On a conjecture of spectral extremal problems*

Jing Wang, Liying Kang†, Yisai Xue
Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China

Abstract

For a simple graph F, let $\mathrm{Ex}(n, F)$ and $\mathrm{Ex}_{\mathrm{sp}}(n, F)$ denote the set of graphs with the maximum number of edges and the set of graphs with the maximum spectral radius in an n-vertex graph without any copy of the graph F, respectively. The Turán graph $T_{n,r}$ is the complete r-partite graph on n vertices where its part sizes are as equal as possible. Cioabă, Desai and Tait [The spectral radius of graphs with no odd wheels, European J. Combin., 99 (2022) 103420] posed the following conjecture: Let F be any graph such that the graphs in $\mathrm{Ex}(n, F)$ are Turán graphs plus $O(1)$ edges. Then $\mathrm{Ex}_{\mathrm{sp}}(n, F) \subset \mathrm{Ex}(n, F)$ for sufficiently large n. In this paper we consider the graph F such that the graphs in $\mathrm{Ex}(n, F)$ are obtained from $T_{n,r}$ by adding $O(1)$ edges, and prove that if G has the maximum spectral radius among all n-vertex graphs not containing F, then G is a member of $\mathrm{Ex}(n, F)$ for n large enough. Then Cioabă, Desai and Tait's conjecture is completely solved.

Key words: Spectral radius; Spectral extremal graph; Turán graph.

1 Introduction

Let F be a simple graph. A graph G is F-free if there is no subgraph of G isomorphic to F. The Turán type extremal problem is to determine the maximum number of edges in a graph on n vertices that is F-free, and the maximum number of edges is called the *Turán number*, denoted by $\mathrm{ex}(n, F)$. Such a graph with $\mathrm{ex}(n, F)$ edges is called an *extremal graph* for F and we denote by $\mathrm{Ex}(n, F)$ the set of all extremal graphs on n vertices for F. The *Turán graph* is the complete r-partite graph on n vertices where each partite set has either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$ vertices and the edge set consists of all pairs joining distinct parts, denoted by $T_{n,r}$. The well-known Turán Theorem [26] states that the extremal graph corresponding to Turán number $\mathrm{ex}(n, K_{r+1})$ is $T_{n,r}$, i.e. $\mathrm{ex}(n, K_{r+1}) = e(T_{n,r})$. Erdős, Stone and Simonovits [12, 11] presented the following result

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\frac{n^2}{2} + o(n^2), \tag{1}$$

where $\chi(F)$ is the vertex-chromatic number of F. There are lots of researches on Turán type extremal problems (such as [2, 13, 3, 17, 24, 16]).

In this paper we focus on spectral analogues of the Turán type problem for graphs, which was proposed by Nikiforov [20]. The spectral Turán type problem is to determine the maximum spectral radius instead of the number of edges among all n-vertex F-free graphs. The graph which attains the maximum spectral radius is called a spectral extremal graph. We denote by $\mathrm{Ex}_{\mathrm{sp}}(n, F)$ the set of all spectral extremal graphs for F. Researches of the spectral Turán type problem have drawn increasing extensive intersect (see [18, 1, 15, 22, 28]). Nikiforov [19] showed that if G is a K_{r+1}-free graph on n vertices, then $\lambda(G) \le \lambda(T_{n,r})$, with equality if and only if $G = T_{n,r}$. This implies that if G attains the maximum spectral radius over all

*This research was partially supported by the National Nature Science Foundation of China (Nos. 11871329, 11971298)
†*Corresponding author.*
Email address: lykang@shu.edu.cn (L. Kang), wj517062214@163.com (J. Wang), xys16720018@163.com (Y. Xue)

n-vertex K_{r+1}-free graphs for sufficiently large n, then $G \in \mathrm{Ex}(n, K_{r+1})$. Cioabă, Feng, Tait and Zhang [4] proved that the spectral extremal graph for F_k belongs to $\mathrm{Ex}(n, F_k)$, where F_k is the graph consisting of k triangles which intersect in exactly one common vertex. In addition, Chen, Gould, Pfender and Wei [3] proved that $\mathrm{ex}(n, F_{k,r+1}) = e(T_{n,r}) + O(1)$, where $F_{k,r+1}$ is the graph consisting of k copies of K_{r+1} which intersect in a single vertex. Naturally, Cioabă, Desai and Tait [5] raised the following conjecture.

Conjecture 1.1 (Cioabă et al. [5]). *Let F be any graph such that the graphs in $\mathrm{Ex}(n, F)$ are Turán graphs plus $O(1)$ edges. Then $\mathrm{Ex}_{\mathrm{sp}}(n, F) \subset \mathrm{Ex}(n, F)$ for sufficiently large n.*

The results of Nikiforov [19], Cioabă, Feng, Tait and Zhang [4], Li and Peng [27], and Desai, Kang, Li, Ni, Tait and Wang [8] tell us that Conjecture 1.1 holds for K_{r+1}, F_k, $H_{s,k}$ and $F_{k,r}$, where $H_{s,k}$ is the graph defined by intersecting s triangles and k odd cycles of length at least 5 in exactly one common vertex. In this paper, we shall prove the following theorem which confirms Conjecture 1.1.

Theorem 1.2. *Let $r \geq 2$ be an integer, and F be any graph such that the graphs in $\mathrm{Ex}(n, F)$ are obtained from $T_{n,r}$ by adding $O(1)$ edges. For sufficiently large n, if G has the maximal spectral radius over all n-vertex F-free graphs, then*

$$G \in \mathrm{Ex}(n, F).$$

2 Notation and Preliminaries

In this section we introduce some notation and give the preparatory lemmas.

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. If $u, v \in V(G)$, and $uv \in E(G)$, then u and v are said to be *adjacent*. For a vertex $v \in V(G)$, the *neighborhood* $N_G(v)$ (or simply $N(v)$) of v is $\{u |\ uv \in E(G)\}$, and the *degree* $d_G(v)$ (or simply $d(v)$) of v is $|N_G(v)|$. The minimum and maximum degrees are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For $S \subseteq V(G)$ and $v \in V(G)$, let $d_S(v) = |N_S(v)| = |N_G(v) \cap S|$. For $V_1, V_2 \subseteq V(G)$, $e(V_1, V_2)$ denotes the number of edges of G between V_1 and V_2. For $S \subseteq V(G)$, denote by $G \setminus S$ the graph obtained from G by deleting all vertices of S and the incident edges. Denote by $G[S]$ the graph induced by S whose vertex set is S and whose edge set consists of all edges of G which have both ends in S.

Let G be a simple graph with n vertices. The *adjacent matrix* of G is $A(G) = (a_{ij})_{n\times n}$ with $a_{ij} = 1$ if $ij \in E(G)$, and $a_{ij} = 0$ otherwise. The *spectral radius* of G is the largest eigenvalue of $A(G)$, denoted by $\lambda(G)$. Let $G_1, \ldots, G_s$ be the components of G, then $\lambda(G) = \max\{\lambda(G_i)|\ i \in [s]\}$. For a connected graph G, let $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ be an eigenvector of $A(G)$ corresponding to $\lambda(G)$. Then $\mathbf{x}$ is a positive real vector, and

$$\lambda(G)x_i = \sum_{ij\in E(G)} x_j, \text{ for any } i \in [n]. \tag{2}$$

The following Rayleigh quotient equation is very useful:

$$\lambda(G) = \max_{\mathbf{x}\in\mathbb{R}^n_+} \frac{\mathbf{x}^{\mathrm{T}}A(G)\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} = \max_{\mathbf{x}\in\mathbb{R}^n_+} \frac{2\sum_{ij\in E(G)} x_ix_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}. \tag{3}$$

We have the following two lemmas from Zhan [29].

