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On a converse inequality for maximal functions in Orlicz spaces

by

H. KITA (Oita)

Abstract. Let $\Phi(t) = \int_0^t a(s)\,ds$ and $\Psi(t) = \int_0^t b(s)\,ds$, where $a(s)$ is a positive continuous function such that $\int_1^\infty a(s)/s\,ds = \infty$ and $b(s)$ is quasi-increasing and $\lim_{s\to\infty} b(s) = \infty$. Then the following statements for the Hardy–Littlewood maximal function $Mf(x)$ are equivalent:

(j) there exist positive constants c_1 and s_0 such that

$$\int_1^s \frac{a(t)}{t}\,dt \ge c_1 b(c_1 s) \quad \text{for all } s \ge s_0;$$

(jj) there exist positive constants c_2 and c_3 such that

$$\int_0^{2\pi} \Psi\Big(\frac{c_2}{|f|_{\mathbb{T}}}|f(x)|\Big)\,dx \le c_3 + c_3 \int_0^{2\pi} \Phi\Big(\frac{1}{|f|_{\mathbb{T}}} Mf(x)\Big)\,dx \quad \text{for all } f \in L^1(\mathbb{T}).$$

1. Introduction. Let $\mathbb{T}$ be the group of real numbers modulo 2π and $f(x)$ be a real-valued integrable function defined on $\mathbb{T}$ with period 2π. The classical Hardy–Littlewood maximal function $Mf(x)$ is defined by

$$Mf(x) := \sup_{x\in I} \frac{1}{|I|} \int_I |f(y)|\,dy, \tag{1.1}$$

where the supremum is taken over all open intervals $I \subseteq \mathbb{T}$ with $x \in I$.

The aim of this paper is to give a necessary and sufficient condition for a function $f(x)$ to be in an Orlicz space L^Ψ if the maximal function $Mf(x)$ defined by (1.1) is in L^Φ. Let us recall the definition of L^Ψ.

DEFINITION 1.1. Let $\Psi(t)$ be a nondecreasing continuous function such that $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. Put

$$L^\Psi := \Big\{f : \int_0^{2\pi} \Psi(\varepsilon|f(x)|)\,dx < \infty \text{ for some } \varepsilon > 0\Big\}. \tag{1.2}$$

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The space L^Ψ is called an *Orlicz space* (see Kita and Yoneda [3], Rao and Ren [6], Zygmund [9]).

We note that if $\Psi(t) = t^p$ $(p > 1)$, then the Orlicz space L^Ψ defined by (1.2) is the usual Lebesgue space.

Let $a(s)$ be a positive continuous function defined on $[0,\infty)$ with the following property:

$$\int_1^\infty \frac{a(s)}{s}\,ds = \infty. \tag{1.3}$$

A function $b(s)$ defined on $[0,\infty)$ is said to be *quasi-increasing* on $[0,\infty)$ if there exists a positive constant $c_0 \ge 1$ such that

$$0 < b(s_1) \le c_0 b(c_0 s_2) \quad \text{for all } 0 < s_1 < s_2. \tag{1.4}$$

Let $b(s)$ be a continuous quasi-increasing function defined on $[0,\infty)$ satisfying

$$\lim_{s\to\infty} b(s) = \infty. \tag{1.5}$$

Put

$$\Phi(t) := \int_0^t a(s)\,ds \quad \text{and} \quad \Psi(t) := \int_0^t b(s)\,ds \quad \text{for } t \ge 0. \tag{1.6}$$

The detailed results on maximal functions in the class $\Phi(L) := \{f : \int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx < \infty\}$ can be found in the book of Kokilashvili and Krbec [4]. In this paper we consider the maximal functions of functions in an Orlicz space L^Ψ, where $\Psi(t)$ does not necessarily satisfy the Δ_2-condition, that is, there exist positive constants c and t_0 such that $\Psi(2t) \le c\Psi(t)$ for all $t \ge t_0$. The following result can be found in Kita [2].

THEOREM 1.2. *Let $a(s)$, $b(s)$, $\Phi(t)$ and $\Psi(t)$ satisfy* (1.3)–(1.6). *Then the following statements are equivalent*:

(i) *there exists a positive constant c_1 such that*

$$\int_1^s \frac{a(t)}{t}\,dt \le c_1 b(c_1 s) \quad \textit{for all } s \ge 1; \tag{1.7}$$

(ii) *there exists a positive constant c_2 such that*

$$\int_0^{2\pi} \Phi(Mf(x))\,dx \le c_2 + c_2 \int_0^{2\pi} \Psi(c_2|f(x)|)\,dx \quad \textit{for all } f \in L^1(\mathbb{T}). \tag{1.8}$$

We consider a converse inequality to (1.8). We say that a measurable function $f(x)$ is in $L\log L(\mathbb{T})$ provided that $\int_0^{2\pi} |f(x)| \log^+ |f(x)|\,dx < \infty$,

where $\log^+ t = \log t$ for $t \ge 1$ and $\log^+ t = 0$ for $0 \le t < 1$. Stein [7] proved the following result (see Torchinsky [8], p. 93).

