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ON A CONSTRUCTION OF THE FUNDAMENTAL SOLUTION FOR THE FREE WEYL EQUATION BY HAMILTONIAN PATH-INTEGRAL METHOD —AN EXACTLY SOLVABLE CASE WITH "ODD VARIABLE COEFFICIENTS"

Dedicated to Professor Takeshi Kotake on his retirement from Tôhoku University

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Abstract. A fundamental solution for the free Weyl equation is easily constructed using the Clifford relation of the Pauli matrices. But, we insist on Feynman's idea of representing a fundamental solution using classical objects. To do this, we first reformulate the usual matrix-valued Weyl equation on the ordinary Euclidian space to the "non-commutative scalar"-valued equation on the superspace, called the super Weyl equation. Then, we may find the classical mechanics corresponding to that super Weyl equation. Using analysis on the superspace, we may associate the classical Hamiltonian with that super Weyl equation. From this mechanics, we define phase and amplitude functions which are solutions of the Hamilton-Jacobi and continuity equations, respectively. Moreover, they are exactly solvable. Then, we define a Fourier integral operator with phase and amplitude given by those functions, which gives a solution to the initial value problem of that super Weyl equation. The method and idea developped here, may be applied not only to the Pauli, Weyl or Dirac equations but also to any system of P.D.E's.

1. Introduction and the result.

Let $\psi(t, q): \boldsymbol{R} \times \boldsymbol{R}^3 \to \boldsymbol{C}^2$ satisfy

$$
(1.1)\qquad \begin{cases} i\hbar \dfrac{\partial}{\partial t}\psi(t, q) = \boldsymbol{H}\psi(t, q), \qquad \boldsymbol{H} = -ic\hbar\boldsymbol{\sigma}_j \dfrac{\partial}{\partial q_j}, \\ \psi(0, q) = \underline{\psi}(q). \end{cases}
$$

Here, $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q))$, c and $\hbar$ are positive constants, the summation with respect to $j = 1, 2, 3$ is abbreviated. And the Pauli matrices $\{\boldsymbol{\sigma}_j\}$ are 2×2 matrices satisfying the following relations ($\boldsymbol{I}_m$ stands for the $m \times m$ identity matrix):

$$
(1.2)\qquad \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k + \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j = 2\delta_{jk}\boldsymbol{I}_2 \quad \text{for} \quad j, k = 1, 2, 3, \quad \text{(Clifford relation)}
$$

$$
(1.3)\qquad \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_3, \quad \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_1, \quad \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = i\boldsymbol{\sigma}_2,
$$

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for example,

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

Applying formally the Fourier transformation (which contains a parameter $\hbar$) with respect to $q\in \boldsymbol{R}^3$ to (1.1), we get

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(t,p)=\hat{\boldsymbol{H}}\hat{\psi}(t,p) \qquad \text{where} \quad \hat{\boldsymbol{H}}=c\sigma_j p_j=c\begin{pmatrix} p_3 & p_1-ip_2 \\ p_1+ip_2 & -p_3 \end{pmatrix}.$$

As $\hat{\boldsymbol{H}}^2=c^2|p|^2\boldsymbol{I}_2$ by (1.2), we easily have

$$e^{-i\hbar^{-1}t\hat{\boldsymbol{H}}}\underline{\hat{\psi}}(p)=[\cos(c\hbar^{-1}t|p|)\boldsymbol{I}_2-ic^{-1}|p|^{-1}\sin(c\hbar^{-1}t|p|)\hat{\boldsymbol{H}}]\underline{\hat{\psi}}(p).$$

Therefore, we have

PROPOSITION 1.1. *For any* $t\in\boldsymbol{R}$,

$$e^{-i\hbar^{-1}t\boldsymbol{H}}\underline{\psi}(q)=(2\pi\hbar)^{-3/2}\int_{\boldsymbol{R}^3}dp\,e^{i\hbar^{-1}qp}e^{-i\hbar^{-1}t\hat{\boldsymbol{H}}}\underline{\hat{\psi}}(p)=\int_{\boldsymbol{R}_3}dq'\boldsymbol{E}(t,q,q')\underline{\psi}(q') \tag{1.5}$$

with

$$\boldsymbol{E}(t,q,q')=(2\pi\hbar)^{-3}\int_{\boldsymbol{R}^3}dp\,e^{i\hbar^{-1}(q-q')p}[\cos(c\hbar^{-1}t|p|)\boldsymbol{I}_2-ic^{-1}|p|^{-1}\sin(c\hbar^{-1}t|p|)\hat{\boldsymbol{H}}]. \tag{1.6}$$

On the other hand, Feyman's main motivation for deriving his notorious measure, is to clarify the so-called "Bohr correspondence" as explicitly as possible. He expressed quantum objects using classical quantities as ingredients of the integral representation with respect to his measure. But it seems difficult to imagine from the above formula that there exist classical objects when $\hbar\to 0$. Therefore, he could not apply his idea to the equation containing "spin" and posed a problem in p. 355 of Feynman & Hibbs [7].

In spite of this, we claim that there exists the classical mechanics corresponding to the Weyl equation and that a fundamental solution of (1.1) is constructed as a Fourier integral operator using phase and amplitude functions defined by that classical mechanics. Therefore, the Weyl equation is obtained by quantizing that classical mechanics after Feynman's procedure. Because the Hamiltonian defined on the superspace is "of first order both in even and odd variables," we should modify Feynman's argument from the Lagrangian formulated "path integral" to the Hamiltonian formulated one.

MAIN THEOREM (Path-integral representation of a solution for the Weyl equation).

(1.7)

$$\psi(t,q)=\flat\left((2\pi\hbar)^{-3/2}\hbar\iint_{\mathfrak{R}^{3|2}} d\underline{\xi}d\underline{\pi}\mathscr{D}^{1/2}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})e^{i\hbar^{-1}\mathscr{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})}\mathscr{F}(\sharp\psi)(\underline{\xi},\underline{\pi})\right)\Bigg|_{x_B=q}.$$

Here, $\mathscr{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ and $\mathscr{D}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ are solutions of the Hamilton-Jacobi and continuity equations, (1.17) *and* (1.19) *respectively.*

REMARK. Unfamiliar notation above is roughly explained in the course of describing the outline of our procedure below. For example, one may intuitively regard odd variables θ_j, π_k as something-like odd forms on "$\boldsymbol{R}^\infty=\prod_{j=1}^\infty \boldsymbol{R}$", where the space $\boldsymbol{R}^\infty$ has the Fréchet-Grassmann structure. See also Appendix A where the fundamentals of superanalysis (=analysis on the superspace $\mathfrak{R}^{m|n}$) is given.

Outline of our procedure (1)–(6). (1) We identify a "spinor" $\psi(t,q)={}^t(\psi_1(t,q), \psi_2(t,q)):\boldsymbol{R}\times\boldsymbol{R}^3\to\boldsymbol{C}^2$ with an even supersmooth function $u(t,x,\theta)=u_0(t,x)+u_1(t,x)\theta_1\theta_2$: $\boldsymbol{R}\times\mathfrak{R}^{3|2}\to\mathfrak{C}_{\mathrm{ev}}$. Here, $\mathfrak{R}^{3|2}$ is the superspace and $u_0(t,x)$, $u_1(t,x)$ are the Grassmann continuation of $\psi_1(t,q)$, $\psi_2(t,q)$, respectively. For example,

$$\boldsymbol{C}^2\ni\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\underset{\flat}{\overset{\sharp}{\rightleftarrows}}u(\theta)=u_0+u_1\theta_1\theta_2\in\mathfrak{C}_{\mathrm{ev}}\qquad\text{with}\quad u_0=\psi_1,\quad u_1=\psi_2.$$

(2) We represent the matrices $\{\sigma_j\}$ satisfying (1.2) and (1.3), which act on $u(t,x,\theta)$ as follows:

(1.8)
$$\begin{aligned}
\sigma_1\left(\theta,\frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)&=i\lambda^{-1}\left(\theta_1\theta_2+\lambda^2\frac{\partial^2}{\partial\theta_1\partial\theta_2}\right),\\
\sigma_2\left(\theta,\frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)&=-\lambda^{-1}\left(\theta_1\theta_2-\lambda^2\frac{\partial^2}{\partial\theta_1\partial\theta_2}\right),\\
\sigma_3\left(\theta,\frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)&=1-\theta_1\frac{\partial}{\partial\theta_1}-\theta_2\frac{\partial}{\partial\theta_2}.
\end{aligned}$$

Here, the symbol λ is an arbitrary parameter in $\boldsymbol{C}^\times=\boldsymbol{C}-\{0\}$.

REMARK. It is easily checked that only when $|\lambda|=1$, $\{\flat\sigma_j(\theta,-i\lambda\partial_\theta)\sharp\}$ are unitary matrices.

(3) Therefore, we may define a differential operator given by

(1.9)
$$\begin{aligned}
\mathscr{H}\left(\frac{\hbar}{i}\frac{\partial}{\partial x},\theta,\frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)&=c\lambda^{-1}\hbar\left(\theta_1\theta_2+\lambda^2\frac{\partial^2}{\partial\theta_1\partial\theta_2}\right)\frac{\partial}{\partial x_1}\\
&+ic\lambda^{-1}\hbar\left(\theta_1\theta_2-\lambda^2\frac{\partial^2}{\partial\theta_1\partial\theta_2}\right)\frac{\partial}{\partial x_2}-ic\hbar\left(1-\theta_1\frac{\partial}{\partial\theta_1}-\theta_2\frac{\partial}{\partial\theta_2}\right)\frac{\partial}{\partial x_3}
\end{aligned}$$

which corresponds to $\boldsymbol{H}$, and we have the superspace version of the Weyl equation

$$(1.10)\qquad \begin{cases} i\hbar\dfrac{\partial}{\partial t}u(t,x,\theta)=\mathscr{H}\left(\dfrac{\hbar}{i}\dfrac{\partial}{\partial x},\theta,\dfrac{\lambda}{i}\dfrac{\partial}{\partial\theta}\right)u(t,x,\theta)\,, \\ u(0,x,\theta)=u(x,\theta)\,. \end{cases}$$

Moreover, the "complete Weyl symbol" of (1.9) (see Appendix B) is given by

$$(1.11)\quad \begin{aligned}\mathscr{H}(\xi,\theta,\pi)&=ic\lambda^{-1}(\theta_1\theta_2-\lambda^2\hbar^{-2}\pi_1\pi_2)\xi_1-c\lambda^{-1}(\theta_1\theta_2+\lambda^2\hbar^{-2}\pi_1\pi_2)\xi_2\\&\quad-ic\hbar^{-1}(\theta_1\pi_1+\theta_2\pi_2)\xi_3\\&=ic\lambda^{-1}(\xi_1+i\xi_2)\theta_1\theta_2-ic\lambda\hbar^{-2}(\xi_1-i\xi_2)\pi_1\pi_2-ic\hbar^{-1}\xi_3(\theta_1\pi_1+\theta_2\pi_2)\,.\end{aligned}$$

Here, $\hbar$ is an arbitrary parameter in $\boldsymbol{R}^\times$ or in $i\boldsymbol{R}^\times$ ($\boldsymbol{R}^\times=\boldsymbol{R}-\{0\}$), related to the Fourier transformation with respect to odd variables.

(4) We consider the classical mechanics corresponding to $\mathscr{H}(\xi,\theta,\pi)$ given by

$$(1.12)\qquad \begin{cases} \dfrac{d}{dt}x_j=\dfrac{\partial\mathscr{H}(\xi,\theta,\pi)}{\partial\xi_j}\,,\quad \dfrac{d}{dt}\xi_k=-\dfrac{\partial\mathscr{H}(\xi,\theta,\pi)}{\partial x_k}=0 & \text{for}\quad j,k=1,2,3\,, \\ \dfrac{d}{dt}\theta_l=-\dfrac{\partial\mathscr{H}(\xi,\theta,\pi)}{\partial\pi_l}\,,\quad \dfrac{d}{dt}\pi_m=-\dfrac{\partial\mathscr{H}(\xi,\theta,\pi)}{\partial\theta_m} & \text{for}\quad l,m=1,2,3\,. \end{cases}$$

PROPOSITION 1.2. *There exists a unique global solution* $(x(t),\xi(t),\theta(t),\pi(t))$ *of* (1.12) *with any initial data* $(x(0),\xi(0),\theta(0),\pi(0))=(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\in\mathfrak{R}^{6|4}$.

REMARKS. (i) We also denote the above solution $x(t)$ by $x(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$, etc., if necessity occurs.

(ii) Instead of $\mathfrak{R}^{3|2}\times\mathfrak{R}^{3|2}$, we regard the space $\mathfrak{R}^{6|4}$ as the cotangent space $\mathscr{T}^*\mathfrak{R}^{3|2}$ of $\mathfrak{R}^{3|2}$.

