AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **29** (2004), Pages 135–142

On a conjecture on k-walks of graphs

Zemin Jin   Xueliang Li

Center for Combinatorics
Nankai University
Tianjin 300071
P.R. China
x.li@eyou.com

Abstract

In this paper we give examples to show that a conjecture on k-walks of graphs, due to B. Jackson and N.C. Wormald, is false. We also give a maximum degree condition for the existence of k-walks and k-trees in 2-connected graphs.

1 Introduction

All graphs considered here are simple and finite. We use G to denote a graph, and use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For any $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v in G, and $|N_G(v)|$ the degree of v in G. Sometimes, we simply use $N(v)$ and $d(v)$ to denote them, respectively, if no confusion occurs. Let $\delta(G) = \min\{d(v) \mid v \in V(G)\}$ and $\Delta(G) = \max\{d(v) \mid v \in V(G)\}$. A k-walk of G is a spanning closed walk of G using each vertex at most k times. When $k = 1$, a k-walk of G is a hamiltonian cycle of G. We say that G is $K_{1,r}$-free if no induced subgraph of G is isomorphic to $K_{1,r}$. A graph G is t-tough if for any $S \subseteq V(G)$, the number of components $c(G-S) \leq |S|/t$. For notations and terminology not defined here, we refer to [1].

A well known conjecture by Chvatál [8] states that every sufficiently tough graph has a hamiltonian cycle. Many results for a $K_{1,3}$-free graph to be hamiltonian have been obtained. Since the concept of a k-walk is a generalization of the concept of a hamiltonian cycle, in [3] B. Jackson and N.C. Wormald investigated k-walks and obtained the following results.

Theorem 1.1. [3] *Let $k \geq 2$ be an integer. If G is connected and for any $S \subseteq V(G)$, $c(G-S) \leq (k-2)|S|+2$, then G has a k-walk.*

As a consequence, the following result is immediate.

Theorem 1.2. [3] *Every $1/(k-2)$-tough graph has a k-walk.*

A well known conjecture related to k-walks is stated as follows, which is still open.

Conjecture A. [3] *Every* $1/(k-1)$*-tough graph has a* k*-walk.*

Theorem 1.3. [3] *If* G *is connected and* $K_{1,k+1}$*-free, then* G *has a* k*-walk.*

Theorem 1.4. [3] *Let* $j \geq 1$, $k \geq 3$ *be integers. If* G *is* j*-connected and* $K_{1,j(k-2)+1}$*-free, then* G *has a* k*-walk.*

The authors of [3] believe that Theorem 1.4 can be sharpened as follows.

Conjecture B. [3] *Let* $j \geq 1$, $k \geq 2$ *be integers. If* G *is* j*-connected and* $K_{1,jk+1}$*-free, then* G *has a* k*-walk.*

Clearly, Conjecture B holds for $j = 1$. But, as we will see in Section 2, it is false for $j \geq 2$. Our counterexamples are based on a result of [4], where the author constructed a family of graphs G_j, $j \geq 3$, which are j-connected, j-regular and non-hamiltonian. From their graphs G_j, we employ a similar technique to construct counterexamples to Conjecture B for $j \geq 3$. Also, we give a minimally 2-connected graph to show that Conjecture B is false for $j = 2$. So, perhaps $1/k$-tough graphs do not have k-walks. In some sense, we feel that Conjecture A, if true, is best possible.

In Section 3, we give a maximum degree condition for the existence of k-walks and k-trees in 2-connected graphs, which is best possible for k-trees. But, we know that under this condition it is impossible for graphs to have a hamiltonian cycle.

2 Negative Answer for Conjecture B

In order to construct our counterexamples for $j \geq 3$, first of all, we need the following lemmas.