Lemma 2.1 (Zhan [29]). *Let A and B be two nonnegative square matrices. If $B < A$ and A is irreducible, then $\lambda(B) < \lambda(A)$.*

Lemma 2.2 (Zhan [29]). *Let A be a nonnegative square matrix. If B is a principal submatrix of A, then $\lambda(B) \leq \lambda(A)$. If A is irreducible and B is a proper principal submatrix of A, then $\lambda(B) < \lambda(A)$.*

Let $A(G)$ be the adjacent matrix of graph G. Then G is connected if and only if $A(G)$ is irreducible. Combining with Lemmas 2.1 and 2.2, we have the following result.

Lemma 2.3. *Let G be a connected graph. If G' is a proper subgraph of G, then $\lambda(G') < \lambda(G)$.*

Recall the classical stability theorem proved by Erdős [9, 10] and Simonovits [23]:

Lemma 2.4 (Erdős [9, 10], Simonovits [23])**.** *For every $r \geq 2, \varepsilon > 0$, and $(r+1)$-chromatic graph F, there exists $\delta > 0$ such that if a graph G of order n satisfies $e(G) > (1 - \frac{1}{r} - \delta)\frac{n^2}{2}$, then either G contains F, or G differs form $T_{n,r}$ in at most εn^2 edges.*

Write $K_r(n_1, \ldots, n_r)$ for the complete r-partite graph with classes of sizes $n_1, \ldots, n_r$. Nikiforov [21] proved the spectral version of Stability Lemma.

Lemma 2.5 (Nikiforov [21])**.** *Let $r \geq 2, 1/\ln n < c < r^{-8(r+21)(r+1)}, 0 < \varepsilon < 2^{-36}r^{-24}$ and G be a graph on n vertices. If $\lambda(G) > (1 - \frac{1}{r} - \varepsilon)n$, then one of the following statements holds:*
(a) G contains a $K_{r+1}(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$;
(b) G differs from $T_{n,r}$ in fewer than $(\varepsilon^{1/4} + c^{1/(8r+8)})n^2$ edges.

From the above theorem, one can easily get the following result.

Corollary 2.6. *Let F be a graph with chromatic number $\chi(F) = r + 1$. For every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that if G is an F-free graph on $n \geq n_0$ vertices with $\lambda(G) \geq (1 - \frac{1}{r} - \delta)n$, then G can be obtained from $T_{n,r}$ by adding and deleting at most εn^2 edges.*

For $K_r(n_1, n_2, \ldots, n_r)$, let $n = \sum_{i=1}^{r} n_i$. For convenience, we assume that $n_1 \geq n_2 \geq \ldots \geq n_r > 0$. It is well-known [6, p. 74] or [7] that the characteristic polynomial of $K_r(n_1, n_2, \ldots, n_r)$ is given as

$$\phi(K_r(n_1, n_2, \ldots, n_r), x) = x^{n-r}\left(1 - \sum_{i=1}^{r} \frac{n_i}{x + n_i}\right)\prod_{j=1}^{r}(x + n_j).$$

So the spectral radius $\lambda(K_r(n_1, n_2, \ldots, n_r))$ satisfies the following equation:

$$\sum_{i=1}^{r} \frac{n_i}{\lambda(K_r(n_1, n_2, \ldots, n_r)) + n_i} = 1 \tag{4}$$

Feng, Li and Zhang [14, Theorem 2.1] proved the following lemma, which can also be seen in Stevanović, Gutnam and Rehman [25].

Lemma 2.7 (Feng et al. [14], Stevanović et al. [25])**.** *If $n_i - n_j \geq 2$, then*

$$\lambda(K_r(n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_r)) > \lambda(K_r(n_1, \ldots, n_i, \ldots, n_j, \ldots, n_r)).$$

The following lemma was given in [4].

Lemma 2.8 (Cioabă et al. [4])**.** *Let $A_1, \ldots, A_p$ be finite sets. Then*

$$|A_1 \cap \ldots \cap A_p| \geq \sum_{i=1}^{p} |A_i| - (p-1)\left|\bigcup_{i=1}^{p} A_i\right|.$$

3 Proof of Theorem 1.2

Let F be any graph such that the graphs in $\mathrm{Ex}(n, F)$ are Turán graphs plus $O(1)$ edges. We may assume that the graphs in $\mathrm{Ex}(n, F)$ are obtained from $T_{n,r}$ by adding a edges. Then $\mathrm{ex}(n, F) = e(T_{n,r}) + a$, which implies that $\chi(F) = r + 1$ by (1) and the fact $(1 - \frac{1}{r})\frac{n^2}{2} - \frac{r}{8} \leq e(T_{n,r}) \leq (1 - \frac{1}{r})\frac{n^2}{2}$. In the sequel, we always assume that G is a graph on n vertices containing no F as a subgraph and attaining the maximum spectral radius. The aim of this section is to prove that G is obtained from $T_{n,r}$ by adding a edges for n large enough.

The sketch of our proof is as follows: Firstly, we give a lower bound on $\lambda(G)$, and determine a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that $\sum_{1\leq i<j\leq r} e(V_i, V_j)$ attains the maximum by using the spectral version of Stability Lemma. Then we show that any vertex except at most $2a$ vertices in V_i is adjacent to all vertices in V_j for any $i, j \in [r]$ and $j \neq i$. Next, we prove that all vertices have eigenvector entry very close to the maximum entry and show that the partition is balanced. Finally, we prove $e(G) = \mathrm{ex}(n, F)$ by contradiction.

Lemma 3.1. *G is connected.*

Proof. Suppose to the contrary that G is not connected. Assume $G_1, \ldots, G_s$ are the components of G and $\lambda(G_1) = \max\{\lambda(G_i)|\ i \in [s]\}$, then $\lambda(G) = \lambda(G_1)$ and $|V(G_1)| \leq n - 1$. For any vertex $u \in V(G_1)$, let G' be the graph obtained from G_1 by adding a pendent edge uv at u and $n - 1 - |V(G_1)|$ isolated vertices. Then $\lambda(G') > \lambda(G_1) = \lambda(G)$ by Lemma 2.3. This implies that G' contains a copy of F as a subgraph, denote it as F_1, then uv is an edge of F_1. Next, we claim that $d_{G_1}(u) < |V(F)|$. Otherwise, $d_{G_1}(u) \geq |V(F)|$. Then there exists a vertex $w \in N_{G_1}(u)$ and $w \notin V(F_1)$. Then $F_1 - uv + uw$ is a copy of F in G_1, this is a contraction. Due to the arbitrary of vertex u, $\Delta(G_1) < |V(F)|$. Thus $\lambda(G) = \lambda(G_1) \leq \Delta(G_1) < |V(F)| < \lambda(T_{n,r})$ and this contradicts the fact that G has the maximum spectral radius among all n-vertex F-free graphs as $T_{n,r}$ is F-free. Therefore, G is connected. □

In the following, let $\lambda(G)$ be the spectral radius of G, $\mathbf{x}$ be a positive eigenvector corresponding to $\lambda(G)$ with $\max\{x_i|\ i \in V(G)\} = 1$. Without loss of generality, we assume that $x_z = 1$. If there are multiple such vertices, we choose and fix z arbitrarily among them.

Lemma 3.2.

$$\lambda(G) \geq \left(1 - \frac{1}{r}\right)n - \frac{r}{4n} + \frac{2a}{n}.$$

Proof. Let H be an F-free graph on n vertices with maximum number of edges. Since G attains the maximum spectral radius over all n-vertex F-free graphs, and $\mathrm{ex}(n, F) = e(T_{n,r}) + a$, by the Rayleigh quotient equation, we have

$$\lambda(G) \geq \lambda(H) \geq \frac{\mathbf{1}^T A(H)\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{2(e(T_{n,r}) + a)}{n} \geq \frac{2}{n}\left(\left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \frac{r}{8} + a\right) \geq \left(1 - \frac{1}{r}\right)n - \frac{r}{4n} + \frac{2a}{n}.$$

□

Let ℓ be an integer satisfying $\ell \gg \max\{a, |V(F)|\}$.