Moreover, we have

PROPOSITION 1.3. *For any fixed* $(t,\underline{\xi},\underline{\pi})$, *the map defined by*

$$(\underline{x},\underline{\theta})\mapsto(\bar{x}=x(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\ \bar{\theta}=\theta(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}))$$

gives a supersmooth diffeomorphism $\mathfrak{R}^{3|2}\to\mathfrak{R}^{3|2}$. *Therefore, there exists the inverse map given by*

$$(\bar{x},\bar{\theta})\to(\underline{x}=y(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}),\ \underline{\theta}=\omega(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}))\,,$$

which satisfies

$$(1.13)\qquad \begin{cases} \bar{x}=x(t,y(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}),\underline{\xi},\omega(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}),\underline{\pi})\,, \\ \bar{\theta}=\theta(t,y(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}),\underline{\xi},\omega(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}),\underline{\pi})\,, \\ \underline{x}=y(t,x(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\xi},\theta(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\pi})\,, \\ \underline{\theta}=\omega(t,x(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\xi},\theta(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\pi})\,. \end{cases}$$

We put

(1.14) $$\mathscr{S}_0(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}) = \int_0^t \{\langle \dot{x}(s) \,|\, \xi(s)\rangle + \langle \dot{\theta}(s) \,|\, \pi(s)\rangle - \mathscr{H}(x(s), \xi(s), \theta(s), \pi(s))\} ds\,,$$

and

(1.15) $$\mathscr{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \langle \underline{x} \,|\, \underline{\xi}\rangle + \hbar ƛ^{-1}\langle \underline{\theta} \,|\, \underline{\pi}\rangle + \mathscr{S}_0(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi})\Big|_{\substack{\underline{x} = y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) \\ \underline{\theta} = \omega(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})}}\,.$$

Proposition 1.4. *$\mathscr{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$ can be expressed as*

(1.16)
$$\mathscr{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \langle \bar{x} \,|\, \underline{\xi}\rangle + [|\underline{\xi}|\cos(cƛ^{-1}t|\underline{\xi}|) - i\underline{\xi}_3 \sin(cƛ^{-1}t|\underline{\xi}|)]^{-1}[\hbar ƛ^{-1}|\underline{\xi}|\langle \bar{\theta} \,|\, \underline{\pi}\rangle$$
$$- i\lambda^{-1}ƛ \sin(cƛ^{-1}t|\underline{\xi}|)(\underline{\xi}_1 + i\underline{\xi}_2)\bar{\theta}_1\bar{\theta}_2 + i\lambda ƛ^{-1}(2\hbar ƛ^{-1} - 1)\sin(cƛ^{-1}t|\underline{\xi}|)(\underline{\xi}_1 - i\underline{\xi}_2)\underline{\pi}_1\underline{\pi}_2]\,.$$

Moreover, if $\hbar = ƛ$, it satisfies the following Hamilton-Jacobi equation:

(1.17) $$\begin{cases} \dfrac{\partial}{\partial t}\mathscr{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) + \mathscr{H}\left(\dfrac{\partial \mathscr{S}}{\partial \bar{x}}, \bar{\theta}, \dfrac{\partial \mathscr{S}}{\partial \bar{\theta}}\right) = 0\,, \\ \mathscr{S}(0, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \langle \bar{x} \,|\, \underline{\xi}\rangle + \langle \bar{\theta} \,|\, \underline{\pi}\rangle\,. \end{cases}$$

Remark. For the meaning of $|\underline{\xi}|$, $|\underline{\xi}|^{-1}$, $\sin|\underline{\xi}|$ and $\cos|\underline{\xi}|$, see Appendix A.

Now, we put

(1.18) $$\mathscr{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \operatorname{sdet}\begin{pmatrix} \dfrac{\partial^2 \mathscr{S}}{\partial \bar{x}\partial \underline{\xi}} & \dfrac{\partial^2 \mathscr{S}}{\partial \bar{x}\partial \underline{\pi}} \\ \dfrac{\partial^2 \mathscr{S}}{\partial \bar{\theta}\partial \underline{\xi}} & \dfrac{\partial^2 \mathscr{S}}{\partial \bar{\theta}\partial \underline{\pi}} \end{pmatrix},$$

where "sdet" stands for the super-determinant, see [15]. Then, we get

Proposition 1.5.

$$\mathscr{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = (\hbar^{-1}ƛ)^2|\underline{\xi}|^{-2}[|\underline{\xi}|\cos(cƛ^{-1}t|\underline{\xi}|) - i\underline{\xi}_3 \sin(cƛ^{-1}t|\underline{\xi}|)]^2\,.$$

If $\hbar = ƛ$, then it satisfies the following continuity equation:

(1.19) $$\begin{cases} \dfrac{\partial}{\partial t}\mathscr{D} + \dfrac{\partial}{\partial \bar{x}}\left(\mathscr{D}\dfrac{\partial \mathscr{H}}{\partial \xi}\right) + \dfrac{\partial}{\partial \bar{\theta}}\left(\mathscr{D}\dfrac{\partial \mathscr{H}}{\partial \pi}\right) = 0\,, \\ \mathscr{D}(0, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = 1\,. \end{cases}$$

In the above, the argument of $\mathscr{D}$ is $(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$, those of $\partial\mathscr{H}/\partial\xi$ and $\partial\mathscr{H}/\partial\pi$ are $(\mathscr{S}_{\bar{x}}, \bar{\theta}, \mathscr{S}_{\bar{\theta}})$, respectively.

From here on, we change the order of variables $(\bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$ to $(\bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})$.

We define an operator

$$(1.20)\quad (\mathscr{U}(t)\underline{u})(\bar{x},\bar{\theta})=(2\pi\hbar)^{-3/2}\bar{k}\iint d\underline{\xi}d\underline{\pi}\mathscr{D}^{1/2}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})e^{i\hbar^{-1}\mathscr{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})}\mathscr{F}\underline{u}(\underline{\xi},\underline{\pi}),$$

where $\mathscr{F}$ stands for the Fourier transformation defined for functions on the superspace. The function $u(t,\bar{x},\bar{\theta})=(\mathscr{U}(t)\underline{u})(\bar{x},\bar{\theta})$ will be shown as a desired solution for (1.10) if $\hbar=\bar{k}$.

(5) On the other hand, using the Fourier transformation, we have readily that

$$(1.21)\quad \mathscr{H}\left(\frac{\hbar}{i}\frac{\partial}{\partial x},\theta,\frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)=\hat{\mathscr{H}}$$

where $\hat{\mathscr{H}}$ is a (Weyl type) pseudo-differential operator with symbol $\mathscr{H}(\xi,\theta,\pi)$ defined by

(1.22)

$$(\hat{\mathscr{H}}u)(x,\theta)=(2\pi\hbar)^{-3}\bar{k}^2\iint d\xi d\pi dy d\omega e^{i\hbar^{-1}\langle x-y|\xi\rangle+i\bar{k}^{-1}\langle\theta-\omega|\pi\rangle}\mathscr{H}\left(\xi,\frac{\theta+\omega}{2},\pi\right)u(y,\omega).$$

THEOREM 1.6. (1) *For* $t\in\boldsymbol{R}$, $\mathscr{U}(t)$ *is a well defined unitary operator in* $\mathscr{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2})$ *if* $\hbar=\bar{k}$ *and* $|\lambda|=1$.

(2) (i) $\boldsymbol{R}\ni t\mapsto\mathscr{U}(t)\in\boldsymbol{B}(\mathscr{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2}),\mathscr{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2}))$ *is continuous.*

(ii) $\mathscr{U}(t)\mathscr{U}(s)=\mathscr{U}(t+s)$ *for any* $t, s\in\boldsymbol{R}$.

(iii) *Put* $\lambda=i$. *For* $\underline{u}\in\mathscr{C}_{\mathrm{SS,ev,0}}(\mathfrak{R}^{3|2})$, *we put* $u(t,\bar{x},\bar{\theta})=(\mathscr{U}(t)\underline{u})(\bar{x},\bar{\theta})$. *Then, it satisfies*

$$(1.23)\quad \begin{cases} i\hbar\dfrac{\partial}{\partial t}u(t,\bar{x},\bar{\theta})=\hat{\mathscr{H}}u(t,\bar{x},\bar{\theta}),\\ u(0,\bar{x},\bar{\theta})=\underline{u}(\bar{x},\bar{\theta}). \end{cases}$$

(6) We interpret the above theorem with $\hbar=\bar{k}$ and $|\lambda|=1$ using the identification maps

$$(1.24)\quad \sharp: L^2(\boldsymbol{R}^3:\boldsymbol{C}^2)\to\mathscr{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2})\quad\text{and}\quad \flat:\mathscr{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2})\to L^2(\boldsymbol{R}^3:\boldsymbol{C}^2).$$

That is, remarking $\flat\hat{\mathscr{H}}\sharp\psi=\boldsymbol{H}\psi$ and putting $\boldsymbol{U}(t)\psi=\flat\mathscr{U}(t)\sharp\psi$, we have

THEOREM 1.7. (1) *For* $t\in\boldsymbol{R}$, $\boldsymbol{U}(t)$ *is a well defined unitary operator in* $L^2(\boldsymbol{R}^3:\boldsymbol{C}^2)$.

(2) (i) $\boldsymbol{R}\ni t\mapsto\boldsymbol{U}(t)\in\boldsymbol{B}(L^2(\boldsymbol{R}^3:\boldsymbol{C}^2),L^2(\boldsymbol{R}^3:\boldsymbol{C}^2))$ *is continuous.*

(ii) $\boldsymbol{U}(t)\boldsymbol{U}(s)=\boldsymbol{U}(t+s)$ *for any* $t, s\in\boldsymbol{R}$.

(iii) *Put* $\lambda=i$. *For* $\underline{\psi}\in C_0^\infty(\boldsymbol{R}^3:\boldsymbol{C}^2)$, *we put* $\psi(t,q)=\flat(\mathscr{U}(t)\sharp\underline{\psi})|_{\bar{x}_{\mathrm{B}}=q}$. *Then, it satisfies*

$$(1.25)\quad \begin{cases} i\hbar\dfrac{\partial}{\partial t}\psi(t,q)=\boldsymbol{H}\psi(t,q),\\ \psi(0,q)=\underline{\psi}(q). \end{cases}$$

COROLLARY 1.8. *$\boldsymbol{H}$ is an essentially self-adjoint operator in $L^2(\boldsymbol{R}^3 : \boldsymbol{C}^2)$.*

REMARK. The main result in this paper is announced in Inoue [11].

2. Proofs of Propositions 1.2–1.5.

2.1. Formulation of the classical mechanics. On the superspace $\mathfrak{R}^{3|2}$, we introduce an even supersmooth function

$$u(x, \theta) = u_0(x) + u_1(x)\theta_1\theta_2 . \tag{2.1}$$

Here, for $u_j(q) \in C^\infty(\boldsymbol{R}^3 : \boldsymbol{C})$, we define its Grassmann continuation as in [8], [14]:

$$u_j(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_q^\alpha u_j(x_B) x_S^\alpha \qquad \text{where} \quad x = x_B + x_S \in \mathfrak{R}^{3|0} , \quad x_B = x^{[0]} = q \in \boldsymbol{R}^3 . \tag{2.2}$$

REMARK. Supersmooth functions should satisfy a Cauchy-Riemann like equation as explained in [14].

For $\psi(q) = {}^t(\psi_1(q), \psi_2(q))$ and $u(x, \theta) = u_0(x) + u_1(x)\theta_1\theta_2$, we make a correspondence defined by

$$\begin{cases} u(x, \theta) = (\sharp\psi)(x, \theta) & \text{where } u_{j-1}(x) = \sum_{|\alpha|=0}^{\infty} (1/\alpha!) \partial_q^\alpha \psi_j(x_B) x_S^\alpha \text{ for } j = 1, 2 , \\ \psi(q) = (b u)(q) & \text{where } \psi_{k+1}(q) = u_k(q) \text{ for } k = 0, 1 . \end{cases} \tag{2.3}$$

Or

$$\psi_1(q) = u(q, 0) , \quad \psi_2(q) = \frac{\partial^2}{\partial\theta_2 \partial\theta_1} u(q, 0) .$$

We define the operators $\{e_j\}$ as

$$\begin{aligned} e_1 &= \sqrt{\frac{i}{\lambda}} \left(\theta_1 - \frac{\lambda}{i} \frac{\partial}{\partial\theta_1} \right), \quad e_2 = \sqrt{\frac{-i}{\lambda}} \left(\theta_1 + \frac{\lambda}{i} \frac{\partial}{\partial\theta_1} \right), \\ e_3 &= \sqrt{\frac{i}{\lambda}} \left(\theta_2 - \frac{\lambda}{i} \frac{\partial}{\partial\theta_2} \right), \quad e_4 = \sqrt{\frac{-i}{\lambda}} \left(\theta_2 + \frac{\lambda}{i} \frac{\partial}{\partial\theta_2} \right). \end{aligned} \tag{2.4}$$

Here, $\lambda \in \boldsymbol{R}^\times$ or $\in i\boldsymbol{R}^\times$, the branch of $\sqrt{\alpha}$ is taken as $|\arg\sqrt{\alpha}| < \pi/2$ for $\alpha \in \boldsymbol{C} \setminus (-\infty, 0]$ and put $\sqrt{\alpha} = i\sqrt{|\alpha|}$ for $\alpha \in (-\infty, 0]$. Then, they satisfy

$$e_j e_k + e_k e_j = -2\delta_{jk} . \tag{2.5}$$

Now, we put

$$\sigma_1\left(\theta, \frac{\lambda}{i} \frac{\partial}{\partial\theta}\right) = \frac{i}{2}(e_2 e_3 + e_1 e_4) = i\lambda^{-1}\left(\theta_1\theta_2 + \lambda^2 \frac{\partial^2}{\partial\theta_1 \partial\theta_2}\right),$$

(2.6)
$$\sigma_2\left(\theta, \frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)=\frac{i}{2}(e_2e_4+e_1e_3)=-\lambda^{-1}\left(\theta_1\theta_2-\lambda^2\frac{\partial^2}{\partial\theta_1\partial\theta_2}\right),$$
$$\sigma_3\left(\theta, \frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)=-\frac{i}{2}(e_3e_4+e_1e_2)=1-\theta_1\frac{\partial}{\partial\theta_1}-\theta_2\frac{\partial}{\partial\theta_2}.$$

Then, it satisfies the relations (1.2) and (1.3). When $\lambda=i$, we have

(2.7)
$$\mathfrak{b}\sigma_1\left(\theta, \frac{\partial}{\partial\theta}\right)\sharp\psi=\boldsymbol{\sigma}_1\psi,$$
$$\mathfrak{b}\sigma_2\left(\theta, \frac{\partial}{\partial\theta}\right)\sharp\psi=\boldsymbol{\sigma}_2\psi,$$
$$\mathfrak{b}\sigma_3\left(\theta, \frac{\partial}{\partial\theta}\right)\sharp\psi=\boldsymbol{\sigma}_3\psi.$$

Therefore, we get the superspace representation of the Weyl equation given by (1.10).