Lemma 2.1. [4] *For any integer* $j \geq 3$, *there always exist* j*-connected and* j*-regular non-hamiltonian graphs.*

The counterexamples are constructed as follows. Let G be a j-connected and j-regular non-hamiltonian graph, $j \geq 3$. For every $x \in V(G)$, we create $jk-1$ new vertices $x^1, x^2, \cdots, x^{jk-1}$, and for every edge $\alpha \in E(G)$ incident to x, we create a new vertex x_α. Denote

$$D(x) = \{x_\alpha \mid \alpha \in E(G) \text{ and is incident to } x\},$$

$$S(x) = \{x^i \mid i = 1, 2, \cdots, jk-1\}.$$

Obviously, $|D(x)| = d_G(x) = j$ and $|S(x)| = jk-1$. We construct a new graph G^* as follows:

$$V(G^*) = \bigcup_{x \in V(G)} (D(x) \cup S(x)),$$

$$E(G^*) = E_1 \cup E_2,$$

in which,

$$E_1 = \{x_\alpha y_\alpha \mid \alpha = xy \in E(G)\},$$

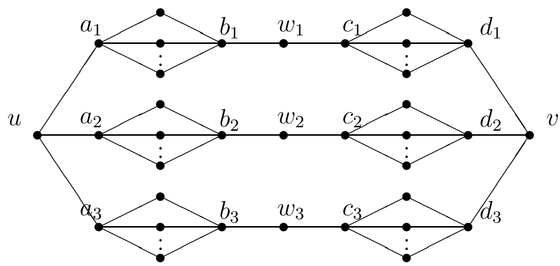

Figure 1: A counterexample graph G

$$E_2 = \{uv \mid u \in D(x),\ v \in S(x) \text{ for some } x \in V(G)\}.$$

From the construction, the following result follows immediately.

Lemma 2.2. *G^* is j-connected and $K_{1,jk+1}$-free.*

Next, we shall show the following result.

Lemma 2.3. *G^* does not have any k-walks.*

Proof. Suppose that G^* has a k-walk W. Then we can show that, for every vertex $x \in V(G)$, there exists a sub-walk $W_x = v_1\ v_2\ \cdots\ v_{2jk-1}$ in W such that $S(x) = \{v_{2i} | 1 \le i \le jk-1\}$ and $D(x) = \bigcup_{i=1}^{jk}\{v_{2i-1}\}$.

Otherwise, in order to meet all vertices of $S(x)$, the sum of the meeting times of vertices in $D(x)$ is at least $|S(x)|+2 = jk+1$. Since $N_G(S(x)) = D(x)$ and both $D(x)$ and $S(x)$ are independent sets in G^*, there exists at least one vertex in $D(x)$ which is met at least $k+1$ times in W, a contradiction.

Then every vertex in $D(x)$ is met exactly k times, since the sum of meeting times of all vertices in $D(x)$ is $|S(x)|+1 = jk$ and $|D(x)| = j$. We can denote W by $x_\alpha W_{x_1} W_{x_2} \cdots W_{x_n} y_\alpha$, where $n = |V(G)|$, $x = x_1$, $y = x_n$, $\alpha = xy \in E(G)$ and $x_i \ne x_l, i \ne l$. Since W is a k-walk, there must exist an edge $e_i \in E(G)$ such that $e_i = x_i x_{i+1}$ for each $1 \le i \le n-1$. Thus, we can obtain a hamiltonian cycle of G, a contradiction. The proof is complete. □

From above, we can see that Conjecture B is false for $j \ge 3$. Now we consider the case $j = 2$. The following Figure 1 shows a 2-connected graph G with $\Delta(G) = 2k$ and without any k-walks.

In fact, as shown in Figure 1, we can see that $|N(a_i) \cap N(b_i)| = |N(c_i) \cap N(d_i)| = 2k-1$, $i = 1,\ 2,\ 3,\ k \ge 2$, and G is 2-connected with $\Delta(G) = 2k$. Both $N(a_i) \cap N(b_i)$ and $N(c_i) \cap N(d_i)$ are independent sets, $i = 1,\ 2,\ 3$. By a proof analogous to that in Lemma 2.3, we know that there exists a walk W_i with ends a_i and d_i which contains only $N(a_i) \cup N(c_i) \cup \{a_i,\ b_i,\ c_i,\ d_i,\ w_i\}$, since $N(w_i) = \{c_i, b_i\}$; whereas $W - W_i$ does not contain any vertex of $N(a_i) \cup N(c_i) \cup \{a_i,\ b_i,\ c_i,\ d_i,\ w_i\}$. So, W can be written as $uW_1vW_2uW_3v$, a contradiction.

Thus, we obtain the following negative answer to Conjecture B of [3].

Theorem 2.1. *Conjecture B is false for $j \geq 2$.*

3 Maximum Degree Condition for the Existence of k-Walks and k-Trees in 2-Connected Graphs

A k-tree of a connected graph G is a spanning tree of G with maximum degree at most k. In this section we consider only 2-connected graphs. A graph G is minimally 2-connected if, for any $e \in E(G)$, $G - e$ has a cut vertex.