Lemma 3.3. *For every $\epsilon > 0$, there exists an integer n_0 such that if $n \geq n_0$, then*

$$e(G) \geq e(T_{n,r}) - \epsilon n^2.$$

Furthermore, G has a partition $V(G) = V_1 \cup \ldots \cup V_r$ such that $\sum_{1\leq i<j\leq r} e(V_i, V_j)$ attains the maximum, and

$$\sum_{i=1}^{r} e(V_i) \leq \epsilon n^2,$$

and for each $i \in [r]$,

$$\left(\frac{1}{r} - 3\sqrt{\epsilon}\right) n < |V_i| < \left(\frac{1}{r} + 3\sqrt{\epsilon}\right) n.$$

Proof. From Lemma 3.2 and Corollary 2.6, it follows that G is obtained from $T_{n,r}$ by adding or deleting at most ϵn^2 edges for large enough n. Then there is a partition of $V(G) = U_1 \cup \ldots \cup U_r$ with $\sum_{i=1}^{r} e(U_i) \leq \epsilon n^2$, $\sum_{1\leq i<j\leq r} e(U_i, U_j) \geq e(T_{n,r}) - \epsilon n^2$ and $\lfloor \frac{n}{r} \rfloor \leq |U_i| \leq \lceil \frac{n}{r} \rceil$ for each $i \in [r]$. So $e(G) \geq e(T_{n,r}) - \epsilon n^2$. Furthermore, G has a partition $V = V_1 \cup \ldots \cup V_r$ such that $\sum_{1\leq i<j\leq r} e(V_i, V_j)$ attains the maximum. In this case, $\sum_{i=1}^{r} e(V_i) \leq \sum_{i=1}^{r} e(U_i) \leq \epsilon n^2$ and $\sum_{1\leq i<j\leq r} e(V_i, V_j) \geq \sum_{1\leq i<j\leq r} e(U_i, U_j) \geq e(T_{n,r}) - \epsilon n^2$. Let $s = \max\left\{\left||V_j| - \frac{n}{r}\right|, j \in [r]\right\}$. Without loss of generality, we assume $\left||V_1| - \frac{n}{r}\right| = s$. Then

$$\begin{aligned}
e(G) &\leq \sum_{1\leq i<j\leq r} |V_i||V_j| + \sum_{i=1}^{r} e(V_i) \\
&\leq |V_1|(n - |V_1|) + \sum_{2\leq i<j\leq r} |V_i||V_j| + \epsilon n^2 \\
&= |V_1|(n - |V_1|) + \frac{1}{2}\Big((\sum_{j=2}^{r} |V_j|)^2 - \sum_{j=2}^{r} |V_j|^2\Big) + \epsilon n^2 \\
&\leq |V_1|(n - |V_1|) + \frac{1}{2}(n - |V_1|)^2 - \frac{1}{2(r-1)}(n - |V_1|)^2 + \epsilon n^2 \\
&< -\frac{r}{2(r-1)}s^2 + \frac{r-1}{2r}n^2 + \epsilon n^2,
\end{aligned}$$

where the last second inequality holds by Hölder's inequality, and the last inequality holds since $\left||V_1| - \frac{n}{r}\right| = s$. On the other hand,

$$e(G) \geq e(T_{n,r}) - \epsilon n^2 \geq \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \frac{r}{8} - \epsilon n^2 > \frac{r-1}{2r}n^2 - 2\epsilon n^2,$$

as n is large enough. Therefore, $\frac{r}{2(r-1)}s^2 < 3\epsilon n^2$, which implies that $s < \sqrt{\frac{6(r-1)\epsilon}{r}n^2} < \sqrt{6\epsilon}n < 3\sqrt{\epsilon}n$. The proof is completed. □

Lemma 3.4. *Let $\theta > 0$ and $\epsilon > 0$ be sufficiently small constants with $\theta < \frac{1}{100r^5\ell}$ and $2\epsilon < \theta^3$. We denote*

$$W := \cup_{i=1}^{r}\{v \in V_i |\ d_{V_i}(v) \geq 2\theta n\}, \tag{5}$$

and

$$L := \left\{v \in V(G) |\ d(v) \leq \left(1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}}\right)n\right\}. \tag{6}$$

Then $|L| \leq \epsilon^{\frac{1}{3}}n$ and $W \subseteq L$.

Proof. We first prove the following claims.

Claim 1. $|W| < \theta n$

Proof. It follows from Lemma 3.3 that $\sum_{i=1}^{r} e(V_i) \leq \epsilon n^2$. On the other hand, let $W_i := W \cap V_i$ for all $i \in [r]$. Then

$$2e(V_i) = \sum_{u\in V_i} d_{V_i}(u) \geq \sum_{u\in W_i} d_{V_i}(u) \geq 2|W_i|\theta n$$

Thus
$$\sum_{i=1}^{r} e(V_i) \geq \sum_{i=1}^{r} |W_i|\theta n = |W|\theta n.$$
Therefore, we have that $|W|\theta n \leq \epsilon n^2$. This proves that $|W| \leq \frac{\epsilon n}{\theta} < \theta n$. □

Claim 2. $|L| \leq \epsilon^{\frac{1}{3}} n$.

Proof. Suppose to the contrary that $|L| > \epsilon^{\frac{1}{3}} n$. Then there exists a subset $L' \subseteq L$ with $|L'| = \lfloor \epsilon^{\frac{1}{3}} n \rfloor$. Therefore,
$$\begin{aligned} e(G[V \setminus L']) &\geq e(G) - \sum_{v \in L'} d(v) \geq e(T_{n,r}) - \epsilon n^2 - \epsilon^{\frac{1}{3}} n^2 \left(1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}}\right) \\ &> \frac{(n - \lfloor \epsilon^{\frac{1}{3}} n \rfloor)^2}{2} \left(1 - \frac{1}{r}\right) + a \\ &\geq e(T_{n',r}) + a = \mathrm{ex}(n', F), \end{aligned}$$
where $n' = n - \lfloor \epsilon^{\frac{1}{3}} n \rfloor$ and n is large enough. However, $e(G[V \setminus L']) > \mathrm{ex}(n', F)$ implies that $G[V \setminus L']$ contains an F, which contradicts that G is F-free. □

Next, we prove that $W \subseteq L$. Otherwise, there exists a vertex $u_0 \in W$ and $u_0 \notin L$. Without loss of generality, let $u_0 \in V_1$. Since $V(G) = V_1 \cup \ldots \cup V_r$ is the partition such that $\sum_{1 \leq i < j \leq r} e(V_i, V_j)$ attains the maximum, $d_{V_1}(u_0) \leq d_{V_i}(u_0)$ for each $i \in [2, r]$. Thus $d(u_0) \geq r d_{V_1}(u_0)$, that is $d_{V_1}(u_0) \leq \frac{1}{r} d(u_0)$. On the other hand, since $u_0 \notin L$, we get $d(u_0) > (1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}})n$. Thus
$$\begin{aligned} d_{V_2}(u_0) &\geq d(u_0) - d_{V_1}(u_0) - (r-2)\left(\frac{1}{r} + 3\sqrt{\epsilon}\right) n \\ &\geq \left(1 - \frac{1}{r}\right) d(u_0) - (r-2)\left(\frac{1}{r} + 3\sqrt{\epsilon}\right) n \\ &> \frac{n}{r^2} - 3(r-1)\epsilon^{\frac{1}{3}} n - 3(r-2)\sqrt{\epsilon} n \qquad (7) \\ &> \frac{n}{r^2} - 6r\epsilon^{\frac{1}{3}} n. \end{aligned}$$

Recall from Claim 1 and Claim 2 that $|W| < \theta n$ and $|L| \leq \epsilon^{\frac{1}{3}} n$, hence, for any $i \in [r]$ and sufficiently large n, we have
$$|V_i \setminus (W \cup L)| \geq \left(\frac{1}{r} - 3\sqrt{\epsilon}\right) n - \theta n - \epsilon^{\frac{1}{3}} n \geq \ell.$$