As is explained in Appendix B, we obtain "Weyl symbol" for operators $\sigma_j(\theta, -i\lambda\partial_\theta)$ as follows:

(2.8)
$$\sigma_1(\theta, \pi)=i\lambda^{-1}(\theta_1\theta_2-\lambda^2\hbar^{-2}\pi_1\pi_2),$$
$$\sigma_2(\theta, \pi)=-\lambda^{-1}(\theta_1\theta_2+\lambda^2\hbar^{-2}\pi_1\pi_2),$$
$$\sigma_3(\theta, \pi)=-i\hbar^{-1}(\theta_1\pi_1+\theta_2\pi_2).$$

Therefore, we have the complete Weyl symbol of (1.9) which gives (1.11).

Now, following [5], [6], we introduce the graded Poisson bracket $\{\{,\}\}$ as

$$\{\{E_1, E_2\}\}=\sum_{j=1}^m\left(\frac{\partial E_1}{\partial x_j}\frac{\partial E_2}{\partial \xi_j}-\frac{\partial E_2}{\partial x_j}\frac{\partial E_1}{\partial \xi_j}\right)+\sum_{k=1}^n\left(\frac{\partial E_1}{\partial \theta_k}\frac{\partial E_2}{\partial \pi_k}-\frac{\partial E_2}{\partial \theta_k}\frac{\partial E_1}{\partial \pi_k}\right),$$

(2.9)
$$\{\{E, O\}\}=-\{\{O, E\}\}=\sum_{j=1}^m\left(\frac{\partial E}{\partial x_j}\frac{\partial O}{\partial \xi_j}-\frac{\partial O}{\partial x_j}\frac{\partial E}{\partial \xi_j}\right)+\sum_{k=1}^n\left(\frac{\partial E}{\partial \theta_k}\frac{\partial O}{\partial \pi_k}+\frac{\partial O}{\partial \theta_k}\frac{\partial E}{\partial \pi_k}\right),$$

$$\{\{O_1, O_2\}\}=\sum_{j=1}^m\left(\frac{\partial O_1}{\partial x_j}\frac{\partial O_2}{\partial \xi_j}+\frac{\partial O_2}{\partial x_j}\frac{\partial O_1}{\partial \xi_j}\right)-\sum_{k=1}^n\left(\frac{\partial O_1}{\partial \theta_k}\frac{\partial O_2}{\partial \pi_k}+\frac{\partial O_2}{\partial \theta_k}\frac{\partial O_1}{\partial \pi_k}\right).$$

Here, E, E_1, $E_2\in\mathscr{C}_{\mathrm{SS,ev}}(\mathscr{T}^*\mathfrak{R}^{m|n})$ and O, O_1, $O_2\in\mathscr{C}_{\mathrm{SS,od}}(\mathscr{T}^*\mathfrak{R}^{m|n})$. Then, the classical mechanics governed by $H\in\mathscr{C}_{\mathrm{SS,ev}}(\mathscr{T}^*\mathfrak{R}^{m|n})$ is given by

(2.10)
$$\begin{cases}\dfrac{d}{dt}\varphi(t)=\{\{\varphi(t), H\}\}, \\ \varphi(0)=\varphi(X, \Xi)\end{cases}\qquad \text{for}\quad \varphi\in\mathscr{C}_{\mathrm{SS}}(\mathscr{T}^*\mathfrak{R}^{m|n}),\quad \varphi(t)=\varphi(X(t), \Xi(t)).$$

Hamilton flows: In our case, we have (1.12). More precisely,

$$\frac{d}{dt}x_1 = ic\lambda^{-1}(\theta_1\theta_2 - \lambda^2\hbar^{-2}\pi_1\pi_2) = c\sigma_1(\theta, \pi),$$

$$\frac{d}{dt}x_2 = -c\lambda^{-1}(\theta_1\theta_2 + \lambda^2\hbar^{-2}\pi_1\pi_2) = c\sigma_2(\theta, \pi),$$

$$\frac{d}{dt}x_3 = -ic\hbar^{-1}(\theta_1\pi_1 + \theta_2\pi_2) = c\sigma_3(\theta, \pi),$$

$$\frac{d}{dt}\xi_j = 0 \qquad \text{for} \quad j = 1, 2, 3,$$

$$\text{(2.11)} \qquad \frac{d}{dt}\theta_1 = ic\lambda\hbar^{-2}(\xi_1 - i\xi_2)\pi_2 - ic\hbar^{-1}\xi_3\theta_1,$$

$$\frac{d}{dt}\theta_2 = -ic\lambda\hbar^{-2}(\xi_1 - i\xi_2)\pi_1 - ic\hbar^{-1}\xi_3\theta_2,$$

$$\frac{d}{dt}\pi_1 = -ic\lambda^{-1}(\xi_1 + i\xi_2)\theta_2 + ic\hbar^{-1}\xi_3\pi_1,$$

$$\frac{d}{dt}\pi_2 = ic\lambda^{-1}(\xi_1 + i\xi_2)\theta_1 + ic\hbar^{-1}\xi_3\pi_2.$$

2.2. Proof of Proposition 1.2. Rewriting the above, we have

$$\text{(2.12)} \qquad \frac{d}{dt}\begin{pmatrix}\theta_1\\ \theta_2\\ \pi_1\\ \pi_2\end{pmatrix} = icX\begin{pmatrix}\theta_1\\ \theta_2\\ \pi_1\\ \pi_2\end{pmatrix} \quad \text{with} \quad \begin{pmatrix}\theta_1(0)\\ \theta_2(0)\\ \pi_1(0)\\ \pi_2(0)\end{pmatrix} = \begin{pmatrix}\underline{\theta}_1\\ \underline{\theta}_2\\ \underline{\pi}_1\\ \underline{\pi}_2\end{pmatrix},$$

where

$$X = \begin{pmatrix} -\hbar^{-1}\xi_3 & 0 & 0 & \lambda\hbar^{-2}(\xi_1 - i\xi_2) \\ 0 & -\hbar^{-1}\xi_3 & -\lambda\hbar^{-2}(\xi_1 - i\xi_2) & 0 \\ 0 & -\lambda^{-1}(\xi_1 + i\xi_2) & \hbar^{-1}\xi_3 & 0 \\ \lambda^{-1}(\xi_1 + i\xi_2) & 0 & 0 & \hbar^{-1}\xi_3 \end{pmatrix}.$$

As $X^2 = \hbar^{-2}|\xi|^2 I_4$, we get

$$\text{(2.13)} \qquad e^{ictX} = \cos(c\hbar^{-1}t|\xi|)I_4 + i\hbar|\xi|^{-1}\sin(c\hbar^{-1}t|\xi|)X.$$

For notational simplicity, we put $\eta = \underline{\xi}_1 + i\underline{\xi}_2$, $\bar{\eta} = \underline{\xi}_1 - i\underline{\xi}_2$. Remarking $\xi_j(t) = \underline{\xi}_j$, we have

(2.14)

$$\theta_1(s)=\cos(c\hbar^{-1}s|\underset{\sim}{\xi}|)\underset{\sim}{\theta}_1+i|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)[-\underset{\sim}{\xi}_3\underset{\sim}{\theta}_1+\lambda\hbar^{-1}\bar{\eta}\underset{\sim}{\pi}_2]$$
$$=[\cos(c\hbar^{-1}s|\underset{\sim}{\xi}|)-i\underset{\sim}{\xi}_3|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)]\underset{\sim}{\theta}_1+i\lambda\hbar^{-1}\bar{\eta}|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)\underset{\sim}{\pi}_2,$$

$$\theta_2(s)=\cos(c\hbar^{-1}s|\underset{\sim}{\xi}|)\underset{\sim}{\theta}_2-i|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)[\underset{\sim}{\xi}_3\underset{\sim}{\theta}_2+\lambda\hbar^{-1}\bar{\eta}\underset{\sim}{\pi}_1]$$
$$=[\cos(c\hbar^{-1}s|\underset{\sim}{\xi}|)-i\underset{\sim}{\xi}_3|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)]\underset{\sim}{\theta}_2-i\lambda\hbar^{-1}\bar{\eta}|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)\underset{\sim}{\pi}_1,$$

$$\pi_1(s)=\cos(c\hbar^{-1}s|\underset{\sim}{\xi}|)\underset{\sim}{\pi}_1+i|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)[-\lambda^{-1}\hbar\eta\underset{\sim}{\theta}_2+\underset{\sim}{\xi}_3\underset{\sim}{\pi}_1]$$
$$=[\cos(c\hbar^{-1}s|\underset{\sim}{\xi}|)+i\underset{\sim}{\xi}_3|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)]\underset{\sim}{\pi}_1-i\lambda^{-1}\hbar\eta|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)\underset{\sim}{\theta}_2,$$

$$\pi_2(s)=\cos(c\hbar^{-1}s|\underset{\sim}{\xi}|)\underset{\sim}{\pi}_2+i|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)[\lambda^{-1}\hbar\eta\underset{\sim}{\theta}_1+\underset{\sim}{\xi}_3\underset{\sim}{\pi}_2]$$
$$=[\cos(c\hbar^{-1}s|\underset{\sim}{\xi}|)+i\underset{\sim}{\xi}_3|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)]\underset{\sim}{\pi}_2+i\lambda^{-1}\hbar\eta|\underset{\sim}{\xi}|^{-1}\sin(c\hbar^{-1}s|\underset{\sim}{\xi}|)\underset{\sim}{\theta}_1.$$

On the other hand, putting

$$\begin{aligned}\sigma_1(t)&=i\lambda^{-1}(\theta_1(t)\theta_2(t)-\lambda^2\hbar^{-2}\pi_1(t)\pi_2(t)),\\ \sigma_2(t)&=-\lambda^{-1}(\theta_1(t)\theta_2(t)+\lambda^2\hbar^{-2}\pi_1(t)\pi_2(t)),\\ \sigma_3(t)&=-i\hbar^{-1}(\theta_1(t)\pi_1(t)+\theta_2(t)\pi_2(t)),\end{aligned}\tag{2.15}$$

and differentiating with respect to t, we get easily

$$\frac{d}{dt}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}=2c\hbar^{-1}\boldsymbol{Y}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}\quad\text{where}\quad \boldsymbol{Y}=\begin{pmatrix}0&-\xi_3&\xi_2\\ \xi_3&0&-\xi_1\\ -\xi_2&\xi_1&0\end{pmatrix}.\tag{2.16}$$

As

$$\boldsymbol{Y}^2=\begin{pmatrix}-\xi_3^2-\xi_2^2&\xi_2\xi_1&\xi_3\xi_1\\ \xi_1\xi_2&-\xi_3^2-\xi_1^2&\xi_3\xi_2\\ \xi_1\xi_3&\xi_2\xi_3&-\xi_2^2-\xi_1^2\end{pmatrix}\quad\text{and}\quad \boldsymbol{Y}^3=-|\xi|^2\boldsymbol{Y},$$

we have

$$e^{2c\hbar^{-1}t\boldsymbol{Y}}=\boldsymbol{I}_3+|\xi|^{-1}\sin(2c\hbar^{-1}t|\xi|)\boldsymbol{Y}+|\xi|^{-2}(1-\cos(2c\hbar^{-1}t|\xi|))\boldsymbol{Y}^2.\tag{2.17}$$

This implies

$$\begin{aligned}\sigma_1(s)&=\underset{\sim}{\sigma}_1+\sin(2c\hbar^{-1}s|\underset{\sim}{\xi}|)|\underset{\sim}{\xi}|^{-1}(-\underset{\sim}{\xi}_3\underset{\sim}{\sigma}_2+\underset{\sim}{\xi}_2\underset{\sim}{\sigma}_3)\\ &\quad+(1-\cos(2c\hbar^{-1}s|\underset{\sim}{\xi}|))|\underset{\sim}{\xi}|^{-2}[-(\underset{\sim}{\xi}_2^2+\underset{\sim}{\xi}_3^2)\underset{\sim}{\sigma}_1+\underset{\sim}{\xi}_1\underset{\sim}{\xi}_2\underset{\sim}{\sigma}_2+\underset{\sim}{\xi}_1\underset{\sim}{\xi}_3\underset{\sim}{\sigma}_3],\\ \sigma_2(s)&=\underset{\sim}{\sigma}_2+\sin(2c\hbar^{-1}s|\underset{\sim}{\xi}|)|\underset{\sim}{\xi}|^{-1}(\underset{\sim}{\xi}_3\underset{\sim}{\sigma}_1-\underset{\sim}{\xi}_1\underset{\sim}{\sigma}_3)\\ &\quad+(1-\cos(2c\hbar^{-1}s|\underset{\sim}{\xi}))|\underset{\sim}{\xi}|^{-2}[\underset{\sim}{\xi}_1\underset{\sim}{\xi}_2\underset{\sim}{\sigma}_1-(\underset{\sim}{\xi}_1^2+\underset{\sim}{\xi}_3^2)\underset{\sim}{\sigma}_2+\underset{\sim}{\xi}_2\underset{\sim}{\xi}_3\underset{\sim}{\sigma}_3],\\ \sigma_3(s)&=\underset{\sim}{\sigma}_3+\sin(2c\hbar^{-1}s|\underset{\sim}{\xi}|)|\underset{\sim}{\xi}|^{-1}(-\underset{\sim}{\xi}_2\underset{\sim}{\sigma}_1+\underset{\sim}{\xi}_1\underset{\sim}{\sigma}_2)\\ &\quad+(1-\cos(2c\hbar^{-1}s|\underset{\sim}{\xi}|))|\underset{\sim}{\xi}|^{-2}[\underset{\sim}{\xi}_1\underset{\sim}{\xi}_3\underset{\sim}{\sigma}_1+\underset{\sim}{\xi}_2\underset{\sim}{\xi}_3\underset{\sim}{\sigma}_2-(\underset{\sim}{\xi}_1^2+\underset{\sim}{\xi}_2^2)\underset{\sim}{\sigma}_3].\end{aligned}\tag{2.18}$$