Lemma 3.1. [2] *If G is a minimally 2-connected graph, then every 2-connected subgraph of G is minimally 2-connected.*

Lemma 3.2. [2] *If G is a minimally 2-connected graph, then for any $e \in E(G)$, e is not a chord of any cycle of G.*

More results on minimally 2-connected graphs can be found in [2].

Let G be a minimally 2-connected graph. We say that G satisfies Ω on a vertex-cut $\{u, v\}$ if one of the following conditions holds

(P_1) $c(G - \{u, v\})$ is even, and for every component G_i of $G - \{u, v\}$, both $|N_G(u) \cap V(G_i)|$ and $|N_G(v) \cap V(G_i)|$ are odd;

(P_2) For every component G_i of $G - \{u, v\}$, every block of $G_i + \{u, v\}$ satisfies (P_1) on the vertex-cut $\{x, y\}$, in which $N_G(x) \cap V(G - B) \neq \emptyset$ and $N_G(y) \cap V(G - B) \neq \emptyset$;

(P_3) $G = G' \cup G''$, $G' \cap G'' = \{u, v\}$, and G' and G'' satisfies (P_1) and (P_2), respectively, on the vertex-cut $\{u, v\}$.

Lemma 3.3. *Let $k \geq 2$ be an integer, G be minimally 2-connected, $\Delta(G) \leq 2k - 2$, and $\{u, v\}$ be a vertex-cut of G. Then, G contains a spanning tree T such that if G satisfies Ω on $\{u, v\}$, then*

(i) $d_T(u) \leq d(u)/2$, $d_T(v) \leq d(v)/2 + 1$ and $d_T(x) \leq k$, $x \in V(G) - \{u, v\}$, or

(ii) $d_T(u) \leq \lceil d(u)/2 \rceil$, $d_T(v) \leq \lceil d(v)/2 \rceil$ and $d_T(x) \leq k$, $x \in V(G) - \{u, v\}$.

Proof. By induction on $|V(G)|$. For $|V(G)| = 3, 4, 5, 6$, the lemma holds obviously. We assume that the lemma holds for graphs with order less than $|V(G)|$. Let $G_1, G_2, \cdots, G_r$ be the components of $G - \{u, v\}$, $r \geq 2$, and let $H_i = G_i + \{u, v\}$, $i = 1, 2, \cdots, r$. Then, H_i has at least two blocks, and each block is minimally 2-connected or a K_2, see [2]. Let $B_{i,1}, B_{i,2}, \cdots, B_{i,s_i}$, be the blocks of H_i such that $B_{i,j} \cap B_{i,j+1} = \{x_{i,j+1}\}$, $u = x_{i,1}$, $v = x_{i,s_i+1}$, and $d_{H_i}(u, B_{i,t}) < d_{H_i}(u, B_{i,j})$ if and only if $t < j$. We distinguish the following two cases to consider H_i, $i = 1, 2, \cdots, r$.

Case 1. If $B_{i,\ j}$, $1 \leq j \leq s_i$, satisfies Ω on $\{x_{i,\ j},\ x_{i,\ j+1}\}$, then by the induction hypothesis, $B_{i,\ j}$ contains a spanning tree $T_{i,\ j}$ such that

$$d_{T_{i,\ j}}(x_{i,\ j}) \leq d_{B_{i,\ j}}(x_{i,\ j})/2,\ d_{T_{i,\ j}}(x_{i,\ j+1}) \leq d_{B_{i,\ j}}(x_{i,\ j+1})/2 + 1$$

and $d_{T_{i,\ j}}(x) \leq k,\ x \in V(B_{i,\ j}) - \{x_{i,\ j},\ x_{i,\ j+1}\}$. Let $T_i = \bigcup_{j=1}^{s_i} T_{i,\ j}$. Then, T_i is a spanning tree of H_i such that

$$d_{T_i}(u) \leq d_{H_i}(u)/2,\ d_{T_i}(v) \leq d_{H_i}(v)/2 + 1$$

and $d_{T_i}(x) \leq k,\ x \in V(G_i)$.