We claim that u_0 is adjacent to at most a vertices in $V_1 \setminus (W \cup L)$. Otherwise, let $u_{1,1}, u_{1,2}, \ldots, u_{1,a+1}$ be the neighbors of u_0 in $V_1 \setminus (W \cup L)$. Let $u_{1,a+2}, \ldots, u_{1,\ell}$ be another $\ell - a - 1$ vertices in $V_1 \setminus (W \cup L)$. For any $j \in [\ell]$, since $u_{1,j} \notin L$ and $u_{1,j} \notin W$, we have $d(u_{1,j}) > \left(1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}}\right) n$, and $d_{V_1}(u_{1,j}) < 2\theta n$. Thus,
$$\begin{aligned} d_{V_2}(u_{1,j}) &\geq d(u_{1,j}) - d_{V_1}(u_{1,j}) - (r-2)\left(\frac{1}{r} + 3\sqrt{\epsilon}\right) n \\ &> \frac{n}{r} - 3r\epsilon^{\frac{1}{3}} n - 2\theta n - 3(r-2)\sqrt{\epsilon} n \qquad (8) \\ &> \frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - 2\theta n. \end{aligned}$$

By Lemma 2.8, we consider the common neighbors of $u_0, u_{1,1}, \ldots, u_{1,\ell}$ in V_2,

$$\begin{aligned}
& |N_{V_2}(u_0) \cap N_{V_2}(u_{1,1}) \cap \cdots \cap N_{V_2}(u_{1,\ell}) \setminus (W \cup L)| \\
\geq\ & d_{V_2}(u_0) + \sum_{j=1}^{\ell} d_{V_2}(u_{1,j}) - \ell\,|V_2| - |W| - |L| \\
>\ & \frac{n}{r^2} - 6r\epsilon^{\frac{1}{3}} n + \ell\left(\frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - 2\theta n\right) - \ell\left(\frac{n}{r} + 3\sqrt{\epsilon} n\right) - \theta n - \epsilon^{\frac{1}{3}} n \\
>\ & \frac{n}{r^2} - 16r\ell\epsilon^{\frac{1}{3}} n - (2\ell+1)\theta n > \ell,
\end{aligned}$$

for sufficiently large n. This implies that there exist ℓ vertices $u_{2,1}, u_{2,2}, \ldots, u_{2,\ell}$ in $V_2 \setminus (W \cup L)$ such that $\{u_0, u_{1,1}, \ldots, u_{1,\ell}\}$ and $\{u_{2,1}, \ldots, u_{2,\ell}\}$ induce a complete bipartite graph. For an integer s with $2 \leq s \leq r-1$, suppose that for any $1 \leq i \leq s$, there exist $u_{i,1}, u_{i,2}, \ldots, u_{i,\ell} \in V_i \setminus (W \cup L)$ such that $\{u_0, u_{1,1}, \ldots, u_{1,\ell}\}$, $\{u_{2,1}, \ldots, u_{2,\ell}\}$, …, $\{u_{s,1}, \ldots, u_{s,\ell}\}$ induce a complete s-partite graph. We next consider the common neighbors of these vertices in V_{s+1}. Similarly, by (7) and (8), we get that for each $i \in [s]$ and $j \in [\ell]$,

$$d_{V_{s+1}}(u_0) > \frac{n}{r^2} - 6r\epsilon^{\frac{1}{3}} n,$$

and

$$d_{V_{s+1}}(u_{i,j}) > \frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - 2\theta n.$$

By Lemma 2.8 again, we can obtain

$$\begin{aligned}
& \left|N_{V_{s+1}}(u_0) \cap \left(\cap_{i\in[s], j\in[\ell]} N_{V_{s+1}}(u_{i,j})\right) \setminus (W \cup L)\right| \\
\geq\ & d_{V_{s+1}}(u_0) + \sum_{i\in[s], j\in[\ell]} d_{V_{s+1}}(u_{i,j}) - s\ell\,|V_{s+1}| - |W| - |L| \\
>\ & \frac{n}{r^2} - 6r\epsilon^{\frac{1}{3}} n + s\ell\left(\frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - 2\theta n\right) - s\ell\left(\frac{n}{r} + 3\sqrt{\epsilon} n\right) - \theta n - \epsilon^{\frac{1}{3}} n \\
>\ & \frac{n}{r^2} - 16sr\ell\epsilon^{\frac{1}{3}} n - (2s\ell+1)\theta n > \ell,
\end{aligned}$$

where n is sufficiently large. Hence there exist ℓ vertices $u_{s+1,1}, u_{s+1,2}, \ldots, u_{s+1,\ell} \in V_{s+1} \setminus (W \cup L)$ such that $\{u_0, u_{1,1}, \ldots, u_{1,\ell}\}$, …, $\{u_{s+1,1}, \ldots, u_{s+1,\ell}\}$ induce a complete $(s+1)$-partite graph. Thus, for each $i \in [2, r]$, there exist $u_{i,1}, u_{i,2}, \ldots, u_{i,\ell}$ in $V_i \setminus (W \cup L)$ such that $\{u_0, u_{1,1}, \ldots, u_{1,\ell}\}$, $\{u_{2,1}, \ldots, u_{2,\ell}\}$, …, $\{u_{r,1}, \ldots, u_{r,\ell}\}$ induce a complete r-partite graph. Let G' be the graph induced by $\{u_0, u_{1,1}, \ldots, u_{1,\ell}\}$, …, $\{u_{r,1}, \ldots, u_{r,\ell}\}$. Since u_0 is adjacent to $u_{1,1}, \ldots, u_{1,a+1}$, then $e(G') > e(T_{r\ell+1,r}) + a$, by the definition of Turán number, G' contains an F, this is a contradiction. Therefore u_0 is adjacent to at most a vertices in $V_1 \setminus (W \cup L)$. Hence

$$\begin{aligned}
d_{V_1}(u_0) &\leq |W| + |L| + a \\
&< \theta n + \epsilon^{\frac{1}{3}} n + a \\
&< 2\theta n,
\end{aligned}$$

for sufficiently large n. This is a contradiction to the fact that $u_0 \in W$. Hence $W \subseteq L$.

□

Lemma 3.5. *For each $i \in [r]$,*

$$e(G[V_i \setminus L]) \leq a.$$

Furthermore, for each $i \in [r]$, there exists an independent set $I_i \subseteq V_i \setminus L$ such that

$$|I_i| \geq |V_i| - \epsilon^{\frac{1}{3}} n - a.$$

Proof. Suppose to the contrary that there exists an $i_0 \in [r]$ such that $e(G[V_{i_0} \setminus L]) > a$. Without loss of generality, we may assume that $e(G[V_1 \setminus L]) > a$. By Lemmas 3.3 and 3.4, we have $|V_i \setminus L| \geq \left(\frac{1}{r} - 3\sqrt{\epsilon}\right) n - \epsilon^{\frac{1}{3}} n \geq \ell$ for any $i \in [r]$. Let $u_{1,1}, u_{1,2}, \ldots, u_{1,\ell}$ be ℓ vertices chosen from $V_1 \setminus L$ such that the induced subgraph of $\{u_{1,1}, u_{1,2}, \ldots, u_{1,\ell}\}$ in G contains at least $a+1$ edges. For any $j \in [\ell]$, $u_{1,j} \notin L$ implies that $u_{1,j} \notin W$ by Lemma 3.4, thus $d(u_{1,j}) > \left(1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}}\right) n$, and $d_{V_1}(u_{1,j}) < 2\theta n$. Then we have

$$\begin{aligned} d_{V_2}(u_{1,j}) &\geq d(u_{1,j}) - d_{V_1}(u_{1,j}) - (r-2)\left(\frac{1}{r} + 3\sqrt{\epsilon}\right) n \\ &> \frac{n}{r} - 3r\epsilon^{\frac{1}{3}} n - 2\theta n - 3(r-2)\sqrt{\epsilon} n \\ &> \frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - 2\theta n. \end{aligned} \tag{9}$$