Putting $\gamma=c\hbar^{-1}t|\underset{\sim}{\xi}|$,

$$\alpha=\int_0^t ds\sin(2c\hbar^{-1}s|\underline{\xi}|)=c^{-1}\hbar|\underline{\xi}|^{-1}\sin^2\gamma\,,$$

$$\beta=\int_0^t ds(1-\cos(2c\hbar^{-1}s|\underline{\xi}|))=t-c^{-1}\hbar|\underline{\xi}|^{-1}\sin\gamma\cos\gamma\,,$$

we have

(2.19)

$$\int_0^t ds\sigma_1(s)=\underline{\sigma}_1t+\alpha|\underline{\xi}|^{-1}(-\underline{\xi}_3\underline{\sigma}_2+\underline{\xi}_2\underline{\sigma}_3)+\beta|\underline{\xi}|^{-2}[-(\underline{\xi}_2^2+\underline{\xi}_3^2)\underline{\sigma}_1+\underline{\xi}_1\underline{\xi}_2\underline{\sigma}_2+\underline{\xi}_1\underline{\xi}_3\underline{\sigma}_3]\,,$$

$$\int_0^t ds\sigma_2(s)=\underline{\sigma}_2t+\alpha|\underline{\xi}|^{-1}(\underline{\xi}_3\underline{\sigma}_1-\underline{\xi}_1\underline{\sigma}_3)+\beta|\underline{\xi}|^{-2}[\underline{\xi}_1\underline{\xi}_2\underline{\sigma}_1-(\underline{\xi}_1^2+\underline{\xi}_3^2)\underline{\sigma}_2+\underline{\xi}_2\underline{\xi}_3\underline{\sigma}_3]\,,$$

$$\int_0^t ds\sigma_3(s)=\underline{\sigma}_3t+\alpha|\underline{\xi}|^{-1}(-\underline{\xi}_2\underline{\sigma}_1+\underline{\xi}_1\underline{\sigma}_2)+\beta|\underline{\xi}|^{-2}[\underline{\xi}_1\underline{\xi}_3\underline{\sigma}_1+\underline{\xi}_2\underline{\xi}_3\underline{\sigma}_2-(\underline{\xi}_1^2+\underline{\xi}_2^2)\underline{\sigma}_3]\,.$$

As $\dot{x}_j=c\sigma_j$, we have

$$x_j(t)=\underline{x}_j+c\int_0^t ds\sigma_j(s)\qquad\text{for}\quad j=1,2,3\,.$$

Therefore, we get

(2.20)

$$x_1(t)=\underline{x}_1+c\underline{\sigma}_1t+c\alpha|\underline{\xi}|^{-1}(-\underline{\xi}_3\underline{\sigma}_2+\underline{\xi}_2\underline{\sigma}_3)+c\beta|\underline{\xi}|^{-2}[-(\underline{\xi}_2^2+\underline{\xi}_3^2)\underline{\sigma}_1+\underline{\xi}_1\underline{\xi}_2\underline{\sigma}_2+\underline{\xi}_1\underline{\xi}_3\underline{\sigma}_3]\,,$$

$$x_2(t)=\underline{x}_2+c\underline{\sigma}_2t+c\alpha|\underline{\xi}|^{-1}(\underline{\xi}_3\underline{\sigma}_1-\underline{\xi}_1\underline{\sigma}_3)+c\beta|\underline{\xi}|^{-2}[\underline{\xi}_1\underline{\xi}_2\underline{\sigma}_1-(\underline{\xi}_1^2+\underline{\xi}_3^2)\underline{\sigma}_2+\underline{\xi}_2\underline{\xi}_3\underline{\sigma}_3]\,,$$

$$x_3(t)=\underline{x}_3+c\underline{\sigma}_3t+c\alpha|\underline{\xi}|^{-1}(-\underline{\xi}_2\underline{\sigma}_1+\underline{\xi}_1\underline{\sigma}_2)+c\beta|\underline{\xi}|^{-2}[\underline{\xi}_1\underline{\xi}_3\underline{\sigma}_1+\underline{\xi}_2\underline{\xi}_3\underline{\sigma}_2-(\underline{\xi}_1^2+\underline{\xi}_2^2)\underline{\sigma}_3]\,.$$

By (2.14) and (2.20), we prove not only the global existence in time of solutions of (1.11) but also their explicit forms.

2.3. Proof of Proposition 1.3. Put

$$\gamma=c\hbar^{-1}t|\underline{\xi}|\quad\text{and}\quad\delta=|\underline{\xi}|\cos\gamma-i\underline{\xi}_3\sin\gamma\,.$$

From (2.14), we get

$$\bar{\theta}_1=|\underline{\xi}|^{-1}\delta\underline{\theta}_1+i\lambda\hbar^{-1}\bar{\eta}|\underline{\xi}|^{-1}\sin\gamma\underline{\pi}_2\,,\qquad(2.21)$$
$$\bar{\theta}_2=|\underline{\xi}|^{-1}\delta\underline{\theta}_2-i\lambda\hbar^{-1}\bar{\eta}|\underline{\xi}|^{-1}\sin\gamma\underline{\pi}_1\,,$$

which yield

$$\underline{\theta}_1=|\underline{\xi}|\delta^{-1}[\bar{\theta}_1-i\lambda\hbar^{-1}\bar{\eta}|\underline{\xi}|^{-1}\sin\gamma\underline{\pi}_2]=\omega_1(t,\underline{\xi},\bar{\theta},\underline{\pi})\,,\qquad(2.22)$$
$$\underline{\theta}_2=|\underline{\xi}|\delta^{-1}[\bar{\theta}_2+i\lambda\hbar^{-1}\bar{\eta}|\underline{\xi}|^{-1}\sin\gamma\underline{\pi}_1]=\omega_2(t,\underline{\xi},\bar{\theta},\underline{\pi})\,.$$

These imply

$$(2.23)\quad \underline{\theta}_1\underline{\theta}_2=|\underline{\xi}|^2\delta^{-2}[\bar{\theta}_1\bar{\theta}_2+i\lambda\hbar^{-1}|\underline{\xi}|^{-1}\bar{\eta}\sin\gamma\langle\bar{\theta}|\underline{\pi}\rangle-\lambda^2\hbar^{-2}|\underline{\xi}|^{-2}\bar{\eta}^2\sin^2\gamma\underline{\pi}_1\underline{\pi}_2],$$

$$(2.24)\quad \langle\underline{\theta}|\underline{\pi}\rangle=|\underline{\xi}|\delta^{-1}[\langle\bar{\theta}|\underline{\pi}\rangle+2i\lambda\hbar^{-1}|\underline{\xi}|^{-1}\bar{\eta}\sin\gamma\underline{\pi}_1\underline{\pi}_2].$$

Therefore, substituting these, we have

$$(2.25)\quad \begin{aligned}\underline{\sigma}_1&=i\lambda^{-1}(\underline{\theta}_1\underline{\theta}_2-\lambda^2\hbar^{-2}\underline{\pi}_1\underline{\pi}_2)=\sigma_1(t,\underline{\xi},\bar{\theta},\underline{\pi}),\\ \underline{\sigma}_2&=-\lambda^{-1}(\underline{\theta}_1\underline{\theta}_2+\lambda^2\hbar^{-2}\underline{\pi}_1\underline{\pi}_2)=\sigma_2(t,\underline{\xi},\bar{\theta},\underline{\pi}),\\ \underline{\sigma}_3&=-i\hbar^{-1}(\underline{\theta}_1\underline{\pi}_1+\underline{\theta}_2\underline{\pi}_2)=\sigma_3(t,\underline{\xi},\bar{\theta},\underline{\pi}).\end{aligned}$$

That is, we have

$$(2.26)\quad \begin{aligned}\underline{x}_1&=\bar{x}_1-c\int_0^t ds\sigma_1(s)=y_1(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}),\\ \underline{x}_2&=\bar{x}_2-c\int_0^t ds\sigma_2(s)=y_2(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}),\\ \underline{x}_3&=\bar{x}_3-c\int_0^t ds\sigma_3(s)=y_3(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}).\end{aligned}$$

From the above (2.22), (2.26), we have proved the existence and explicit form for $\underline{x}_j=y_j(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})$ and $\underline{\theta}_j=\omega(t,\underline{\xi},\bar{\theta},\underline{\pi})$.

2.4. Proof of Proposition 1.4. By simple calculation, we have

$$(2.27)\quad \begin{aligned}\mathscr{S}_0(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})&=\int_0^t ds[\langle\dot{x}(s)|\xi(s)\rangle+\langle\dot{\theta}(s)|\pi(s)\rangle-\mathscr{H}(\xi(s),\theta(s),\pi(s))]\\ &=\int_0^t ds\langle\dot{\theta}(s)|\pi(s)\rangle=c\eta\int_0^t ds\sigma_1(s)+ic\bar{\eta}\int_0^t ds\sigma_2(s)+c\underline{\xi}_3\int_0^t ds\sigma_3(s).\end{aligned}$$

Here, we used

$$\begin{aligned}\langle\dot{\theta}|\pi\rangle&=[-ic\hbar^{-1}\xi_3\theta_1+ic\lambda\hbar^{-2}\bar{\eta}\pi_2]\pi_1-[ic\hbar^{-1}\xi_3\theta_2+ic\lambda\hbar^{-2}\bar{\eta}\pi_1]\pi_2\\ &=c\xi_j\sigma_j+ic(\xi_1\sigma_2-\xi_2\sigma_1)=c\bar{\eta}\sigma_1+ic\bar{\eta}\sigma_2+c\xi_3\sigma_3.\end{aligned}$$

Now, we define

$$(2.28)\quad \mathscr{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})=\mathscr{S}(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\big|_{\underline{x}=y(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}),\underline{\theta}=\omega(t,\underline{\xi},\bar{\theta},\underline{\pi})},$$

where

$$\mathscr{S}(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})=\langle\underline{x}|\underline{\xi}\rangle+\hbar\hbar^{-1}\langle\underline{\theta}|\underline{\pi}\rangle+\mathscr{S}_0(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}).$$

By (2.26) and (2.27), we put $\mathscr{S}(t,\bar{x},\underline{\theta},\underline{\xi},\underline{\pi})=\mathscr{S}(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\big|_{\underline{x}=y(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})}$ and

$$(2.29)\quad \mathscr{S}(t, \bar{x}, \underline{\theta}, \underline{\xi}, \underline{\pi}) = \left(\bar{x}_j - c\int_0^t ds\sigma_j(s)\right)\underline{\xi}_j + \hbar k\!\!\!^{-}{}^{-1}\langle\underline{\theta}\,|\,\underline{\pi}\rangle + c\bar{\eta}\int_0^t ds\sigma_1(s)$$
$$+ ic\bar{\eta}\int_0^t ds\sigma_2(s) + c\underline{\xi}_3\int_0^t ds\sigma_1(s)$$
$$= \langle\bar{x}\,|\,\underline{\xi}\rangle + \hbar\bar{k}^{-1}\langle\underline{\theta}\,|\,\underline{\pi}\rangle + ic\left(\underline{\xi}_1\int_0^t ds\sigma_2(s) - \underline{\xi}_2\int_0^t ds\sigma_1(s)\right).$$

Remarking

$$\underline{\xi}_1\int_0^t ds\sigma_2(s) - \underline{\xi}_2\int_0^t ds\sigma_1(s) = c^{-1}\bar{k}|\,\underline{\xi}\,|^{-1}\sin\gamma\cos\gamma(\underline{\xi}_1\underline{\sigma}_2 - \underline{\xi}_2\underline{\sigma}_1)t$$
$$+ c^{-1}\bar{k}|\,\underline{\xi}\,|^{-2}\sin^2\gamma[\underline{\xi}_3(\underline{\xi}_1\underline{\sigma}_1 + \underline{\xi}_2\underline{\sigma}_2) - |\,\eta\,|^2\underline{\sigma}_3]$$

and substituting this into (2.29), we get

$$\mathscr{S}(t, \bar{x}, \underline{\theta}, \underline{\xi}, \underline{\pi}) = \langle\bar{x}\,|\,\underline{\xi}\rangle + \hbar\bar{k}^{-1}\langle\underline{\theta}\,|\,\underline{\pi}\rangle + i\bar{k}|\,\underline{\xi}\,|^{-2}\sin^2\gamma(-|\,\underline{\xi}\,|^2\underline{\sigma}_3 + \underline{\xi}_3\underline{\xi}_j\underline{\sigma}_j)$$
$$- i\bar{k}|\,\underline{\xi}\,|^{-1}\sin\gamma\cos\gamma(\underline{\xi}_2\underline{\sigma}_1 - \underline{\xi}_1\underline{\sigma}_2)\,.$$

On the other hand, as

$$\underline{\xi}_1\underline{\sigma}_2 - \underline{\xi}_2\underline{\sigma}_1 = -\lambda^{-1}\eta\underline{\theta}_1\underline{\theta}_2 - \lambda\bar{k}^{-2}\bar{\eta}\underline{\pi}_1\underline{\pi}_2\,,$$
$$\underline{\xi}_3(\underline{\xi}_1\underline{\sigma}_1 + \underline{\xi}_2\underline{\sigma}_2) - |\,\eta\,|^2\underline{\sigma}_3 = i\bar{k}^{-1}|\,\eta\,|^2\langle\underline{\theta}\,|\,\underline{\pi}\rangle + i\lambda^{-1}\underline{\xi}_3\eta\underline{\theta}_1\underline{\theta}_2 - i\lambda\bar{k}^{-2}\underline{\xi}_3\bar{\eta}\underline{\pi}_1\underline{\pi}_2\,,$$

we get

$$\mathscr{S}(t, \bar{x}, \underline{\theta}, \underline{\xi}, \underline{\pi}) = \langle\bar{x}\,|\,\underline{\xi}\rangle + (\hbar\bar{k}^{-1} - |\,\underline{\xi}\,|^{-2}|\,\eta\,|^2\sin^2\gamma)\langle\underline{\theta}\,|\,\underline{\pi}\rangle$$
$$- i|\,\underline{\xi}\,|^{-2}\sin\gamma[\lambda^{-1}\bar{k}\eta(|\,\underline{\xi}\,|\cos\gamma - i\underline{\xi}_3\sin\gamma)\underline{\theta}_1\underline{\theta}_2 + \lambda\bar{k}^{-1}\bar{\eta}(|\,\underline{\xi}\,|\cos\gamma + i\underline{\xi}_3\sin\gamma)\underline{\pi}_1\underline{\pi}_2]\,.$$

