JOURNAL OF COMBINATORIAL THEORY, Series B 26, 346–348 (1979)

On a Conjecture of Tutte Concerning Minimal Tree Numbers

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Communicated by the Editors

Received October 22, 1976

A counterexample is given to a conjecture by Tutte on the minimum number of spanning trees that a 3-connected planar graph with a prescribed number of edges may have.

1. Introduction and Summary

Tutte [3] has stated the following conjecture.

> Among all 3-connected planar graphs with $2m$ edges, the graph with the smallest number of spanning trees is the wheel W_{m+1}.

The conjecture appears in erroneous form as problem 16 in appendix IV of the recent book by Bondy and Murty [1]. In this note we disprove Tutte's conjecture by giving an infinite sequence of graphs for which the tree number is smaller, even of a smaller order of magnitude than that for the corresponding wheels. The smallest counterexample graph in the sequence has 30 edges.

2. Definitions

A network is a 3-connected planar (simple) graph. If G is a multigraph, then $\kappa(G)$ is the tree number, also called complexity, of G, that is: the number of spanning trees of G; $\kappa_e(G)$ is the number of spanning trees of G containing a given edge e of G. Let P_k be the path on k points. Let A_n ($n \geqslant 3$) be defined as $P_2 \circ P_{n-2}$, that is the graph consisting of disjoint copies of P_2 and P_{n-2} with additional edges joining both vertices of P_2 to every vertex of P_{n-2}. Then A_n is a network with n vertices and $3n - 6$ edges ($n \geqslant 4$). See also figure 1.

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3. Results

THEOREM. *$\kappa(A_n) = nh_{n-1}/2$, where h_n is defined by $h_2 = 2$, $h_3 = 8$, $h_n = 4h_{n-1} - h_{n-2} (n \geqslant 4)$.*

Proof. Let S_n and H_{n-1} be the graph and the multigraph obtained from A_n by deleting or contracting, respectively, the edge $\{1, 2\}$ (see Fig. 1). Now each spanning tree of A_n

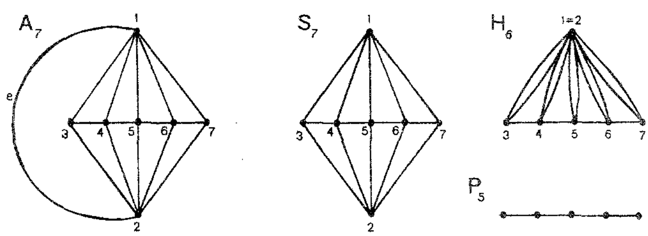

FIG. 1. Some example graphs.

either contains or does not contain e, hence

$$\kappa(A_n) = \kappa_e(A_n) + \kappa(S_n) = \kappa(H_{n-1}) + \kappa(S_n). \tag{1}$$

On the other hand, the quotient $\theta_e = \kappa_e(A_n)/\kappa(A_n)$ can be interpreted as the resistance of A_n viewed as a two-terminal electrical network with the edges as branches of unit resistance, the vertices as nodes, and vertices 1 and 2 as terminal nodes. (See e.g. Mayeda's book [2].) It follows that $\theta_e^{-1} = 1 + (n-2)/2 = n/2$, whence

$$\kappa(A_n) = \frac{n}{2}\,\kappa(H_{n-1}),\ \kappa(S_n) = \frac{n-2}{2}\,\kappa(H_{n-1}). \tag{2}$$

Using the matrix-tree-theorem, we may express $\kappa(H_{n-1})$ as an $(n-2) \times (n-2)$ determinant:

$$\kappa(H_{n-1}) = \begin{vmatrix} 3 & -1 & & & & \\ -1 & 4 & -1 & & \bigcirc & \\ & -1 & 4 & & & \\ & & & \ddots & & \\ & \bigcirc & & & 4 & -1 \\ & & & & -1 & 3 \end{vmatrix}.$$

Let d_n be the determinant of the tridiagonal $n \times n$ matrix $P = (p_{ij})$ with $p_{ii} = 4$, and $p_{ij} = -1$ if $|i - j| = 1$. Then $\kappa(H_{n-1}) = 9d_{n-4} - 6d_{n-5} +$

d_{n-6}, and, since $(d_n)_{n=0}^{\infty}$ satisfies the linear difference equation $d_{n+2} = 4d_{n+1} - d_n$, so does $\kappa(H_{n-1})$:

$$\kappa(H_{n-1}) = 4\kappa(H_{n-2}) - \kappa(H_{n-3}). \tag{3}$$

Since $h_2 = \kappa(H_2)$ and $h_3 = \kappa(H_3)$, as can be verified directly, it follows from (3) that $h_n = \kappa(H_n)$ for all $n \geqslant 3$. The theorem then follows from the first half of (2). ∎

COROLLARY 1. *With the aid of the theorem, one easily finds* $\kappa(A_{12}) = 1815792$. *On the other hand, it is well-known that* $\kappa(W_{n+1}) = L_n^2 - 2 - 2(-1)^n = L_{2n} - 2$ *where* $(L_n)_{n=1}^{\infty}$ *is the sequence of Lucas numbers*: $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ $(n \geqslant 3)$. *This yields* $\kappa(W_{16}) = 1860496$. *Since both* A_{12} *and* W_{16} *have* 30 *edges, we have the announced counterexample.*

COROLLARY 2. *One may solve the equation* (3) *to obtain an explicite formula for* $\kappa(H_{n-1})$, *hence for* $\kappa(A_n)$. *The result is*

$$\kappa(A_n) = n\{(2 + \sqrt{3})^{n-2} - (2 - \sqrt{3})^{n-2}\}/2\sqrt{3}.$$

Let $r = 6s$. *Then both* A_{2s+2} *and* W_{3s+1} *have* r *edges, and*

$$\kappa(A_{2s+2}) \sim cr(2 + \sqrt{3})^{r/3}$$

whereas

$$\kappa(W_{3s+1}) \sim \left(\frac{1 + \sqrt{5}}{2}\right)^r.$$

Since $(2 + \sqrt{3})^{1/3} < 1.5512 < 1.6180 < (1 + \sqrt{5})/2$, *it follows that* $\kappa(A_{2s+2}) < \kappa(W_{3s+1})$ *for all sufficiently large* s.

Remark. Let G be a network with r edges and let

$$\alpha = \liminf_{r \to \infty} \kappa(G)^{1/r}.$$

Then we have seen that $\alpha \leqslant (2 + \sqrt{3})^{1/3}$, but no reasonable lower bound for α seems to be known.

REFERENCES

1. J. A. BONDY AND U. S. R. MURTY, "Graph Theory and its Applications," Macmillan & Co., London, 1975.
2. W. MAYEDA, "Graph Theory," Wiley, New York, 1972.
3. W. T. TUTTE, written communication.