Applying Lemma 2.8, we get

$$\begin{aligned} & |N_{V_2}(u_{1,1}) \cap N_{V_2}(u_{1,2}) \cap \cdots \cap N_{V_2}(u_{1,\ell}) \setminus L| \\ \geq\ & \sum_{j=1}^{\ell} d_{V_2}(u_{1,j}) - (\ell - 1)\,|V_2| - |L| \\ \geq\ & \ell\left(\frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - 2\theta n\right) - (\ell-1)\left(\tfrac{1}{r} + 3\sqrt{\epsilon}\right) n - \epsilon^{\frac{1}{3}} n \\ >\ & \frac{n}{r} - 10r\ell\epsilon^{\frac{1}{3}} n - 2\ell\theta n > \ell, \end{aligned}$$

for sufficiently large n. So there exist ℓ vertices $u_{2,1}, u_{2,2}, \ldots, u_{2,\ell} \in V_2$ such that $\{u_{1,1}, \ldots, u_{1,\ell}\}$ and $\{u_{2,1}, \ldots, u_{2,\ell}\}$ induce a complete bipartite graph. For an integer s with $2 \leq s \leq r-1$, suppose that for any $1 \leq i \leq s$, there exist $u_{i,1}, u_{i,2}, \ldots, u_{i,\ell} \in V_i \setminus L$ such that $\{u_{1,1}, \ldots, u_{1,\ell}\}$, $\{u_{2,1}, \ldots, u_{2,\ell}\}$, …, $\{u_{s,1}, \ldots, u_{s,\ell}\}$ induce a complete s-partite subgraph in G. We next consider the common neighbors of these vertices in V_{s+1}. Similarly, by (9), we get that for each $i \in [s]$ and $j \in [\ell]$,

$$d_{V_{s+1}}(u_{i,j}) \geq \frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - 2\theta n.$$

By Lemma 2.8 again, we can obtain

$$\begin{aligned} & \left|\left(\cap_{i\in[s],j\in[\ell]} N_{V_{s+1}}(u_{i,j})\right) \setminus L\right| \\ \geq\ & \sum_{i\in[s],j\in[\ell]} d_{V_{s+1}}(u_{i,j}) - (s\ell - 1)\,|V_{s+1}| - |L| \\ \geq\ & s\ell\left(\frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - 2\theta n\right) - (s\ell-1)\left(\tfrac{1}{r} + 3\sqrt{\epsilon}\right) n - \epsilon^{\frac{1}{3}} n \\ >\ & \frac{n}{r} - 10rs\ell\epsilon^{\frac{1}{3}} n - 2s\ell\theta n > \ell, \end{aligned}$$

for sufficiently large n. Thus there exist ℓ vertices $u_{s+1,1}, u_{s+1,2}, \ldots, u_{s+1,\ell} \in V_{s+1} \setminus L$ such that $\{u_{1,1}, \ldots, u_{1,\ell}\}$, $\{u_{2,1}, \ldots, u_{2,\ell}\}$, …, $\{u_{s+1,1}, \ldots, u_{s+1,\ell}\}$ induce a complete $(s+1)$-partite subgraph in G. Therefore, for each $i \in [2, r]$, there exist $u_{i,1}, u_{i,2}, \ldots, u_{i,\ell}$ in $V_i \setminus L$ such that $\{u_{1,1}, \ldots, u_{1,\ell}\}$, $\{u_{2,1}, \ldots, u_{2,\ell}\}$, …, $\{u_{r,1}, \ldots, u_{r,\ell}\}$ induce a complete r-partite graph. Let G' be the graph induced by $\{u_{1,1}, \ldots, u_{1,\ell}\}$, …, $\{u_{r,1}, \ldots, u_{r,\ell}\}$. Then $e(G') > e(T_{r\ell,r}) + a$, which implies that G' contains a copy of F, this is a contradiction. Thus for each $i \in [r]$, $e(G[V_i \setminus L]) \leq a$.

Therefore, the subgraph obtained from $G[V_i \setminus L]$ by deleting one vertex of each edge in $G[V_i \setminus L]$ contains no edges, which is an independent set of $G[V_i \setminus L]$. Therefore, for each $i \in [r]$, there exists an independent set $I_i \subseteq V_i$ such that

$$|I_i| \geq |V_i \setminus L| - a \geq |V_i| - \epsilon^{\frac{1}{3}} n - a.$$

□

Lemma 3.6. *L is empty and $e(G[V_i]) \leq a$ for each $i \in [r]$.*

Proof. We first prove that $L = \varnothing$. Otherwise, let v be a vertex in L. Then $d(v) \leq (1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}})n$. Recall that $x_z = \max\{x_i | \ i \in [n]\}$, then $\lambda(G) = \lambda(G)x_z = \sum_{wz \in E(G)} x_w \leq d(z)$. Hence

$$d(z) \geq \lambda(G) \geq \left(1 - \frac{1}{r} - \frac{r}{4n^2} + \frac{2a}{n^2}\right) n > \left(1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}}\right) n,$$

as n is large enough. Hence $z \notin L$. Without loss of generality, we may assume that $z \in V_1$. Let G' be the graph with $V(G') = V(G)$ and edge set $E(G') = E(G \setminus \{v\}) \cup \{vw | \ w \in N(z) \cap (\cup_{i=2}^{r} I_i)\}$. We claim that G' is F-free. Otherwise, G' contains a copy of F, denoted as F', as a subgraph, then $v \in V(F')$. Let $N_{G'}(v) \cap V(F') = \{w_1, \ldots, w_s\}$. Obviously, $w_i \notin V_1$ and $w_i \notin L$ for any $i \in [s]$. If $z \notin V(F')$, then $(F' \setminus \{v\}) \cup \{z\}$ is a copy of F in G, which is a contradiction. Thus $z \in V(F')$. For any $i \in [s]$,

$$\begin{aligned} d_{V_1}(w_i) &= d(w_i) - d_{V \setminus V_1}(w_i) \\ &\geq \left(1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}}\right) n - a - \epsilon^{\frac{1}{3}} n - (r-2)\left(\frac{n}{r} + 3\sqrt{\epsilon} n\right) \\ &> \frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - a, \end{aligned}$$

where the last second inequality holds as $w_i \notin L$ and $e(G[V_j \setminus L]) \leq a$ for $w_i \in V_j$. Using Lemma 2.8, we get

$$\begin{aligned} & \left| \bigcap_{i=1}^{s} N_{V_1}(w_i) \setminus L \right| \\ \geq \quad & \sum_{i=1}^{s} d_{V_1}(w_i) - (s-1)|V_1| - |L| \\ > \quad & s\left(\frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - a\right) - (s-1)\left(\frac{n}{r} + 3\sqrt{\epsilon} n\right) - \epsilon^{\frac{1}{3}} n \\ > \quad & \frac{n}{r} - 10sr\epsilon^{\frac{1}{3}} n - sa > 1. \end{aligned}$$

Thus there exists $v' \in V_1 \setminus L$ such that v' is adjacent to $w_1, \ldots, w_s$. Then $(F' \setminus \{v\}) \cup \{v'\}$ is a copy of F in G, which is a contradiction. Thus G' is F-free.