Using (2.22) and putting $\mathscr{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \mathscr{S}(t, \bar{x}, \underline{\theta}, \underline{\xi}, \underline{\pi})\,|_{\underline{\theta} = \omega(t, \underline{\xi}, \bar{\theta}, \underline{\pi})}$, we have

$$(2.30)\quad \mathscr{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$$
$$= \langle\bar{x}\,|\,\underline{\xi}\rangle + \delta^{-1}[\hbar\bar{k}^{-1}|\,\underline{\xi}\,|\langle\bar{\theta}\,|\,\underline{\pi}\rangle - i\lambda^{-1}\bar{k}\eta\sin\gamma\bar{\theta}_1\bar{\theta}_2 + i\lambda\bar{k}^{-1}(2\hbar\bar{k}^{-1} - 1)\bar{\eta}\sin\gamma\underline{\pi}_1\underline{\pi}_2]\,.$$

It is easily checked that if $\hbar = \bar{k}$, then $\mathscr{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$ satisfies the Hamilton-Jacobi equation (1.17). Indeed, as

$$\mathscr{S}_{\bar{x}_j} = \underline{\xi}_j \qquad \text{for} \quad j = 1, 2, 3\,,$$
$$\mathscr{S}_{\bar{\theta}_1} = \delta^{-1}[\hbar\bar{k}^{-1}|\,\underline{\xi}\,|\underline{\pi}_1 - i\lambda^{-1}\bar{k}\eta\sin\gamma\bar{\theta}_2]\,,\quad \mathscr{S}_{\bar{\theta}_2} = \delta^{-1}[\hbar\bar{k}^{-1}|\,\underline{\xi}\,|\underline{\pi}_2 + i\lambda^{-1}\bar{k}\eta\sin\gamma\bar{\theta}_1]\,,$$

we get

$$\mathscr{S}_{\bar{\theta}_1}\mathscr{S}_{\bar{\theta}_2} = \delta^{-2}[(\hbar\bar{k}^{-1}|\,\underline{\xi}\,|)^2\underline{\pi}_1\underline{\pi}_2 - i\hbar\lambda^{-1}|\,\underline{\xi}\,|\eta\sin\gamma\langle\bar{\theta}\,|\,\underline{\pi}\rangle - (\lambda^{-1}\bar{k}\eta\sin\gamma)^2\bar{\theta}_1\bar{\theta}_2]\,,$$
$$\bar{\theta}_1\mathscr{S}_{\bar{\theta}_1} + \bar{\theta}_2\mathscr{S}_{\bar{\theta}_2} = \delta^{-1}[\hbar\bar{k}^{-1}|\,\underline{\xi}\,|\langle\bar{\theta}\,|\,\underline{\pi}\rangle - 2i\lambda^{-1}\bar{k}\eta\sin\gamma\bar{\theta}_1\bar{\theta}_2]\,.$$

Substituting these into $\mathscr{H}(\mathscr{S}_{\bar{x}}, \bar{\theta}, \mathscr{S}_{\bar{\theta}})$, we have

$$(2.31)\quad \mathscr{H}(\mathscr{S}_{\bar{x}}, \bar{\theta}, \mathscr{S}_{\bar{\theta}}) = ic\lambda^{-1}\eta\bar{\theta}_1\bar{\theta}_2 - ic\lambda\bar{k}^{-2}\bar{\eta}\mathscr{S}_{\bar{\theta}_1}\mathscr{S}_{\bar{\theta}_2} - ic\bar{k}^{-1}\xi_3(\bar{\theta}_1\mathscr{S}_{\bar{\theta}_1} + \bar{\theta}_2\mathscr{S}_{\bar{\theta}_2})$$
$$= c|\underline{\xi}|^2\delta^{-2}[i\lambda^{-1}\eta\bar{\theta}_1\bar{\theta}_2 - \hbar\bar{k}^{-2}(|\underline{\xi}|\sin\gamma + i\xi_3\cos\gamma)\langle\bar{\theta}|\underline{\pi}\rangle - i\hbar^2\lambda\bar{k}^{-4}\bar{\eta}\underline{\pi}_1\underline{\pi}_2].$$

On the other hand,

(2.32)
$$\mathscr{S}_t = \hbar\bar{k}^{-1}|\underline{\xi}|\partial_t(\delta^{-1})\langle\bar{\theta}|\underline{\pi}\rangle + [-i\lambda^{-1}\bar{k}\eta\bar{\theta}_1\theta_2 + i\lambda\bar{k}^{-1}(2\hbar\bar{k}^{-1} - 1)\bar{\eta}\underline{\pi}_1\underline{\pi}_2]\partial_t(\delta^{-1}\sin\gamma).$$

As we get easily

$$\delta_t(\delta^{-1}) = c\bar{k}^{-1}|\underline{\xi}|\delta^{-2}(|\underline{\xi}|\sin\gamma + i\xi_3\cos\gamma) \quad \text{and} \quad \delta_t(\delta^{-1}\sin\gamma) = c\bar{k}^{-1}|\underline{\xi}|^2\delta^{-2},$$

we have

$$(2.33)\quad \mathscr{S}_t = c\hbar\bar{k}^{-2}|\underline{\xi}|^2\delta^{-2}(|\underline{\xi}|\sin\gamma + i\xi_3\cos\gamma)\langle\bar{\theta}|\underline{\pi}\rangle$$
$$+ c\bar{k}^{-1}|\underline{\xi}|^2\delta^{-2}[-i\lambda^{-1}\bar{k}\eta\bar{\theta}_1\bar{\theta}_2 + i\lambda\bar{k}^{-1}(2\hbar\bar{k}^{-1} - 1)\bar{\eta}\underline{\pi}_1\underline{\pi}_2].$$

Comparing (2.31) and (2.33), we proved that the Hamilton-Jacobi equation is satisfied under the condition $(\hbar\bar{k}^{-1})^2 - 2\hbar\bar{k}^{-1} + 1 = 0$.

2.5. Proof of proposition 1.5. From (2.30), we get easily

$$(2.34)\quad \mathscr{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \operatorname{sdet}\begin{pmatrix} \dfrac{\partial^2\mathscr{S}}{\partial\bar{x}\partial\underline{\xi}} & \dfrac{\partial^2\mathscr{S}}{\partial\bar{x}\partial\underline{\pi}} \\ \dfrac{\partial^2\mathscr{S}}{\partial\bar{\theta}\partial\underline{\xi}} & \dfrac{\partial^2\mathscr{S}}{\partial\bar{\theta}\partial\underline{\pi}} \end{pmatrix}$$
$$= (\hbar\bar{k}^{-1}|\underline{\xi}|)^{-2}[|\underline{\xi}|\cos(c\bar{k}^{-1}t|\underline{\xi}|) - i\xi_3\sin(c\bar{k}^{-1}t|\underline{\xi}|)]^2,$$

and it is checked easily that it satisfies the continuity equation (1.19).

For future use, we derive the continuity equation from the Hamilton-Jacobi equation in a more general situation, that is, without resorting to the concrete expression of $\mathscr{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$.

PROPOSITION 2.1. *Let $\mathscr{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$ satisfy the Hamilton-Jacobi equation below where $(\bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) \in \mathscr{T}^*\mathfrak{R}^{m|n} = \mathfrak{R}^{2m|2n}$.*

$$(2.35)\quad \mathscr{S}_t(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) + \mathscr{H}(\bar{x}, \mathscr{S}_{\bar{x}}, \bar{\theta}, \mathscr{S}_{\bar{\theta}}) = 0.$$

Putting

$$\mathscr{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \operatorname{sdet}\begin{pmatrix} \dfrac{\partial^2\mathscr{S}}{\partial\bar{x}\partial\underline{\xi}} & \dfrac{\partial^2\mathscr{S}}{\partial\bar{x}\partial\underline{\pi}} \\ \dfrac{\partial^2\mathscr{S}}{\partial\bar{\theta}\partial\underline{\xi}} & \dfrac{\partial^2\mathscr{S}}{\partial\bar{\theta}\partial\underline{\pi}} \end{pmatrix},$$

we have

(2.36) $$\mathcal{D}_t+\partial_{\bar{x}}(\mathcal{D}\mathcal{H}_{\xi})+\partial_{\bar{\theta}}(\mathcal{D}\mathcal{H}_{\pi})=0\,,$$

where $\mathcal{H}_*=\mathcal{H}_*(\bar{x},\mathcal{S}_{\bar{x}},\bar{\theta},\mathcal{S}_{\bar{\theta}})$.

PROOF. Let $(\bar{X}_B)=(\bar{x}_j,\bar{\theta}_k)$ and $(\Xi_A)=(\xi_j,\pi_k)$. By differentiating the Hamilton-Jacobi equation with respect to Ξ_A, we get

$$\frac{\partial\mathcal{S}_t}{\partial\Xi_A}+\frac{\partial\mathcal{S}_{\bar{X}_C}}{\partial\Xi_A}\mathcal{H}_{\Xi_C}=0\qquad\text{for}\quad A,C=1,\dots,m,m+1,\dots,m+n\,.$$

Differentiating this once more with respect to $\bar{X}_B$, we have

$$\frac{\partial^2\mathcal{S}_t}{\partial\bar{X}_B\partial\Xi_A}+\frac{\partial^3\mathcal{S}}{\partial\bar{X}_B\partial\Xi_A\partial\bar{X}_C}\mathcal{H}_{\Xi_C}+(-1)^{p(B)(p(A)+p(C))}\frac{\partial^2\mathcal{S}}{\partial\Xi_A\partial\bar{X}_C}\frac{\partial\mathcal{H}_{\Xi_C}}{\partial\bar{X}_B}=0\,.$$

Here, $p(A)$ denotes the parity of the variable indexed by A. Putting $\mathcal{S}_{BA}=\partial^2\mathcal{S}/\partial\bar{X}_B\partial\Xi_A$ and rewriting the above, we get

(2.37) $$\frac{\partial\mathcal{S}_{BA}}{\partial t}+(-1)^{p(C)(p(A)+p(B))}\frac{\partial\mathcal{S}_{BA}}{\partial\bar{X}_C}\mathcal{H}_{\Xi_C}+(-1)^{p(A)p(B)+p(B)p(C)+p(C)p(A)}\mathcal{S}_{CA}\frac{\partial\mathcal{H}_{\Xi_C}}{\partial\bar{X}_B}=0\,.$$

In general, for any invertible $(m+n)\times(m+n)$ even supermatrix $\mathcal{X}$ depending on a parameter τ (regardless of whether τ is even or odd), we have

(2.38) $$\frac{\partial}{\partial\tau}\operatorname{sdet}\mathcal{X}=\operatorname{str}(\mathcal{X}^{-1}\mathcal{X}_\tau)\operatorname{sdet}\mathcal{X}=\operatorname{sdet}\mathcal{X}\operatorname{str}(\mathcal{X}_\tau\mathcal{X}^{-1})\,.$$

Here, we use the following convention:

$$\left(\frac{\partial\mathcal{X}}{\partial\tau}\right)_{AB}=(-1)^{p(\tau)p(A)}\frac{\partial\mathcal{X}_{AB}}{\partial\tau}\,.$$

(See Berezin [3, pp. 109–110] and Leites [16, p. 44].)

Defining $\mathcal{D}=\operatorname{sdet}(\mathcal{S}_{BA})=\operatorname{sdet}\mathcal{S}$ and using the first equality above, we have

$$\mathcal{D}^{-1}\frac{\partial\mathcal{D}}{\partial t}=(-1)^{p(A)}\mathcal{S}_{AB}^{-1}\frac{\partial\mathcal{S}_{BA}}{\partial t}\,.$$

Multiplying $(-1)^{p(A)}\mathcal{S}_{AB}^{-1}$ to (2.37) and remarking

(2.39) $$(-1)^{p(A)}\mathcal{S}_{AB}^{-1}(-1)^{p(C)(p(A)+p(B))}\frac{\partial\mathcal{S}_{BA}}{\partial\bar{X}_C}=\operatorname{str}(\mathcal{S}_{\bar{X}_C}\mathcal{S}^{-1})=\mathcal{D}^{-1}\frac{\partial\mathcal{D}}{\partial\bar{X}_C}\,,$$

(2.40) $$(-1)^{p(A)}\mathcal{S}_{AB}^{-1}(-1)^{p(A)p(B)+p(B)p(C)+p(C)p(A)}\mathcal{S}_{CA}\frac{\partial\mathcal{H}_{\Xi_C}}{\partial\bar{X}_B}=\frac{\partial\mathcal{H}_{\Xi_C}}{\partial\bar{X}_C}$$

we get

$$\mathcal{D}^{-1}\frac{\partial\mathcal{D}}{\partial t}+\mathcal{D}^{-1}\frac{\partial\mathcal{D}}{\partial\bar{X}_C}\mathcal{H}_{\Xi_C}+\frac{\partial\mathcal{H}_{\Xi_C}}{\partial\bar{X}_C}=0\,,$$

$$\frac{\partial \mathscr{D}}{\partial t}+\frac{\partial}{\partial \bar{X}_C}(\mathscr{D}\mathscr{H}_{\Xi_C})=0 .$$

□

REMARK. The above proof is due to Mañes & Zumino [17].