THEOREM 1.3. *Let $f \in L^1(\mathbb{T})$. If $Mf \in L^1(\mathbb{T})$, then $f \in L\log L(\mathbb{T})$.*

2. Main theorem. In this section we give a generalization of Theorem 1.3 to functions in an Orlicz space L^Ψ, which is also a generalization of the result of Moscariello [5].

THEOREM 2.1. *Let $a(s)$, $b(s)$, $\Phi(t)$ and $\Psi(t)$ satisfy* (1.3)–(1.6). *Then the following statements are equivalent*:

(j) *there exist positive constants c_3 and $s_0 > 1$ such that*

$$\int_1^s \frac{a(t)}{t}\,dt \ge c_3 b(c_3 s) \quad \textit{for all } s \ge s_0; \tag{2.1}$$

(jj) *there exist positive constants c_4 and c_5 such that*

$$\int_0^{2\pi} \Psi\Big(\frac{c_4}{|f|_{\mathbb{T}}}|f(x)|\Big)\,dx \le c_5 + c_5 \int_0^{2\pi} \Phi\Big(\frac{1}{|f|_{\mathbb{T}}} Mf(x)\Big)\,dx \quad \textit{for all } f \in L^1(\mathbb{T}), \tag{2.2}$$

where $|f|_{\mathbb{T}} := \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\,dx$.

Proof. (j)$\Rightarrow$(jj). Without loss of generality, we may assume that $|f|_{\mathbb{T}} = 1$. Put $\Psi_1(t) = \Psi(c_3 t)$ for $t \ge 0$. Then it is easy to see that

$$\begin{aligned} I &:= \int_0^{2\pi} \Phi(c_3|f(x)|)\,dx = \int_0^\infty |\{|f| > s\}|\Psi_1'(s)\,ds \\ &= \int_0^\infty |\{|f| > s\}| c_3 b(c_3 s)\,ds \\ &= \Big(\int_0^{s_0} + \int_{s_0}^\infty\Big)|\{|f| > s\}| c_3 b(c_3 s)\,ds =: I_0 + I_1. \end{aligned}$$

Since $b(s)$ is quasi-increasing, it follows from (1.4) that

$$0 < b(c_3 s) \le c_0 b(c_0 c_3 s_0) \quad \text{for } 0 < s < s_0. \tag{2.3}$$

From (2.3) we have

$$I_0 \le 2\pi s_0 c_0 c_3 b(c_0 c_3 s_0). \tag{2.4}$$

By (2.1) we have

$$I_1 \le \int_{s_0}^{\infty} |\{|f|>s\}| \left(\int_1^s \frac{a(t)}{t}\,dt \right) ds \le \int_1^{\infty} |\{|f|>s\}| \left(\int_1^s \frac{a(t)}{t}\,dt \right) ds$$
$$= \int_1^{\infty} \frac{a(t)}{t} \left(\int_t^{\infty} |\{|f|>s\}|\,ds \right) dt.$$

For any $t>0$, put

$$f_t(x) = \begin{cases} f(x) & \text{if } |f(x)|>t, \\ 0 & \text{if } |f(x)| \le t. \end{cases}$$

Then it is easy to see that

$$\int_t^{\infty} |\{|f|>s\}|\,ds \le \int_{|f|>t} |f(x)|\,dx \quad \text{for all } t>0. \tag{2.5}$$

Indeed,

$$\int_{|f|>t} |f(x)|\,dx = \int_0^{2\pi} |f_t(x)|\,dx = \int_0^{\infty} |\{|f_t|>s\}|\,ds$$
$$\ge \int_t^{\infty} |\{|f_t|>s\}|\,ds = \int_t^{\infty} |\{|f|>s\}|\,ds.$$

Therefore it follows from (2.5) that

$$I_1 \le \int_1^{\infty} \frac{a(t)}{t} \left(\int_{|f|>t} |f(x)|\,dx \right) dt.$$

By the converse inequality for the maximal function (see Guzmán [1], Torchinsky [8], p. 93), it follows from $|f|_{\mathbb{T}}=1$ that

$$I_1 \le \int_1^{\infty} \frac{a(t)}{t} \cdot \frac{t}{c_6} \cdot |\{Mf>t\}|\,dt \le \frac{1}{c_6} \int_0^{2\pi} \Phi(Mf(x))\,dx. \tag{2.6}$$

Therefore it follows from (2.4) and (2.6) that

$$\int_0^{2\pi} \Psi(c_3|f(x)|)\,dx \le 2\pi s_0 c_0 c_3 b(c_0 c_3 s_0) + \frac{1}{c_6} \int_0^{2\pi} \Phi(Mf(x))\,dx,$$

which is nothing but (2.2).