By Lemma 3.5, we have $e(G[V_1 \setminus L]) \leq a$, then the maximum degree in the induced subgraph $G[V_1 \setminus L]$ is at most a. Combining this with Lemma 3.4, we get

$$d_{V_1}(z) = d_{V_1 \cap L}(z) + d_{V_1 \setminus L}(z) \leq \epsilon^{\frac{1}{3}} n + a.$$

Therefore, by Lemma 3.5, we have

$$\begin{aligned}\lambda(G) &= \lambda(G)x_z = \sum_{v\sim z} x_v = \sum_{v\in V_1, v\sim z} x_v + \sum_{i=2}^{r}\left(\sum_{v\in V_i, v\sim z} x_v\right)\\ &= \sum_{v\in V_1, v\sim z} x_v + \sum_{i=2}^{r}\left(\sum_{v\in I_i, v\sim z} x_v + \sum_{v\in V_i\setminus I_i, v\sim z} x_v\right)\\ &\le d_{V_1}(z) + \sum_{i=2}^{r}\left(\sum_{v\in I_i, v\sim z} x_v\right) + \sum_{i=2}^{r}|V_i\setminus I_i|\\ &\le \epsilon^{\frac{1}{3}}n + a + \sum_{i=2}^{r}\left(\sum_{v\in I_i, v\sim z} x_v\right) + (r-1)(\epsilon^{\frac{1}{3}}n + a).\end{aligned}$$

By Lemma 3.2, we have

$$\sum_{i=2}^{r}\left(\sum_{v\in I_i, v\sim z} x_v\right) \ge \left(1-\frac{1}{r}\right)n - \frac{r}{4n} + \frac{2a}{n} - r\epsilon^{\frac{1}{3}}n - ra. \tag{10}$$

By the Rayleigh quotient equation,

$$\begin{aligned}\lambda(G') - \lambda(G) \ge \frac{\mathbf{x}^T\left(A(G') - A(G)\right)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} &= \frac{2x_v}{\mathbf{x}^T\mathbf{x}}\left(\sum_{i=2}^{r}\left(\sum_{w\in I_i, v\sim z} x_w\right) - \sum_{uv\in E(G)} x_u\right)\\ &\ge \frac{2x_v}{\mathbf{x}^T\mathbf{x}}\left(\left(1-\frac{1}{r}\right)n - \frac{r}{4n} + \frac{2a}{n} - r\epsilon^{\frac{1}{3}}n - ra - \left(1-\frac{1}{r}-3r\epsilon^{\frac{1}{3}}\right)n\right) > 0,\end{aligned}$$

where the last second inequality holds since (10) and $\sum_{uv\in E(G)} x_u \le d(v)$, and the last inequality holds for n large enough. This contradicts the fact that G has the largest spectral radius over all F-free graphs, so L must be empty. Furthermore, by Lemma 3.5, we have $e(G[V_i]) \le a$ for each $i \in [r]$. □

Lemma 3.7. *For any $i \in [r]$, let $B_i = \{u \in V_i | \ d_{V_i}(u) \ge 1\}$ and $C_i = V_i \setminus B_i$. Then*
(1) $|B_i| \le 2a$;
(2) For every vertex $u \in C_i$, u is adjacent to all vertices of $V \setminus V_i$.

Proof. We prove the assertions by contradiction.

(1) If there exists a $j \in [r]$ such that $|B_j| > 2a$, then $\sum_{u\in B_j} d_{V_j}(u) > 2a$. On the other hand, $e(G[V_j]) \le a$ by Lemma 3.6 . Therefore,

$$2a < \sum_{u\in B_j} d_{V_j}(u) = \sum_{u\in V_j} d_{V_j}(u) = 2e(G[V_j]) \le 2a,$$

which is a contradiction.

(2) If there exists a vertex $v \in C_{i_0}$ such that there is a vertex $w_1 \notin V_{i_0}$ and $vw_1 \notin E(G)$, where $i_0 \in [r]$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{vw_1\}$. We claim that G' is F-free. Otherwise, G' contains a copy of F, denoted as F', as a subgraph, then $vw_1 \in E(F')$. Let

$N_{G'}(v) \cap V(F') = \{w_1, \dots, w_s\}$. Obviously, $w_i \notin V_{i_0}$ for any $i \in [s]$, then we have,

$$\begin{aligned} d_{V_{i_0}}(w_i) &= d(w_i) - d_{V\setminus V_{i_0}}(w_i) \\ &\geq \left(1 - \frac{1}{r} - 3r\epsilon^{\frac{1}{3}}\right) n - a - (r-2)\left(\frac{n}{r} + 3\sqrt{\epsilon} n\right) \\ &> \frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - a, \end{aligned} \tag{11}$$

where the last second inequality holds as $L = \emptyset$, and $e(G[V_j]) \leq a$ for $w_i \in V_j$. Using Lemma 2.8, we consider the common neighbors of $w_1, \dots, w_s$ in C_{i_0},

$$\begin{aligned} & \left|\bigcap_{i=1}^{s} N_{V_{i_0}}(w_i) \setminus B_{i_0}\right| \\ \geq\ & \sum_{i=1}^{s} d_{V_{i_0}}(w_i) - (s-1)|V_{i_0}| - |B_{i_0}| \\ >\ & s\left(\frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - a\right) - (s-1)\left(\frac{n}{r} + 3\sqrt{\epsilon} n\right) - 2a \\ >\ & \frac{n}{r} - 9rs\epsilon^{\frac{1}{3}} n - (s+2)a > 1. \end{aligned}$$

Then there exists $v' \in C_{i_0}$ such that v' is adjacent to $w_1, \dots, w_s$. Then $(F' \setminus \{v\}) \cup \{v'\}$ is a copy of F in G, which is a contradiction. Thus G' is F-free. From the construction of G', we see that $\lambda(G') > \lambda(G)$, which contradicts the assumption that G has the maximum spectral radius among all F-free graphs on n vertices. □

Lemma 3.8. *For any $u \in V(G)$, $x_u \geq 1 - \frac{20a^2r^2}{n}$.*

Proof. We will prove this lemma by contradiction. Suppose that there is a vertex $v \in V(G)$ with $x_v < 1 - \frac{20a^2r^2}{n}$. Recall that $x_z = \max\{x_i |\ i \in V(G)\} = 1$. Without loss of generality, we may assume that $z \in V_1$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G \setminus \{v\}) \cup \{vw |\ w \in N(z) \cap (\cup_{i=2}^{r} C_i)\}$. We claim that G' is F-free. Otherwise, G' contains a copy of F, denoted by F', as a subgraph, then $v \in V(F')$. Let $N_{G'}(v) \cap V(F') = \{w_1, \dots, w_s\}$. Obviously, $w_i \notin V_1$ for any $i \in [s]$. If $z \notin V(F')$, then $(F' \setminus \{v\}) \cup \{z\}$ is a copy of F in G, which is a contradiction. Thus $z \in V(F')$. By using the similar method as in Lemma 3.7, we get

$$d_{V_1}(w_i) > \frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - a,$$

for any $i \in [s]$. Using Lemma 2.8, we consider the common neighbors of $w_1, \dots, w_s$ in C_1,

$$\begin{aligned} & \left|\bigcap_{i=1}^{s} N_{V_1}(w_i) \setminus B_1\right| \\ \geq\ & \sum_{i=1}^{s} d_{V_1}(w_i) - (s-1)|V_1| - |B_1| \\ >\ & s\left(\frac{n}{r} - 6r\epsilon^{\frac{1}{3}} n - a\right) - (s-1)\left(\frac{n}{r} + 3\sqrt{\epsilon} n\right) - 2a \\ >\ & \frac{n}{r} - 9rs\epsilon^{\frac{1}{3}} n - (s+2)a > 1. \end{aligned}$$

Then there exists $v' \in C_1$ such that v' is adjacent to $w_1, \dots, w_s$. Then $(F' \setminus \{v\}) \cup \{v'\}$ is a copy of F in G, which is a contradiction. Thus G' is F-free.