COROLLARY 2.2. *Putting $\mu(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})=\mathscr{D}^{1/2}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$, we have*

$$\mu_t+\frac{1}{2}\partial_{\bar{x}}(\mu\mathscr{H}_{\xi})+\frac{1}{2}\partial_{\bar{\theta}}(\mu\mathscr{H}_{\pi})=0 , \tag{2.41}$$

where the argument of $\mathscr{H}_*$ is $(\bar{x}, \mathscr{S}_{\bar{x}}, \bar{\theta}, \mathscr{S}_{\bar{\theta}})$.

3. Proofs of the Main Theorem.

It is easy to have

PROPOSITION 3.1. *Operators $\{\sigma_j(\theta, -i\lambda\partial_\theta)\}$ are unitary in $\mathscr{L}^2_{SS,ev}(\Re^{0|2})$ only when $|\lambda|=1$.*

Therefore, we assume that $\hbar=k$ and $|\lambda|=1$.

3.1. Unitarity. In order to prove the unitarity of the operator $\mathscr{U}(t)$, we rewrite it as follows:

$$\mathscr{U}(t)u(x, \theta)=(2\pi\hbar)^{-3/2}\hbar\iint d\xi d\pi \mu(t, x, \theta, \xi, \pi)e^{i\hbar^{-1}\mathscr{S}(t,x,\theta,\xi,\pi)}\mathscr{F}u(\xi, \pi) .$$

As $\mathscr{F}u(\xi, \pi)=\hbar\hat{u}_1(\xi)+\hbar^{-1}\hat{u}_0(\xi)\pi_1\pi_2$, we get

$$\mathscr{U}(t)u(x, \theta)=v_0(t, x)+v_1(t, x)\theta_1\theta_2 , \tag{3.1}$$

where

$$\begin{aligned} v_0(t, x)&=(2\pi\hbar)^{-3/2}\int d\xi\mu(t)[\hat{u}_0(\xi)-\delta(t)^{-1}\lambda\bar{\eta}\sin\gamma(t)\hat{u}_1(\xi)] , \\ v_1(t, \xi)&=(2\pi\hbar)^{-3/2}\int d\xi\mu(t)[\delta(t)^{-1}\lambda^{-1}\eta\sin\gamma(t)\hat{u}_0(\xi)+\delta(t)^{-1}\bar{\delta}(t)\hat{u}_1(\xi)] . \end{aligned} \tag{3.2}$$

Here, we put

$$\delta(t)=|\xi|\cos\gamma(t)-i\xi_3\sin\gamma(t) , \quad \bar{\delta}(t)=|\xi|\cos\gamma(t)+i\xi_3\sin\gamma(t) ,$$

$$\gamma(t)=c\hbar^{-1}t|\xi| , \quad \eta=\xi_1+i\xi_2 , \quad \bar{\eta}=\xi_1-i\xi_2 ,$$

$$|\delta(t)|^2=|\xi|^2\cos^2\gamma(t)+\xi_3^2\sin^2\gamma(t)=|\eta|^2\cos^2\gamma(t)+\xi_3^2 .$$

Simple but lengthy calculations using the Parseval equality lead us to

PROPOSITION 3.2.

$$\|\mathscr{U}(t)u\| = \|u\| \quad in \quad \mathscr{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2}).$$

3.2. Regularity.

PROPOSITION 3.3. *When* $u(x, \theta) \in \mathscr{C}_{\mathrm{SS,ev,0}}(\mathfrak{R}^{3|2})$, $\mathscr{U}(t)u$ *is differentiable with respect to* $(t, x, \theta) \in \boldsymbol{R} \times \mathfrak{R}^{3|2}$.

PROOF. From (3.1) with (3.2), the differentiability with respect to θ_j is clear. Differentiation with respect to x is formally formulated in (3.2), and Lebesgue's dominated convergence theorem guarantees that procedure. Same for the time derivatives. □

3.3. Calculation of infinitesimal generator. Applying the Hamilton-Jacobi and continuity equations and remarking $\partial_{\bar{x}}(\mu(t, \underline{\xi})\mathscr{H}_{\underline{\xi}}(\mathscr{S}_{\bar{x}}, \bar{\theta}, \mathscr{S}_{\theta})) = 0$, we have

$$\frac{\partial}{\partial t}(\mu e^{i\hbar^{-1}\mathscr{S}}) = (\mu_t + i\hbar^{-1}\mu\mathscr{S}_t)e^{i\hbar^{-1}\mathscr{S}} = -\mu[\cdots]e^{i\hbar^{-1}\mathscr{S}}, \tag{3.3}$$

where

(3.4)

$$\begin{aligned}[\cdots] = \left[\frac{1}{2}\partial_{\bar{\theta}_j}\mathscr{H}_{\pi_j} + i\hbar^{-1}\mathscr{H}\right] &= c\hbar^{-1}\delta^{-1}|\underline{\xi}|(i\underline{\xi}_3\cos\gamma + |\underline{\xi}|\sin\gamma) \\ &+ i\hbar^{-1}c|\underline{\xi}|^2\delta^{-2}(i\lambda^{-1}\eta\bar{\theta}_1\bar{\theta}_2 - \hbar^{-1}(|\underline{\xi}|\sin\gamma + i\underline{\xi}_3\cos\gamma)\langle\bar{\theta}|\underline{\pi}\rangle - i\hbar^{-2}\lambda\bar{\eta}\underline{\pi}_1\underline{\pi}_2).\end{aligned}$$

Here, we substituted quantities before (2.31) and itself into $\mathscr{H}_\pi$ for deriving (3.4).

On the other hand, by simple calculation, we have

$$\begin{aligned}&\mu\mathscr{H}(-i\hbar\partial_{\bar{x}}, \bar{\theta}, -i\lambda\partial_{\bar{\theta}})e^{i\hbar^{-1}\mathscr{S}} \\ &= c\mu\left(i\lambda^{-1}\eta\bar{\theta}_1\bar{\theta}_2 + i\lambda\bar{\eta}\frac{\partial^2}{\partial\bar{\theta}_1\partial\bar{\theta}_2} + \underline{\xi}_3\left(1 - \bar{\theta}_1\frac{\partial}{\partial\bar{\theta}_1} - \bar{\theta}_2\frac{\partial}{\partial\bar{\theta}_2}\right)\right)e^{i\hbar^{-1}\mathscr{S}} \\ &= c\mu\{\cdots\}e^{i\hbar^{-1}\mathscr{S}}.\end{aligned}$$

In the above, we put

$$\begin{aligned}\{\cdots\} &= |\underline{\xi}|\delta^{-1}(\underline{\xi}_3\cos\gamma - i|\underline{\xi}|\sin\gamma) \\ &+ \delta^{-2}|\underline{\xi}|^2[i\lambda^{-1}\eta\bar{\theta}_1\bar{\theta}_2 - i\hbar^{-2}\lambda\bar{\eta}\underline{\pi}_1\underline{\pi}_2 + \hbar^{-1}(-\underline{\xi}_3\cos\gamma + i|\underline{\xi}|\sin\gamma)\langle\bar{\theta}|\underline{\pi}\rangle].\end{aligned} \tag{3.5}$$

Comparing (3.4) and (3.5), we have $-i\hbar[\cdots] = c\{\cdots\}$, which implies

$$i\hbar\frac{\partial}{\partial t}\mathscr{U}(t)u(x, \theta) = \mathscr{H}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}, \theta, \frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)\mathscr{U}(t)u(x, \theta).$$

□

REMARK. In calculating the righ-hand side, here we used the exact form. But, in general, we must establish the composition formula of pseudo-differential operators of Weyl type and Fourier integral operators of above type, by which the "infinitesimal generator" is calculated. This will be given in a forthcoming paper.

3.4. Evolutional property.

PROPOSITION 3.4.

$$\mathcal{U}(t)\mathcal{U}(s)u=\mathcal{U}(t+s)u \qquad \textit{for any} \quad u\in\mathscr{C}_{\mathrm{SS},0}(\mathfrak{R}^{3|2})\,.$$

PROOF. As $\mathcal{U}(s)u=u(s,x,\theta)=v_0(s,x)+v_1(s,x)\theta_1\theta_2$,

$$\begin{aligned}\mathcal{U}(t)u(s,x,\theta)&=(2\pi\hbar)^{-3/2}\int d\xi e^{i\hbar^{-1}\langle x|\xi\rangle}\mu(t)\mu(s)[(\hat{u}_0-\delta(s)^{-1}\lambda\bar{\eta}\sin\gamma(s)\hat{u}_1)\\&\quad-\delta(t)^{-1}\lambda\bar{\eta}\sin\gamma(t)\delta(s)^{-1}(\lambda^{-1}\eta\sin\gamma(s)\hat{u}_0+\bar{\delta}(s)\hat{u}_1)]\\&\quad+(2\pi\hbar)^{-3/2}\int d\xi e^{i\hbar^{-1}\langle x|\xi\rangle}\mu(t)\mu(s)[\delta(t)^{-1}\lambda^{-1}\eta\sin\gamma(t)(\hat{u}_0-\delta(s)^{-1}\lambda\bar{\eta}\sin\gamma(s)\hat{u}_1)\\&\quad+\delta(t)^{-1}\bar{\delta}(t)\delta(s)^{-1}(\lambda^{-1}\eta\sin\gamma(s)\hat{u}_0+\bar{\delta}(s)\hat{u}_1)]\theta_1\theta_2\,.\end{aligned}$$

By simple calculation, we get

$$|\xi|^{-2}(\delta(t)\delta(s)-|\eta|^2\sin\gamma(t)\sin\gamma(s))\hat{u}_0=|\xi|^{-1}\delta(t+s)\hat{u}_0\,,$$
$$-\lambda(\delta(t)\bar{\eta}\sin\gamma(s)+\bar{\delta}(s)\bar{\eta}\sin\gamma(t))\hat{u}_1=-\lambda\bar{\eta}|\xi|^{-1}\sin\gamma(t+s)\hat{u}_1\,,$$

and

$$|\xi|^{-1}\delta(t+s)\hat{u}_0-\lambda\bar{\eta}|\xi|^{-1}\sin\gamma(t+s)\hat{u}_1=\mu(t+s)[\hat{u}_0-\lambda\bar{\eta}\delta(t+s)^{-1}\sin\gamma(t+s)\hat{u}_1]\,.$$

Analogously, the coefficient of $\theta_1\theta_2$ is calculated as

$$|\xi|^{-1}[\lambda^{-1}\eta\sin\gamma(t+s)\hat{u}_0+\bar{\delta}(t+s)\hat{u}_1]\,.$$

Therefore, we have the evolutional property. □

4. Concluding remarks.

There are many problems stemming from physicists' saying:

(1) They say that a neutrino does not interact with electro-magnetic field.

It is well-known that the Weyl equation itself is introduced as a model equation following Dirac's derivation of the Dirac equation. The Weyl equation is considered as meaningless because it does not preserve parity, until neutrino is discovered as an elementary particle without parity conservation. Therefore, we mathematicians should have a confidence of our intrinsic ability of recognizing the beauty of the equation itself. But to do so, we pose a question whether we can interpret physicists' saying on the insensibility of a neutrino with respect to any electro-magnetic field. One candidate for this will be to construct an intertwining operator $\boldsymbol{W}_A(t)$ such that

$$W_A(t)U(t)=U_A(t;0)W_A(0) \qquad \text{for any} \quad t\in \boldsymbol{R}\,.$$

Here, $U_A(t;0)$ is the fundamental solution of the following problem with a given external electro-magnetic potential $A(t,q)=(A_0(t,q), A_1(t,q), A_2(t,q), A_3(t,q))$:

$$(4.1)\quad \begin{cases} i\hbar\dfrac{\partial}{\partial t}\psi(t,q)=H_A(t)\psi(t,q)\,, \quad H_A(t)=\displaystyle\sum_{k=1}^{3} c\sigma_k\left(\frac{\hbar}{i}\frac{\partial}{\partial q_k}-\frac{e}{c}A_k(t,q)\right)+eA_0(t,q)\,, \\ \psi(0,q)=\underline{\psi}(q)\,. \end{cases}$$

In this case, we will have no "explicit" formula for $U_A(t;0)$ or $\mathscr{U}_A(t;0)$, in general, but we will have

$$\frac{\partial}{\partial t}\mathscr{U}_A(t;0)u\bigg|_{t=0}=\mathscr{H}_A(0)u$$

and

$$U_A(t;s)=\flat\left(\lim_{n\to\infty}\mathscr{U}_A(t,t_{n-1})\cdots\mathscr{U}_A(t_1,s)\right)\sharp \qquad \text{with} \quad t_j=s+\frac{j(t-s)}{n}\,.$$

(2) Another very important problem: Whether a neutrino has or aquires a mass? To answer this, it is useless to consider the Weyl equation itself, because it is derived as a simplified version of the Dirac equation without mass term. But we have an experience, though not proved mathematically, that the quantization of a Lagrangian in a curved space aquires a mass term caused by the curvature (the problem of $(1/12)R$, see [12]). Therefore, we propose to do the above treatment in case where there exists an external gravitational background. That is, for a given Minkowski metric $dt^2+2h_j(q,t)dq^j dt-g_{ij}(t,q)dq^i dq^j$ on $\boldsymbol{R}\times M$, we take the square root of it using the Pauli matrices and the frame bundle, formulate and solve that Weyl equation in the same way as above. Here, M is a Riemannian 3-dimensional manifold with metric $g_{ij}(t,q)dq^i dq^j$. See, for example, Antoine, Comtet and Knecht [1].

(3) The Weyl equation in the domain with suitable boundary conditions.

Berry and Mondragon [4] proposed to study "a Dirac Hamiltonian describing massless spin-half particles ('neutrino') moving in a finite domain of the plane $\boldsymbol{r}=(x,y)$ under the action of a 4-scalar (not electric) potential $V(r)$". See also [1].