(jj)⇒(j). Let (jj) hold and assume that (j) does not hold. Then there exists a sequence $\{s_k : k \ge 0\}$ of numbers such that

$$1 = s_0 < s_1 < \ldots \uparrow \infty \quad \text{as } k \uparrow \infty; \tag{2.7}$$

$$\int_1^{s_k} \frac{a(s)}{s} < \frac{1}{2^k} b\left(\frac{s_k}{2^k}\right) \quad \text{for all } k \ge 1; \tag{2.8}$$

$$b\left(\frac{s_k}{2^k}\right) \ge k2^k \quad \text{for all } k \ge 1; \tag{2.9}$$

$$s_{k+1} \ge 4c_0 s_k \quad \text{for all } k \ge 0, \tag{2.10}$$

where c_0 is the constant appearing in (1.4).

We choose a collection $\{I_k : k \ge 1\}$, $I_k \subseteq \mathbb{T}$, of disjoint open intervals such that

$$|I_k| = \frac{1}{(s_k/2^k) b(s_k/2^k)} \quad \text{for } k \ge 1. \tag{2.11}$$

From (2.7), (2.9) and (2.10) it follows that

$$\sum_{k=1}^{\infty} |I_k| \le \sum_{k=1}^{\infty} \frac{1}{(s_k/2^k)k2^k} < \sum_{k=1}^{\infty} \frac{1}{s_k} \le \sum_{k=1}^{\infty} \frac{1}{4^k} < 2\pi.$$

Put

$$f(x) := \alpha_0 \sum_{k=1}^{\infty} \frac{ks_k}{2^k} \chi_{I_k}(x) \quad \text{for } x \in \mathbb{T}, \tag{2.12}$$

where χ_{I_k} is the characteristic function of I_k and the positive constant α_0 will be defined later. It is easy to see that $f \in L^1(\mathbb{T})$. Indeed, by (2.9), (2.11) and (2.12) we have

$$\int_0^{2\pi} |f(x)|\,dx = \alpha_0 \sum_{k=1}^{\infty} \frac{ks_k}{2^k} |I_k| = \alpha_0 \sum_{k=1}^{\infty} \frac{k}{b(s_k/2^k)}$$
$$\le \alpha_0 \sum_{k=1}^{\infty} \frac{1}{2^k} = \alpha_0 < \infty.$$

Put $\alpha_0 = 2\pi/(\sum_{k=1}^{\infty} k/b(s_k/2^k))$. Then $|f|_{\mathbb{T}} = 1$. We will prove that for each $0<\varepsilon<1$,

$$\int_0^{2\pi} \Psi(\varepsilon|f(x)|)\,dx = \infty. \tag{2.13}$$

From (2.11) and (2.12) it follows that

$$\int_0^{2\pi} \Psi(\varepsilon|f(x)|)\,dx = \sum_{k=1}^{\infty} \Psi\left(\frac{\varepsilon\alpha_0 ks_k}{2^k}\right) \frac{1}{(s_k/2^k) b(s_k/2^k)}. \tag{2.14}$$

The formula (1.6) implies that

$$\Psi\left(\frac{\varepsilon\alpha_0 ks_k}{2^k}\right) = \int_0^{\varepsilon\alpha_0 ks_k/2^k} b(s)\,ds = \int_0^{\varepsilon\alpha_0 ks_k/(c_0 2^k)} c_0 b(c_0 t)\,dt.$$

We can choose a positive integer k_0 sufficiently large so that $\varepsilon\alpha_0 k/c_0 \ge 2$ for all $k \ge k_0$. Then we get

$$\Psi\Big(\frac{\varepsilon\alpha_0 k s_k}{2^k}\Big) \ge \int\limits_0^{2s_k/2^k} c_0 b(c_0 t)\,dt \tag{2.15}$$

$$\ge \int\limits_{s_k/2^k}^{2s_k/2^k} c_0 b(c_0 t)\,dt \quad \text{for all } k \ge k_0.$$