By Lemma 3.6, $e(G[V_1]) \le a$, then $d_{V_1}(z) \le a$. By (2), we have

$$\lambda(G)x_z = \sum_{w\sim z} x_w = \sum_{w\sim z, w\in V_1} x_w + \sum_{i=2}^{r}\Big(\sum_{w\sim z, w\in V_i} x_w\Big)$$
$$= \sum_{w\sim z, w\in V_1} x_w + \sum_{i=2}^{r}\Big(\sum_{w\sim z, w\in B_i} x_w + \sum_{w\sim z, w\in C_i} x_w\Big),$$

which implies that

$$\begin{aligned}
\sum_{i=2}^{r}\Big(\sum_{w\sim z, w\in C_i} x_w\Big) &= \lambda(G) - \sum_{w\sim z, w\in V_1} x_w - \sum_{i=2}^{r}\Big(\sum_{w\sim z, w\in B_i} x_w\Big)\\
&\ge \lambda(G) - d_{V_1}(z) - \sum_{i=2}^{r}\Big(\sum_{w\in B_i} 1\Big)\\
&\ge \lambda(G) - a - (r-1)2a, \qquad (12)\\
&= \lambda(G) - (2r-3)a,
\end{aligned}$$

where (12) holds as $e(G[V_1]) \le a$, and $|B_i| \le 2a$ for any $i \in [r]$.

By Rayleigh quotient equation, we have

$$\begin{aligned}
\lambda(G') - \lambda(G) &\ge \frac{\mathbf{x}^T(A(G') - A(G))\mathbf{x}}{\mathbf{x}^T\mathbf{x}}\\
&= \frac{2x_v}{\mathbf{x}^T\mathbf{x}}\left(\sum_{i=2}^{r}\Big(\sum_{w\sim z, w\in C_i} x_w\Big) - \sum_{uv\in E(G)} x_u\right)\\
&= \frac{2x_v}{\mathbf{x}^T\mathbf{x}}\left(\sum_{i=2}^{r}\Big(\sum_{w\sim z, w\in C_i} x_w\Big) - \lambda(G)x_v\right)\\
&> \frac{2x_v}{\mathbf{x}^T\mathbf{x}}\left(\lambda(G) - (2r-3)a - \lambda(G)\Big(1 - \frac{20a^2r^2}{n}\Big)\right)\\
&\ge \frac{2x_v}{\mathbf{x}^T\mathbf{x}}\left(\frac{r-1}{r}20a^2r^2 - \frac{r}{4n}\frac{20a^2r^2}{n} + \frac{2a}{n}\frac{20a^2r^2}{n} - (2r-3)a\right) > 0,
\end{aligned}$$

where the last second inequality holds as (12), and the last inequality follows by $\lambda(G) \ge \left(1 - \frac{1}{r}\right)n - \frac{r}{4n} + \frac{2a}{n}$. This contradicts the assumption that G has the maximum spectral radius among all F-free graphs on n vertices. Thus $x_u \ge 1 - \frac{20a^2r^2}{n}$ for any $u \in V(G)$.

□

Let $G_{in} = \cup_{i=1}^{r} G[V_i]$. For any $i \in [r]$, let $|V_i| = n_i$, $K = K_r(n_1, n_2, \dots, n_r)$ be the complete r-partite graph on $V_1, V_2, \dots, V_r$, and G_{out} be the graph with $V(G_{out}) = V(G)$ and $E(G_{out}) = E(K) \setminus E(G)$.

Lemma 3.9. $e(G_{in}) - e(G_{out}) \le a$.

Proof. Suppose to the contrary that $e(G_{in}) - e(G_{out}) > a$. For each $i \in [r]$, let S_i be the vertex set satisfying $B_i \subseteq S_i \subseteq V_i$ and $|S_i| = \ell$. Let $S = \cup_{i=1}^{r} S_i$, $G' = G[S]$. By Lemma 3.7, we have $e(G') \geq e(T_{r\ell,r}) + e(G_{in}) - e(G_{out}) > e(T_{r\ell,r}) + a$, which implies that G' contains an F, this is a contradiction. So $e(G_{in}) - e(G_{out}) \leq a$.

□

Lemma 3.10. *For any* $1 \leq i < j \leq r$, $|n_i - n_j| \leq 1$.

Proof. We prove this lemma by contradiction. Without loss of generality, suppose that $n_1 \geq n_2 \geq \ldots \geq n_r$. Assume that there exist i_0, j_0 with $1 \leq i_0 < j_0 \leq r$ such that $n_{i_0} - n_{j_0} \geq 2$.

Claim 1. There exists a constant $c_1 > 0$ such that $\lambda(T_{n,r}) - \lambda(K) \geq \frac{c_1}{n}$.

Proof. Let $K' = K_r(n_1, \ldots, n_{i_0} - 1, \ldots, n_{j_0} + 1, \ldots, n_r)$. Assume $K' \cong K_r(n'_1, n'_2, \ldots, n'_r)$, where $n'_1 \geq n'_2 \geq \ldots \geq n'_r$. By (4), we have

$$1 = \sum_{i=1}^{r} \frac{n_i}{\lambda(K) + n_i} = \frac{n_{i_0}}{\lambda(K) + n_{i_0}} + \frac{n_{j_0}}{\lambda(K) + n_{j_0}} + \sum_{i \in [r] \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(K) + n_i}, \tag{13}$$

and

$$1 = \sum_{i=1}^{r} \frac{n'_i}{\lambda(K') + n'_i} = \frac{n_{i_0} - 1}{\lambda(K') + n_{i_0} - 1} + \frac{n_{j_0} + 1}{\lambda(K') + n_{j_0} + 1} + \sum_{i \in [r] \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(K') + n_i}. \tag{14}$$

Subtracting (14) from (13), we get

$$\begin{aligned}
&\frac{2(n_{i_0} - n_{j_0} - 1)\lambda^2(K) + (n_{i_0} + n_{j_0})(n_{i_0} - n_{j_0} - 1)\lambda(K)}{(\lambda(K) + n_{i_0} - 1)(\lambda(K) + n_{i_0})(\lambda(K) + n_{j_0} + 1)(\lambda(K) + n_{j_0})} \\
= \;& \sum_{i \in [r] \setminus \{i_0, j_0\}} \frac{n_i(\lambda(K') - \lambda(K))}{(\lambda(K) + n_i)(\lambda(K') + n_i)} + \frac{(n_{i_0} - 1)(\lambda(K') - \lambda(K))}{(\lambda(K) + n_{i_0} - 1)(\lambda(K') + n_{i_0} - 1)} \\
& + \frac{(n_{j_0} + 1)(\lambda(K') - \lambda(K))}{(\lambda(K) + n_{j_0} + 1)(\lambda(K') + n_{j_0} + 1)} \\
\leq \;& \frac{\lambda(K') - \lambda(K)}{\lambda(K) + n'_r} \Big(\sum_{i \in [r] \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(K') + n_i} + \frac{n_{i_0} - 1}{\lambda(K') + n_{i_0} - 1} + \frac{n_{j_0} + 1}{\lambda(K') + n_{j_0} + 1} \Big) \\
= \;& \frac{\lambda(K') - \lambda(K)}{\lambda(K) + n'_r},
\end{aligned}$$

where the inequality holds as $n'_r \leq \min\{n_1, \ldots, n_{i_0} - 1, \ldots, n_{j_0} + 1, \ldots, n_r\}$, and the last equality holds by (14). Combining with the assumption $n_{i_0} - n_{j_0} \geq 2$, we obtain

$$\frac{2\lambda^2(K) + (n_{i_0} + n_{j_0})\lambda(K)}{(\lambda(K) + n_{i_0} - 1)(\lambda(K) + n_{i_0})(\lambda(K) + n_{j_0} + 1)(\lambda(K) + n_{j_0})} \leq \frac{\lambda(K') - \lambda(K)}{\lambda(K) + n'_r}. \tag{15}$$

In view of the construction of K, we see that

$$n - \Big(\frac{n}{r} + 3\sqrt{\epsilon} n\Big) \leq \delta(K) \leq \lambda(K) \leq \Delta(K) \leq n - \Big(\frac{n}{r} - 3\sqrt{\epsilon} n\Big),$$

thus $\lambda(K) = \Theta(n)$. From (15), it follows that there exists a constant $c_1 > 0$ such that $\lambda(K') - \lambda(K) \geq \frac{c_1}{n}$. Therefore, by Lemma 2.7, $\lambda(T_{n,r}) - \lambda(K) \geq \lambda(K') - \lambda(K) \geq \frac{c_1}{n}$.