Appendix A. Fundamentals of superanalysis. For symbols $\{\sigma_j\}_{j=1}^{\infty}$ satisfying the Grassmann relation

$$\sigma_j\sigma_k+\sigma_k\sigma_j=0\,, \qquad j,k=1,2,\dots,$$

we put

$$\mathfrak{C}=\left\{X=\sum_{I\in\mathscr{I}}X_I\sigma^I \mid X_I\in\boldsymbol{C}\right\},$$

where

$$\mathscr{I}=\left\{I=(i_k)\in\{0,1\}^{N}\,\middle|\,|I|=\sum_k i_k<\infty\right\},$$

$$\sigma^I=\sigma_1^{i_1}\sigma_2^{i_2}\dots,\quad I=(i_1,i_2,\dots),\quad \sigma^{\tilde{0}}=1,\quad \tilde{0}=(0,0,\dots)\in\mathscr{I}.$$

Besides trivially defined linear operations of sums and scalar multiplications, we have a product operation in $\mathfrak{C}$: For

$$X=\sum_{J\in\mathscr{T}}X_J\sigma^J,\quad Y=\sum_{K\in\mathscr{I}}Y_K\sigma^K,$$

we put

$$XY=\sum_{I\in\mathscr{I}}(XY)_I\sigma^I\qquad\text{with}\quad (XY)_I=\sum_{I=J+K}(-1)^{\tau(I;J,K)}X_JY_K.$$

Here, $\tau(I;J,K)$ is an integer defined by

$$\sigma^J\sigma^K=(-1)^{\tau(I;J,K)}\sigma^I,\quad I=J+K.$$

PROPOSITION A.1 ([15]). *$\mathfrak{C}$ forms an ∞-dimensional Fréchet-Grassmann algebra over $\mathbf{C}$, that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.*

REMARK. (1) Degree in $\mathfrak{C}$ is defined by introducing subspaces

$$\mathfrak{C}^{[j]}=\left\{X=\sum_{I\in\mathscr{I},|I|=j}X_I\sigma^I\right\}\qquad\text{for}\quad j=0,1,\dots$$

which satisfy

$$\mathfrak{C}=\bigoplus_{j=0}^{\infty}\mathfrak{C}^{[j]},\quad \mathfrak{C}^{[j]}\cdot\mathfrak{C}^{[k]}\subset\mathfrak{C}^{[j+k]}.$$

(2) Define

$$\mathrm{proj}_I(X)=X_I\qquad\text{for}\quad X=\sum_{I\in\mathscr{I}}X_I\sigma^I\in\mathfrak{C}.$$

The topology in $\mathfrak{C}$ is given by $X\to0$ in $\mathfrak{C}$ if and only if $\mathrm{proj}_I(X)\to0$ in $\mathbf{C}$, for any $I\in\mathscr{I}$.

This topology is equivalent to the one introduced by the metric $\mathrm{dist}(X,Y)=\mathrm{dist}(X-Y)$ where $\mathrm{dist}(X)$ is defined by

$$\mathrm{dist}(X)=\sum_{I\in\mathscr{I}}\frac{1}{2^{r(I)}}\frac{|\mathrm{proj}_I(X)|}{1+|\mathrm{proj}_I(X)|}\qquad\text{with}\quad r(I)=1+\frac{1}{2}\sum_{k=1}^{\infty}2^k i_k\quad\text{for } I\in\mathscr{I}.$$

(3) We introduce parity in $\mathfrak{C}$ by setting

$$p(X)=\begin{cases}0 & \text{if } X=\sum_{I\in\mathscr{I},|I|=\mathrm{ev}}X_I\sigma^I,\\ 1 & \text{if } X=\sum_{I\in\mathscr{I},|I|=\mathrm{od}}X_I\sigma^I,\\ \text{undefined} & \text{otherwise}.\end{cases}$$

We put

$$\begin{cases} \mathfrak{C}_{ev} = \bigoplus_{j=0}^{\infty} \mathfrak{C}^{[2j]} = \{X \in \mathfrak{C} \mid p(X) = 0\}\,, \\ \mathfrak{C}_{od} = \bigoplus_{j=0}^{\infty} \mathfrak{C}^{[2j+1]} = \{X \in \mathfrak{C} \mid p(X) = 1\}\,, \\ \mathfrak{C} \cong \mathfrak{C}_{ev} \oplus \mathfrak{C}_{od} \cong \mathfrak{C}_{ev} \times \mathfrak{C}_{od}\,. \end{cases}$$

Analogous to $\mathfrak{C}$, we define

$$\begin{cases} \mathfrak{R} = \{X \in \mathfrak{C} \mid \pi_B X \in \boldsymbol{R}\}\,, \qquad \mathfrak{R}^{[j]} = \mathfrak{R} \cap \mathfrak{C}^{[j]}\cdot \\ \mathfrak{R}_{ev} = \mathfrak{R} \cap \mathfrak{C}_{ev}\,, \qquad \mathfrak{R}_{od} = \mathfrak{R} \cap \mathfrak{C}_{od} = \mathfrak{C}_{od}\,, \\ \mathfrak{R} \cong \mathfrak{R}_{ev} \oplus \mathfrak{R}_{od} \cong \mathfrak{R}_{ev} \times \mathfrak{R}_{od}\,. \end{cases}$$

We introduced the body (projection) map π_B by

$$\pi_B X = \mathrm{proj}_{\tilde{0}}(X) = X_{\tilde{0}} = X^{[0]} = X_B \qquad \text{for any} \quad X \in \mathfrak{C}\,,$$

and the soul part X_S of X as

$$X_S = X - X_B = \sum_{|I| \geq 1} X_I \sigma^I\,.$$

We define the (real) superspace $\mathfrak{R}^{m|n}$ by

$$\mathfrak{R}^{m|n} = \mathfrak{R}_{ev}^m \times \mathfrak{R}_{od}^n\,.$$

The distance between X, $Y \in \mathfrak{R}^{m|n}$ is defined by

$$\mathrm{dist}_{m|n}(X, Y) = \mathrm{dist}_{m|n}(X - Y)$$

with

$$\mathrm{dist}_{m|n}(X) = \sum_{j=1}^{m} \left(\sum_{I \in \mathscr{I}} \frac{1}{2^{r(I)}} \frac{|\mathrm{proj}_I(x_j)|}{1 + |\mathrm{proj}_I(x_j)|} \right) + \sum_{k=1}^{n} \left(\sum_{I \in \mathscr{I}} \frac{1}{2^{r(I)}} \frac{|\mathrm{proj}_I(\theta_k)|}{1 + |\mathrm{proj}_I(\theta_k)|} \right).$$

We use the following notation:

$$X = (X_A)_{A=1}^{m+n} = (x, \theta) \in \mathfrak{R}^{m|n} \qquad \text{with}$$

$$x = (X_A)_{A=1}^m = (x_j)_{j=1}^m \in \mathfrak{R}^{m|0}\,, \quad \theta = (X_A)_{A=m+1}^{m+n} = (\theta_k)_{k=1}^n \in \mathfrak{R}^{0|n}\,.$$

We generalize the body map π_B from $\mathfrak{R}^{m|n}$ or $\mathfrak{R}^{m|0}$ to $\boldsymbol{R}^m$ by putting

$$X = (x, \theta) \in \mathfrak{R}^{m|n} \to \pi_B X = X_B = (x_B, 0) \cong x_B = \pi_B x = (\pi_B x_1, \ldots, \pi_B x_m) \in \boldsymbol{R}^m\,.$$

We call $x_j \in \mathfrak{R}_{ev}$ and $\theta_k \in \mathfrak{R}_{od}$ as even and odd (alias bosonic and ferminionic) variable, respectively.

REMARK. For $\xi = (\xi_1, \ldots, \xi_m) \in \mathfrak{R}^{m|0} = \mathfrak{R}_{ev}^m$, we define $|\xi| \in \mathfrak{R}_{ev}$ as follows: Putting

$$|\xi| = |\xi|_B + |\xi|_S \qquad \text{with} \quad |\xi|_S = \sum_{|I| = \text{even} \geq 2} |\xi|_I \sigma^I\,, \quad |\xi|_B \geq 0\,, \quad |\xi|_I \in \boldsymbol{R}\,,$$

we should have

$$|\xi|^2=\sum_{j=1}^m (\xi_{j,B}+\xi_{j,S})(\xi_{j,B}+\overline{\xi_{j,S}})=\sum_{j=1}^m \xi_{j,B}^2+\sum_{j=1}^m \xi_{j,B}(\xi_{j,S}+\overline{\xi_{j,S}})+\sum_{j=1}^m \xi_{j,S}\overline{\xi_{j,S}},$$

$$\xi_{j,S}=\sum_{|I|=\mathrm{even}\geq 2}\xi_{j,I}\sigma^I,\quad \overline{\xi_{j,S}}=\sum_{|I|=\mathrm{even}\geq 2}\overline{\xi_{j,I}}\sigma^I$$

with $\overline{\xi_{j,I}}$ being the complex conjugate of $\xi_{j,I}$ in $\boldsymbol{C}$. Therefore,

$$|\xi|_B=\left\{\sum_{j=1}^m \xi_{j,B}^2\right\}^{1/2},$$

$$2|\xi|_K|\xi|_B+\sum_{I+J=K}|\xi|_I\overline{|\xi|_J}(-1)^{\tau(K;I,J)}=\sum_{j=1}^m 2\xi_{j,B}\Re\xi_{j,K}+\sum_{I+J=K}\sum_{j=1}^m \xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)},$$

which are solved by induction with respect to the length $|K|$. For example, if $|K|=2$, we have

$$|\xi|_K=|\xi|_B^{-1}\sum_{j=1}^m \xi_{j,B}\Re\xi_{j,K}.$$

If $|K|=4$,

$$2|\xi|_K=|\xi|_B^{-1}\Bigg(2\sum_{j=1}^m \xi_{j,B}\Re\xi_{j,K}+\sum_{I+J=K}\sum_{j=1}^m \xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)}$$
$$-\sum_{I+J=K}\sum_{j=1}^m |\xi|_I|\xi|_J(-1)^{\tau(K;I,J)}\Bigg),$$

etc.

Supersmooth functions: For $u_a(q)\in C^\infty(\boldsymbol{R}^m:\boldsymbol{C})$, we put

$$u_a(x)=\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}\partial_q^\alpha u_a(x_B)x_S^\alpha \qquad \text{for}\quad x=x_B+x_S,$$

which is called the Grassmann continuation of $u_a(q)$. We define a function $u\in\mathscr{C}_{SS,ev}(\Re^{m|n})$ by

$$u(X)=u(x,\theta)=\sum_{|a|\leq n}u_a(x)\theta^a,$$

called a supersmooth function on $\Re^{m|n}$. For example, we define $\sin|\xi|$, $\cos|\xi|$ as

$$\sin|\xi|=\sum_{n=0}^\infty\frac{1}{n!}\sin\left(|\xi|_B+\frac{n\pi}{2}\right)|\xi|_S^n,\quad \cos|\xi|=\sum_{n=0}^\infty\frac{1}{n!}\cos\left(|\xi|_B+\frac{n\pi}{2}\right)|\xi|_S^n.$$

We may characterize this function as a solution of a certain Cauchy-Riemann type partial differential equation. See more exactly [15].

Derivations: For a given supersmooth function $u(X)$ on $\mathfrak{R}^{m|n}$, we define its derivatives as follows: For $j=1,2,\ldots,m$ and $k=1,2,\ldots,n$, we put

$$\begin{cases} U_j(X)=\sum_{|\mathfrak{a}|\le n} \partial_{x_j} u_a(x)\theta^a\,, \\ U_{k+m}(X)=\sum_{|\mathfrak{a}|\le n} (-1)^{l_k(a)} u_a(x)\theta_1^{a_1}\cdots\theta_k^{a_k-1}\cdots\theta_n^{a_n}\,, \end{cases}$$

where $l_k(a)=\sum_{j=1}^{k-1} a_j$ and $\theta_k^{-1}=0$. $U_\kappa(X)$ are called the partial derivatives of u with respect to X_κ at $X=(x,\theta)$ and are denoted by

$$\begin{cases} U_j(X)=\dfrac{\partial}{\partial x_j}u(x,\theta)=\partial_{x_j}u(x,\theta) & \text{for}\quad j=1,2,\ldots,m\,, \\ U_{m+s}(X)=\dfrac{\partial}{\partial \theta_s}u(x,\theta)=\partial_{\theta_s}u(x,\theta) & \text{for}\quad s=1,2,\ldots,n\,, \end{cases}$$

or simply by

$$U_\kappa(X)=\partial_{X_\kappa}u(X) \qquad \text{for}\quad \kappa=1,\ldots,m+n\,.$$

For

$$\mathfrak{a}=(\alpha,a)\,,\quad \alpha=(\alpha_1,\ldots,\alpha_m)\in\boldsymbol{N}^m\,,\quad a=(a_1,\ldots,a_n)\in\{0,1\}^n\,,$$
$$|\alpha|=\sum_{j=1}^m \alpha_j\,,\quad |a|=\sum_{k=1}^n a_k\,,\quad |\mathfrak{a}|=|\alpha|+|a|\,,$$

we put

$$\partial_X^{\mathfrak{a}}=\partial_x^\alpha\partial_\theta^a \qquad \text{with}\quad \partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_m}\,,\quad \partial_\theta^a=\partial_{\theta_1}^{a_1}\cdots\partial_{\theta_n}^{a_n}\,.$$

EXAMPLE. $\partial_{\theta_2}\theta_1\theta_2\theta_3=-\theta_1\theta_3$, $\partial_{\theta_1}\partial_{\theta_3}\theta_1\theta_2\theta_3=\theta_2\neq-\theta_2=\partial_{\theta_3}\partial_{\theta_1}\theta_1\theta_2\theta_3$, etc.