Case 2. There exists a subset $I \subseteq \{1,\ 2,\ \cdots,\ s_i\}$ and $I \neq \emptyset$ such that $B_{i,\ t},\ t \in I$, does not satisfy Ω on $\{x_i,\ x_{i+1}\}$. Let $T_{i,\ t} = K_2$, if $B_{i,\ t} = K_2$. If $B_{i,\ t},\ t \in I$, is minimally 2-connected, then by the induction hypothesis, it contains a spanning tree $T_{i,\ t}$ such that

$$d_{T_{i,\ t}}(x_{i,\ t}) \leq \lceil d_{B_{i,\ t}}(x_{i,\ t})/2\rceil,\ d_{T_{i,\ t}}(x_{i,\ t+1}) \leq \lceil d_{B_{i,\ t}}(x_{i,\ t+1})/2\rceil$$

and $d_{T_{i,\ t}}(x) \leq k,\ x \in V(B_{i,\ t}) - \{x_{i,\ t},\ x_{i,\ t+1}\}$. Let $t_0 = max\{t \mid t \in I\}$. Note that for every $j \in \{1,\ 2,\ \cdots,\ s_i\} - I$, $B_{i,\ j}$ satisfies Ω on $\{x_{i,\ j},\ x_{i,\ j+1}\}$. Then,
(1) If $j < t_0$, then $B_{i,\ j}$ contains a spanning tree $T_{i,\ j}$ such that

$$d_{T_{i,\ j}}(x_{i,\ j}) \leq d_{B_{i,\ j}}(x_{i,\ j})/2,\ d_{T_{i,\ j}}(x_{i,\ j+1}) \leq d_{B_{i,\ j}}(x_{i,\ j+1})/2 + 1$$

and $d_{T_{i,\ j}}(x) \leq k,\ x \in V(B_{i,\ j}) - \{x_{i,\ j},\ x_{i,\ j+1}\}$.
(2) If $j > t_0$, then by the symmetry of $x_{i,\ j+1}$ and $x_{i,\ j}$, we have that $B_{i,\ j}$ has a spanning tree $T_{i,\ j}$ such that

$$d_{T_{i,\ j}}(x_{i,\ j}) \leq d_{B_{i,\ j}}(x_{i,\ j})/2 + 1,\ d_{T_{i,\ j}}(x_{i,\ j+1}) \leq d_{B_{i,\ j}}(x_{i,\ j+1})/2$$

and $d_{T_{i,\ j}}(x) \leq k,\ x \in V(B_{i,\ j}) - \{x_{i,\ j},\ x_{i,\ j+1}\}$.

Next, let $T_i = \bigcup_{j=1}^{s_i} T_{i,\ j}$. Then, T_i is a spanning tree of H_i such that

$$d_{T_i}(u) \leq \lceil d_{H_i}(u)/2\rceil,\ d_{T_i}(v) \leq \lceil d_{H_i}(v)/2\rceil$$

and $d_{T_i}(x) \leq k,\ x \in V(G_i)$. In both cases, we use e_i and f_i to denote the edges incident to u and v, respectively, on the u-v path in T_i. Now we distinguish two cases to consider G.

Case a. G satisfies Ω on $\{u, v\}$.

Subcase a.1. (P_1) is true.

Then, r is even, $d_{H_i}(u)$ and $d_{H_i}(v)$ are odd, and

$$d_{T_i}(u) \leq (d_{H_i}(u) + 1)/2,\ d_{T_i}(v) \leq (d_{H_i}(v) + 1)/2$$

and $d_{T_i}(x) \leq k,\ x \in V(G_i)$. Let $T = \bigcup_{i=1}^{r} T_i - \bigcup_{i=1}^{r/2} e_{2i} - \bigcup_{i=1}^{(r-2)/2} f_{2i+1}$.

Subcase a.2. (P_2) is true.

Then,

$$d_{T_i}(u) \leq d_{H_i}(u)/2, \ d_{T_i}(v) \leq d_{H_i}(v)/2 + 1$$

and $d_{T_i}(x) \leq k, \ x \in V(G_i)$. Let $T = \bigcup_{i=1}^{r} T_i - \bigcup_{i=2}^{r} f_i$.

Subcase a.3. (P_3) is true.