□

Claim 2. There exists a constant $c_2 > 0$ such that $\lambda(T_{n,r}) - \lambda(K) \leq \frac{c_2}{n^2}$.

Proof. According to the definition of K, we have $e(G) = e(G_{in}) + e(K) - e(G_{out})$. By Lemma 3.7, for any $i \in [r]$, and every vertex $u \in C_i$, u is adjacent to all vertices of $V \setminus V_i$. Thus

$$e(G_{out}) \leq \sum_{1\leq i<j\leq r} |B_i||B_j| \leq \binom{r}{2}(2a)^2 \leq 2a^2r^2.$$

Therefore

$$\begin{aligned}
\lambda(G) &= \frac{\mathbf{x}^{\mathrm{T}}A(G)\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&= \frac{2\sum_{ij\in E(K)} x_ix_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} + \frac{2\sum_{ij\in E(G_{in})} x_ix_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{2\sum_{ij\in E(G_{out})} x_ix_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&\leq \lambda(K) + \frac{2e(G_{in})}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{2e(G_{out})(1-\frac{20a^2r^2}{n})^2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&\leq \lambda(K) + \frac{2(e(G_{in})-e(G_{out}))}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} + \frac{e(G_{out})\frac{40a^2r^2}{n}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&\leq \lambda(K) + \frac{2a}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} + \frac{\frac{80a^4r^4}{n}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}, \qquad (16)
\end{aligned}$$

where (16) holds by Lemma 3.9 and $e(G_{out}) \leq 2a^2r^2$.

On the other hand, let $\mathbf{y}$ be an eigenvector of $T_{n,r}$ corresponding to $\lambda(T_{n,r})$, $k = n - r\lfloor\frac{n}{r}\rfloor$. Since $T_{n,r}$ is a complete r-partite graph on n vertices where each partite set has either $\lfloor\frac{n}{r}\rfloor$ or $\lceil\frac{n}{r}\rceil$ vertices, we may assume $\mathbf{y} = (\underbrace{y_1,\ldots,y_1}_{k\lceil\frac{n}{r}\rceil}, \underbrace{y_2,\ldots,y_2}_{n-k\lceil\frac{n}{r}\rceil})^{\mathrm{T}}$. Thus by (2), we have

$$\lambda(T_{n,r})y_1 = (r-k)\lfloor\frac{n}{r}\rfloor y_2 + (k-1)\lceil\frac{n}{r}\rceil y_1, \qquad (17)$$

and

$$\lambda(T_{n,r})y_2 = (r-k-1)\lfloor\frac{n}{r}\rfloor y_2 + k\lceil\frac{n}{r}\rceil y_1. \qquad (18)$$

Combining (17) and (18), we obtain

$$\left(\lambda(T_{n,r}) + \lceil\frac{n}{r}\rceil\right)y_1 = \left(\lambda(T_{n,r}) + \lfloor\frac{n}{r}\rfloor\right)y_2.$$

Without loss of generality, we assume that $y_2 = 1$. Then

$$y_2 \geq y_1 = \frac{\lambda(T_{n,r}) + \lfloor\frac{n}{r}\rfloor}{\lambda(T_{n,r}) + \lceil\frac{n}{r}\rceil} \geq 1 - \frac{1}{\lambda(T_{n,r}) + \lceil\frac{n}{r}\rceil}.$$

Since $\lambda(T_{n,r}) \geq \delta(T_{n,r}) \geq n - \lceil\frac{n}{r}\rceil$, $y_1 \geq 1-\frac{1}{n}$. Let $H \in \mathrm{Ex}(n,F)$. Then $e(H) = \mathrm{ex}(n,F) = e(T_{n,r}) + a$. Therefore

$$\begin{aligned}
\lambda(G) \geq \lambda(H) &\geq \frac{\mathbf{y}^{\mathrm{T}}A(H)\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} \\
&\geq \frac{\mathbf{y}^{\mathrm{T}}A(T_{n,r})\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} + \frac{2a}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}\left(1-\frac{1}{n}\right)^2 \\
&\geq \lambda(T_{n,r}) + \frac{2a}{n}\left(1-\frac{2}{n}\right). \qquad (19)
\end{aligned}$$

Combining (16), (19) and $\mathbf{x}^{\mathrm{T}}\mathbf{x} \geq n(1 - \frac{20a^2r^2}{n})^2 \geq n - 40a^2r^2$, we get

$$\begin{aligned}
& \lambda(T_{n,r}) - \lambda(K) \\
\leq\ & \frac{2a}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{2a}{n} + \frac{4a}{n^2} + \frac{\frac{80a^4r^4}{n}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
\leq\ & \frac{2a}{n - 40a^2r^2} - \frac{2a}{n} + \frac{4a}{n^2} + \frac{\frac{80a^4r^4}{n}}{n - 40a^2r^2} \\
\leq\ & \frac{80a^3r^2}{n(n - 40a^2r^2)} + \frac{4a}{n^2} + \frac{80a^4r^4}{n(n - 40a^2r^2)} \\
\leq\ & \frac{c_2}{n^2},
\end{aligned}$$

where c_2 is a positive constant. □

Combining Claim 1 and Claim 2, we have

$$\frac{c_1}{n} \leq \lambda(T_{n,r}) - \lambda(K) \leq \frac{c_2}{n^2},$$

which is a contradiction when n is sufficiently large. Thus $|n_i - n_j| \leq 1$ for any $1 \leq i < j \leq r$.

□

Proof of Theorem 1.2. Now we prove that $e(G) = \mathrm{ex}(n, F)$. Otherwise, we assume that $e(G) \leq \mathrm{ex}(n, F) - 1$. Let $H \in \mathrm{Ex}(n, F)$. Then $|E(H)| = e(T_{n,r}) + a$. By Lemma 3.10, we may assume that $V_1 \cup \ldots \cup V_r$ is a vertex partition of H. Let $E_1 = E(G) \setminus E(H)$, $E_2 = E(H) \setminus E(G)$, then $E(H) = (E(G) \cup E_2) \setminus E_1$, and

$$|E(G) \cap E(H)| + |E_1| = e(G) < e(H) = |E(G) \cap E(H)| + |E_2|,$$

which implies that $|E_2| \geq |E_1| + 1$. Furthermore, by Lemma 3.7, we have

$$|E_2| \leq a + \sum_{1 \leq i < j \leq r} |B_i||B_j| \leq a + \binom{r}{2}(2a)^2 \leq 3a^2r^2. \tag{20}$$

According to (3) and (20), for sufficiently large n, we have

$$\begin{aligned}\lambda(H) &\geq \frac{\mathbf{x}^{\mathrm{T}}A(H)\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\ &= \frac{\mathbf{x}^{\mathrm{T}}A(G)\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} + \frac{2\sum_{ij\in E_2} x_ix_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{2\sum_{ij\in E_1} x_ix_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\ &= \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\Big(\sum_{ij\in E_2} x_ix_j - \sum_{ij\in E_1} x_ix_j\Big) \\ &\geq \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\Big(|E_2|(1-\frac{20a^2r^2}{n})^2 - |E_1|\Big) \\ &\geq \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\Big(|E_2| - \frac{40a^2r^2}{n}|E_2| - |E_1|\Big) \\ &\geq \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\Big(1 - \frac{40a^2r^2}{n}|E_2|\Big) \\ &\geq \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\Big(1 - \frac{40a^2r^2}{n}3a^2r^2\Big) \\ &> \lambda(G),\end{aligned}$$

which contradicts the assumption that G has the maximum spectral radius among all F-free graphs on n vertices. Hence $e(G) = \mathrm{ex}(n, F)$. □

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