Integration: We define

$$\begin{aligned}\int_{\mathfrak{R}^{m|n}} dxd\theta u(x,\theta)&=\int_{\mathfrak{R}^{m|0}} dx\left\{\int_{\mathfrak{R}^{0|n}} d\theta u(x,\theta)\right\}\\ &=\int_{\boldsymbol{R}^m} dX_{\mathrm{B}}(\partial_{\theta_n}\cdots\partial_{\theta_1}u)(X_{\mathrm{B}}) \quad (\pi_{\mathrm{B}}(\mathfrak{R}^{m|0})=\boldsymbol{R}^m)\\ &=\int_{\mathfrak{R}^{0|n}} d\theta\left\{\int_{\mathfrak{R}^{m|0}} dxu(x,\theta)\right\}=\int_{\mathfrak{R}^{m|n}} d\theta dxu(x,\theta)\,.\end{aligned}$$

Especially for odd integration, we have the following curious looking but well-known relations

$$\int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1 \theta_1 \cdots \theta_n = 1 \quad \text{and} \quad \int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1 1 = 0 \quad \text{(Berezin integral)}.$$

Remarks for the need of ∞ number of Grassmann generators. (i) Though $\mathfrak{C}$ does not form a field because $X^2=0$ for any $X \in \mathfrak{C}_{\rm od}$, but if $X, Y \in \mathfrak{C}$ satisfy $XY=0$ for any $Y \in \mathfrak{C}_{\rm od}$, then $X=0$. This property holds only when the number of generators is infinite. By this, we may determine the derivative $\partial_X^\alpha u(X)$ uniquely.

(ii) In general, we need at least countable number of operations in doing analysis. If the number of Grassmann generators is finite, then the effect of odd variables may vanish after finitely many operations.

Scalar products and norms: Following [8], we introduce

$$\mathscr{C}_{\rm SS,ev}(\mathfrak{R}^{m|n}) = \left\{ u(X) = \sum_{|a|={\rm even}\le n} u_a(x)\theta^a \,|\, u_a(q) \in C^\infty(\boldsymbol{R}^m : \boldsymbol{C}) \text{ for any } a \right\},$$

$$\mathscr{C}_{\rm SS,ev,0}(\mathfrak{R}^{m|n}) = \left\{ u(X) = \sum_{|a|={\rm even}\le n} u_a(x)\theta^a \,|\, u_a(q) \in C_0^\infty(\boldsymbol{R}^m : \boldsymbol{C}) \text{ for any } a \right\}, \quad \text{etc.}$$

We define the conjugation $\overline{u(x,\theta)} = \sum_a \overline{u_a(x)\theta^a}$, where $\overline{\theta^a} = \bar\theta_n^{a_n} \cdots \bar\theta_1^{a_1}$, $\bar{\bar\theta}_j = \theta_j$ and $\overline{u_a(x)}$ being the complex conjugate of $u_a(x)$. Then, we define

$$(u, v) = \int_{\mathfrak{R}^{m|2n}} dx d\theta d\bar\theta e^{\langle \bar\theta | \theta \rangle} \overline{u(x,\theta)} v(x,\theta) = \sum_{|a|\le n} \int_{\mathfrak{R}^{m|0}} dx \overline{u_a(x)} v_a(x),$$

$$((u, v))_k = \sum_{|\mathfrak{a}|\le k} (\partial_X^{\mathfrak{a}} u, \partial_X^{\mathfrak{a}} v) = \sum_{|\alpha|+|a|\le k} (\partial_x^\alpha u_a, \partial_x^\alpha v_a),$$

$$(((u, v)))_k = \sum_{|\mathfrak{a}|+l\le k} ((1+|X_{\rm B}|^2)^{l/2} \partial_X^{\mathfrak{a}} u, (1+|X_{\rm B}|^2)^{l/2} \partial_X^{\mathfrak{a}} v)$$

with

$$\|u\|^2 = (u, u), \quad \|u\|_k^2 = ((u, u))_k, \quad |||u|||_k^2 = (((u, u)))_k.$$

The space $\mathscr{L}^2_{\rm SS,ev}(\mathfrak{R}^{m|n})$ is the completion of $\mathscr{C}_{\rm SS,ev,0}(\mathfrak{R}^{m|n})$ in the norm $\|\cdot\|$. In our case, we may identify

$$\mathscr{L}^2_{\rm SS,ev}(\mathfrak{R}^{3|2}) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} L^2(\boldsymbol{R}^3 : \boldsymbol{C}^2).$$

(See, more precisely, [8].)

Fourier transformations:

$$(F_e v)(\xi)=(2\pi\hbar)^{-m/2}\int_{\mathfrak{R}^{m|0}} dx e^{-i\hbar^{-1}\langle x|\xi\rangle}v(x),$$

$$(\bar{F}_e w)(x)=(2\pi\hbar)^{-m/2}\int_{\mathfrak{R}^{m|0}} d\xi e^{i\hbar^{-1}\langle x|\xi\rangle}w(\xi),$$

$$(F_o v)(\pi)=\hbar\!\!\!/^{n/2}\iota_n\int_{\mathfrak{R}^{0|n}} d\theta e^{-i\hbar\!\!\!/^{-1}\langle\theta|\pi\rangle}v(\theta),$$

$$(\bar{F}_o w)(\theta)=\hbar\!\!\!/^{n/2}\iota_n\int_{\mathfrak{R}^{0|n}} d\pi e^{i\hbar\!\!\!/^{-1}\langle\theta|\pi\rangle}w(\pi),$$

where

$$\langle\eta\,|\,y\rangle=\sum_{j=1}^{m}\eta_j y_j,\quad \langle\rho\,|\,\omega\rangle=\sum_{k=1}^{n}\rho_k\omega_k,\quad \iota_n=e^{-\pi i n(n-2)/4}.$$

We put

$$(\mathscr{F}u)(\xi,\pi)=c_{m,n}\int_{\mathfrak{R}^{m|n}} dX e^{-i\hbar^{-1}\langle X|\Xi\rangle}u(X)=\sum_a[(F_e u_a)(\xi)][F_o\theta^a)(\pi)],$$

$$(\bar{\mathscr{F}}v)(x,\theta)=c_{m,n}\int_{\mathfrak{R}^{m|n}} d\Xi e^{i\hbar^{-1}\langle X|\Xi\rangle}v(\Xi)=\sum_a[(\bar{F}_e v_a)(x)][(\bar{F}_o\pi^a)(\theta)],$$

where

$$\langle X\,|\,\Xi\rangle=\langle x\,|\,\xi\rangle+\hbar\hbar\!\!\!/^{-1}\langle\theta\,|\,\pi\rangle\in\mathfrak{R}_{\mathrm{ev}},\qquad c_{m,n}=(2\pi\hbar)^{-m/2}\hbar\!\!\!/^{n/2}\iota_n,$$

REMARK. Though the differential calculus on Fréchet spaces has some difficulties in general, such calculus on Fréchet-Grassmann algebra holds safely in our case. For example, the implicit and inverse function theorems, and the chain rule for differentiation are established as in the standard case.

Appendix B. Derivation of the Weyl symbol. Let a function $a(q,p)\in C^\infty(T^*\boldsymbol{R}^m)$ be given. We define a pseudo-differential operator $a(q,-i\hbar\partial_q)$ with symbol $a(q,p)$ as

$$a(q,-i\hbar\partial_q)u(q)=(2\pi\hbar)^{-m}\iint_{\boldsymbol{R}^{2m}} dp\,dq' e^{i\hbar^{-1}(q-q')p}a(q,p)u(q').$$

Then, the Schwartz kernel $K_a(q,q')$ of $a(q,-i\hbar\partial_q)$ is defined by

$$K_a(q,q')=(2\pi\hbar)^{-m}\int_{\boldsymbol{R}^m} dp\,e^{i\hbar^{-1}(q-q')p}a(q,p),$$

which gives

$$a(q, -i\hbar\partial_q)u(q)=\int_{\mathbf{R}^m} dq' K_a(q, q')u(q').$$

We introduce the Weyl symbol as

$$a^w(q, p)=\int_{\mathbf{R}^m} dq' e^{-i\hbar^{-1}q'p} K_a\left(q+\frac{q'}{2}, q-\frac{q'}{2}\right).$$

Conversely, for any (pseudo-)differential operator $P(q, -i\hbar\partial_q)$, we define its ordinary symbol as

(B.1) $$P(q, p)=e^{-i\hbar^{-1}qp} P\left(q, \frac{\hbar}{i}\frac{\partial}{\partial q}\right) e^{i\hbar^{-1}qp}.$$

The Schwartz kernel is given formally as

(B.2) $$K_P(q, q')=(2\pi\hbar)^{-m}\int_{\mathbf{R}^m} dp e^{i\hbar^{-1}(q-q')p} P(q, p).$$

Therefore, we have

(B.3) $$P^w(q, p)=(2\pi\hbar)^{-m}\iint_{\mathbf{R}^{2m}} dq' dp' e^{i\hbar^{-1}q'(p'-p)} P\left(q+\frac{q'}{2}, p'\right).$$

EXAMPLE. *Let* $P(q, -i\hbar\partial_q)=(-i\hbar\partial_q-A(q))^2=-\hbar^2\partial_q^2+2i\hbar A(q)\partial_q+i\hbar\partial_q A(q)+A^2(q)$. Then

$$P(q, p)=p^2-2A(q)p+A^2(q)+i\hbar\partial_q A(q) \quad \text{and} \quad P^w(q, p)=(p-A(q))^2.$$

These procedures are extended to pseudo-differential operators on the superspace without any serious change: For any differential operator $Q(\theta, \partial/\partial\theta)$, we define its symbol as

(B.4) $$Q(\theta, \pi)=e^{-ik^{-1}\langle\theta|\pi\rangle} Q\left(\theta, \frac{\partial}{\partial\theta}\right) e^{ik^{-1}\langle\theta|\pi\rangle}.$$

The Schwartz kernel and Weyl symbol are given as

(B.5) $$K_Q(\theta, \theta')=k^2\int_{\mathfrak{R}^{0|2}} d\pi e^{ik^{-1}\langle\theta-\theta'|\pi\rangle} Q(\theta, \pi),$$

and

(B.6) $$Q^w(\theta, \pi)=k^2\iint_{\mathfrak{R}^{0|4}} d\theta' d\pi' e^{ik^{-1}\langle\theta'|\pi'-\pi\rangle} Q\left(\theta+\frac{\theta'}{2}, \pi'\right).$$

Therefore, for differential operators with respect to odd variables, we have

$$
\text{(B.7)}\quad \begin{cases}
\sigma_1(\theta,\pi)=e^{-i\hbar^{-1}\langle\theta|\pi\rangle}\sigma_1\left(\theta,\dfrac{\lambda}{i}\dfrac{\partial}{\partial\theta}\right)e^{i\hbar^{-1}\langle\theta|\pi\rangle}=i\lambda^{-1}(\theta_1\theta_2-\lambda^2\hbar^{-2}\pi_1\pi_2)\,,\\[2ex]
\sigma_2(\theta,\pi)=e^{-i\hbar^{-1}\langle\theta|\pi\rangle}\sigma_2\left(\theta,\dfrac{\lambda}{i}\dfrac{\partial}{\partial\theta}\right)e^{i\hbar^{-1}\langle\theta|\pi\rangle}=-\lambda^{-1}(\theta_1\theta_2+\lambda^2\hbar^{-2}\pi_1\pi_2)\,,\\[2ex]
\sigma_3(\theta,\pi)=e^{-i\hbar^{-1}\langle\theta|\pi\rangle}\sigma_3\left(\theta,\dfrac{\lambda}{i}\dfrac{\partial}{\partial\theta}\right)e^{i\hbar^{-1}\langle\theta|\pi\rangle}=1-i\hbar^{-1}(\theta_1\pi_1+\theta_2\pi_2)\,.
\end{cases}
$$

Moreover, the Schwartz kernels are given by

$$
\text{(B.8)}\quad \begin{cases}
K_{\sigma_1}(\theta,\theta')=i\lambda^{-1}(\theta_1\theta_2\theta'_1\theta'_2-\lambda^2)\,,\\
K_{\sigma_2}(\theta,\theta')=-\lambda^{-1}(\theta_1\theta_2\theta'_1\theta'_2+\lambda^2)\,,\\
K_{\sigma_3}(\theta,\theta')=\theta'_1\theta'_2-\theta_1\theta_2\,,
\end{cases}
$$

and then

$$
\text{(B.9)}\quad \begin{cases}
\sigma_1^w(\theta,\pi)=i\lambda^{-1}(\theta_1\theta_2-\lambda^2\hbar^{-2}\pi_1\pi_2)=\sigma_1(\theta,\pi)\,,\\
\sigma_2^w(\theta,\pi)=-\lambda^{-1}(\theta_1\theta_2+\lambda^2\hbar^{-2}\pi_1\pi_2)=\sigma_2(\theta,\pi)\,,\\
\sigma_3^w(\theta,\pi)=-i\hbar^{-1}(\theta_1\pi_1+\theta_2\pi_2)\,.
\end{cases}
$$

For any differential operator $P(x,-i\hbar\partial_x,\theta,-i\lambda\partial_\theta)$ on the superspace $\mathfrak{R}^{3|2}$ represented by

$$
P\left(x,\frac{\hbar}{i}\frac{\partial}{\partial x},\theta,\frac{\lambda}{i}\frac{\partial}{\partial\theta}\right)=a_0\left(x,\frac{\hbar}{i}\frac{\partial}{\partial x}\right)+\sum_{j=1}^{3}a_j\left(x,\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\sigma_j\left(\theta,\frac{\lambda}{i}\frac{\partial}{\partial\theta}\right),
$$

we define its complete Weyl symbol as follows:

$$
P^w(x,\xi,\theta,\pi)=a_0^w(x,\xi)+\sum_{j=1}^{3}a_j^w(x,\xi)\sigma_j^w(\theta,\pi)\,.
$$

REMARK. In the context, we use $\sigma_j(\theta,\pi)$ abbreviating the upper index of $\sigma_j^w(\theta,\pi)$.

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