Since $b(s)$ is quasi-increasing, it follows from (2.15) that

$$\Psi\Big(\frac{\varepsilon\alpha_0 k s_k}{2^k}\Big) \ge \frac{s_k}{2^k} b\Big(\frac{s_k}{2^k}\Big) \quad \text{for all } k \ge k_0. \tag{2.16}$$

From (2.14) and (2.16) we get

$$\int\limits_0^{2\pi} \Psi(\varepsilon|f(x)|)\,dx \ge \sum_{k=k_0}^{\infty} \frac{s_k}{2^k} b\Big(\frac{s_k}{2^k}\Big) \frac{1}{(s_k/2^k) b(s_k/2^k)} = \sum_{k=k_0}^{\infty} 1 = \infty.$$

Thus (2.13) holds.

Now we prove that $\int_0^{2\pi} \Phi(Mf(x))\,dx < \infty$. Put

$$F(x) = \Phi(Mf(x))\chi_{\{Mf>1\}}(x).$$

Then

$$J_1 := \int\limits_{Mf>1} \Phi(Mf(x))\,dx = \int\limits_0^{2\pi} F(x)\,dx = \int\limits_0^{\infty} |\{F > \lambda\}|\,d\lambda$$

$$\le 2\pi\Phi(1) + \int\limits_{\Phi(1)}^{\infty} |\{F > \lambda\}|\,d\lambda = 2\pi\Phi(1) + \int\limits_{\Phi(1)}^{\infty} |\{\Phi(Mf) > \lambda\}|\,d\lambda$$

$$= 2\pi\Phi(1) + \int\limits_1^{\infty} |\{Mf > \Phi^{-1}(\lambda)\}|\,d\lambda$$

$$= 2\pi\Phi(1) + \int\limits_1^{\infty} |\{Mf > t\}| a(t)\,dt.$$

Since the operator M is simultaneously of weak-type (1,1) and of type (∞,∞), it follows by the well known result (see Torchinsky [8], p. 92) that there exist positive constants c_7 and c_8 such that

$$|\{Mf > t\}| \le \frac{c_7}{t} \int\limits_{t/c_8}^{\infty} |\{|f| > s\}|\,ds \quad \text{for all } t > 0. \tag{2.17}$$

Therefore

$$J_1 \le 2\pi\Phi(1) + c_7 \int\limits_1^{\infty} \frac{a(t)}{t} \Big(\int\limits_{t/c_8}^{\infty} |\{|f| > s\}|\,ds\Big)\,dt$$

$$= 2\pi\Phi(1) + c_7 \int\limits_{1/c_8}^{\infty} |\{|f| > s\}| \Big(\int\limits_1^{c_8 s} \frac{a(t)}{t}\,dt\Big)\,ds$$

$$:= 2\pi\Phi(1) + c_7 J_2.$$

It remains to estimate J_2. By (2.10) we have

$$0 < \frac{\alpha_0 s_1}{2} < \ldots < \frac{\alpha_0 k s_k}{2^k} < \ldots \uparrow \infty \quad \text{as } k \uparrow \infty; \tag{2.18}$$

$$0 < \frac{s_k}{2^k} < \frac{1}{c_0} \cdot \frac{s_{k+1}}{2^{k+1}} \quad \text{for all } k \ge 1. \tag{2.19}$$

Since $b(s)$ is quasi-increasing, it follows from (2.19) that

$$b\Big(\frac{s_k}{2^k}\Big) \le c_0 b\Big(\frac{s_{k+1}}{2^{k+1}}\Big) \quad \text{for all } k \ge 1. \tag{2.20}$$

Therefore, by (2.10) and (2.20), it follows that

$$|I_{k+1}| = \frac{1}{\frac{s_{k+1}}{2^{k+1}} b\Big(\frac{s_{k+1}}{2^{k+1}}\Big)} \le \frac{1}{\frac{s_{k+1}}{2^{k+1}} \cdot \frac{1}{c_0} b\Big(\frac{s_k}{2^k}\Big)}$$

$$\le \frac{1}{\frac{4c_0 s_k}{2^{k+1}} \cdot \frac{1}{c_0} b\Big(\frac{s_k}{2^k}\Big)} = \frac{1}{2} \cdot \frac{1}{\frac{s_k}{2^k} b\Big(\frac{s_k}{2^k}\Big)} = \frac{1}{2}|I_k|.$$

Hence

$$|I_{k+1}| \le \frac{1}{2}|I_k| \quad \text{for all } k \ge 1. \tag{2.21}$$

If $\alpha_0(k-1)s_{k-1}/2^{k-1} \le s < \alpha_0 k s_k/2^k$, then it follows from (2.18) and (2.21) that

$$|\{|f| > s\}| = \sum_{j=k}^{\infty} |I_j| \le 2|I_k| \quad \text{for all } k \ge 1. \tag{2.22}$$