Then, $G = G' \cup G''$, $G' \cap G'' = \{u, v\}$, G' and G'' satisfies (P_1) and (P_2), respectively, on the vertex-cut $\{u, v\}$. Without loss of generality, let $G' = \bigcup_{i=1}^{2l} H_i, \ G'' = \bigcup_{i=2l+1}^{r} H_i, \ 2l < r$. Now, let $T = \bigcup_{i=1}^{r} T_i - \bigcup_{i=1}^{l} e_{2i-1} - \bigcup_{i=1}^{l} f_{2i} - \bigcup_{i=2l+1}^{r-1} f_i$.

Thus, in all the above subcases we have obtained a tree T which is a spanning tree of G such that

$$d_T(u) \leq d(u)/2, \ d_T(v) \leq d(v)/2 + 1$$

and $d_T(x) \leq k, \ x \in V(G) - \{u, v\}$.

Case b. G does not satisfy Ω on $\{u, v\}$. Without loss of generality, let $G = G^* \cup G^{**}$, in which $G^*(G^{**})$ satisfies (does not satisfy) Ω on $\{u, \ v\}$. Clearly $G^{**} \neq \emptyset$.

Subcase b.1. $G^* = \emptyset$.

Then, G^* has a spanning tree T^* such that

$$d_{T^*}(u) \leq d_{G^*}(u)/2, \ d_{T^*}(v) \leq d_{G^*}(v)/2 + 1$$

and $d_{T^*}(x) \leq k, \ x \in V(G^*) - \{u, v\}$.

(1) If G^{**} is minimally 2-connected, then G^{**} has a spanning tree T^{**} such that

$$d_{T^{**}}(u) \leq \lceil d_{G^{**}}(u)/2 \rceil, \ d_{T^{**}}(v) \leq \lceil d_{G^{**}}(v)/2 \rceil$$

and $d_{T^{**}}(x) \leq k, x \in V(G^{**}) - \{u, v\}$.

(2) If G^{**} contains a vertex-cut, then G^{**} has at least two blocks, each of which is a K_2 or minimally 2-connected. From Case 1 and Case 2 we know that G^{**} has a spanning tree T^{**} such that

$$d_{T^{**}}(u) \leq \lceil d_{G^{**}}(u)/2 \rceil, \ d_{T^{**}}(v) \leq \lceil d_{G^{**}}(v)/2 \rceil$$

and $d_{T^{**}}(x) \leq k, x \in V(G^{**}) - \{u, v\}$.

In both (1) and (2), let $T = T^* \cup T^{**} - f^*$, in which f^* is the edge incident to v on the u-v path in T^*. Then, T is a spanning tree of G such that (ii) holds.

Subcase b.2. $G^* = \emptyset$.

By an analogous analysis, we can show that (ii) holds, and the details are omitted.

The proof is now complete. □

Lemma 3.4. [3] *If G has a k-tree, then G has a k-walk.*

Lemma 3.5. [2] *Every 2-connected graph contains a minimally 2-connected spanning subgraph.*

Thus, we get our main results as follows.

Theorem 3.1. *Let $k \geq 2$ be an integer and G be a 2-connected graph with $\Delta(G) \leq 2k-2$. Then, G contains a k-tree. And, for $k \geq 3$ the result is best possible.*

Theorem 3.2. *Let $k \geq 2$ be an integer and G be a 2-connected graph with $\Delta(G) \leq 2k-2$. Then, G contains a k-walk.*

Now we construct an example to show that Theorem 3.1 is best possible. Let $K_{2,\ 2k-3} = K(X,\ Y)$, $X = \{x, y\}$. Add four new vertices $a_1,\ b_1,\ a_2,\ b_2$, and connect a_i with x and b_i with y, respectively. Denote thus obtained graph by H. Take $k-1$ copies of H. Let $u,\ v$ be two new vertices and connect u with all a_i and v with all b_i, respectively. Denote thus obtained graph by G. Obviously, G is a 2-connected graph with $\Delta(G) = 2k-1$. However, G does not have any k-trees. But, interestingly, G contains k-walks.

4 Concluding Remark

We have obtained a maximum degree condition for the existence of k-walks in 2-connected graphs. The problem to find an analogous condition for the existence of k-walks in j-connected graphs is still left for further investigation. In [3] the authors proved that the k-walk problem is NP-complete. In fact, using the technique in our Section 2, we can also prove it.

Acknowledgement: The authors would like to thank an anonymous referee for his/her comments and suggestions. One of the authors, Z.M. Jin, would like to thank Professors Z.H. Liu and L.M. Xiong for their discussion.

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(Received 26 Oct 2002)