of the elements of X , then the system $P(\mathcal{A}) = \{P(A_1), \dots, P(A_v)\}$ satisfies the conditions (i), (ii), (iii), (iv) as well. ($P(A)$ is the set consisting of the images of the elements of A by the permutation P .) We count the number of pairs (P, F) , where P is a permutation of the elements of X , F an element of \mathcal{F} and $F \in P(\mathcal{A})$. From properties (i) to (iv) it is clear that $P(\mathcal{A})$ is m -intersecting and that if A_1, \dots, A_q is an $(m+1)$ -intersecting sub-family of $P(\mathcal{A})$ then $A_1 \cap \dots \cap A_q \neq \emptyset$, entailing $q \leq k$. Thus the number of pairs (P, F) is at most $kn!$.

On the other hand to any $F \in \mathcal{F}$ and any $A \in \mathcal{A}$, there are exactly $h!(n-h)!$ different permutations P such that $F = P(A)$. Hence the number of pairs (P, F) is exactly: $h!(n-h)!v|\mathcal{F}|$. Thus we have $|\mathcal{F}|h!(n-h)!v \leq kn!$. Thus taking into

account that $nk = v h$, we obtain $|\mathcal{F}| \leq \binom{n-1}{h-1}$.

COROLLARY. *The conjecture of Chvátal is true for the following triples of integers:*

- (a) m arbitrary, $n = \binom{v}{m}$, $h = \binom{v-1}{m-1}$ for any integer $v > m$.

- (b) $m = 2, h = p^\alpha + 1, n = h^2 - h + 1$ or $n = h^2 - h$ (p is a prime and α an integer).
 - (c) $m = 2, h$ and $n = h(2h + 1)/3$ ($h \geq 3$)
 - (d) $m = 2, h$ and $n = h(3h + 1)/4$ ($h \geq 4$)
 - (e) $m = 2, h$ and $n = h(4h + 1)/5$ ($h \geq 5$)
 - (f) $m = 2, h$ and $n = h(((k-1)h + 1)/k)$ ($h \geq h_0(k)$)
- } n is an integer
- (g) $m = 3, n = \binom{v}{4}, h = \binom{v-1}{3}$ for any integer $v \equiv 2$ or $4 \pmod{6}, v > 4$.

Proof. It follows from the existence of: for (a) trivial t -designs; for (b), (c), (d), (e), (f), known $(v, k, 1)$ 2-designs (see [4] or [7]): (b) corresponds to projective and affine planes, (c), (d), (e) to $k = 3, 4, 5$ and (f) follows from the results of Wilson [7] for large v . (g) is just the case $k = 4, t = 3$ of [3] and follows from [5].

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