We choose a positive integer k_1 such that $c_8\alpha_0 k/2^k \le 1$ for all $k \ge k_1$. If $\alpha_0(k-1)s_{k-1}/2^{k-1} \le s < \alpha_0 k s_k/2^k$, then it follows from (2.8) that

$$\int\limits_1^{c_8 s} \frac{a(t)}{t}\,dt \le \int\limits_1^{c_8\alpha_0 k s_k/2^k} \frac{a(t)}{t}\,dt \tag{2.23}$$

$$\le \int\limits_1^{s_k} \frac{a(t)}{t}\,dt < \frac{1}{2^k} b\Big(\frac{s_k}{2^k}\Big) \quad \text{for } k \ge k_1.$$

From (2.22) and (2.23) we get

$$\int\limits_{\alpha_0(k_1-1)s_{k_1-1}/2^{k_1-1}}^{\infty} |\{|f|>s\}|\Big(\int\limits_1^{c_8 s}\frac{a(t)}{t}\,dt\Big)\,ds$$
$$= \sum_{k=k_1}^{\infty}\int\limits_{\alpha_0(k-1)s_{k-1}/2^{k-1}}^{\alpha_0 k s_k/2^k} |\{|f|>s\}|\Big(\int\limits_1^{c_8 s}\frac{a(t)}{t}\,dt\Big)\,ds$$
$$\le \sum_{k=k_1}^{\infty}\int\limits_{\alpha_0(k-1)s_{k-1}/2^{k-1}}^{\alpha_0 k s_k/2^k} 2|I_k|\cdot\frac{1}{s_k|I_k|}\,ds$$
$$\le \sum_{k=k_1}^{\infty}\frac{2}{s_k}\cdot\frac{\alpha_0 k s_k}{2^k} = \sum_{k=k_1}^{\infty}\frac{2\alpha_0 k}{2^k} < \infty,$$

which implies that $\int_0^{2\pi}\Phi(Mf(x))\,dx < \infty$. We arrive at a contradiction and the theorem is proved.

COROLLARY 2.2. *Let $a(s)$, $b(s)$, $\Phi(t)$ and $\Psi(t)$ satisfy* (1.3)–(1.6). *Then the following statements are equivalent*:

(j) *there exist positive constants c_3 and $s_0 > 1$ such that*

$$\int\limits_1^{s}\frac{a(t)}{t}\,dt \ge c_3 b(c_3 s) \quad \textit{for all } s \ge s_0;$$

(jjj) *if $Mf \in L^\Phi$ for $f \in L^1(\mathbb{T})$, then $f \in L^\Psi$.*

Proof. (j)⇒(jjj). Let $f \in L^1(\mathbb{T})$. We can choose s_0 such that $s_0 > |f|_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get

$$\int\limits_0^{2\pi}\Psi(c_3|f(x)|)\,dx \le c(|f|_{\mathbb{T}}) + \frac{1}{c_6}\int\limits_0^{2\pi}\Phi(Mf(x))\,dx, \tag{2.24}$$

where $c(|f|_{\mathbb{T}})$ is a constant depending on $|f|_{\mathbb{T}}$.

Now suppose that $f \in L^1(\mathbb{T})$ and $Mf \in L^\Phi$. Then there exists $0 < \varepsilon_1 < 1$ such that $\int_0^{2\pi}\Phi(\varepsilon_1 Mf(x))\,dx < \infty$. Therefore it follows from (2.24) that $\int_0^{2\pi}\Psi(c_3\varepsilon_1|f(x)|)\,dx < \infty$, which implies that $f \in L^\Psi$.

(jjj)⇒(j). Let (jjj) hold and assume that (j) does not hold. In the proof of Theorem 2.1, we constructed a function $f \in L^1(\mathbb{T})$ such that

$$\int\limits_0^{2\pi}\Phi(Mf(x))\,dx < \infty \quad\text{and}\quad \int\limits_0^{2\pi}\Psi(\varepsilon|f(x)|)\,dx = \infty \quad\text{for each } 0 < \varepsilon < 1,$$

which contradicts our assumption.

COROLLARY 2.3. *Let $a(s)$, $b(s)$, $\Phi(t)$ and $\Psi(t)$ satisfy* (1.3)–(1.6). *Then the following statements are equivalent*:

(1) *there exist positive constants c_1, c_2 and $s_0 > 1$ such that*

$$c_2 b(c_2 s) \le \int\limits_1^{s}\frac{a(t)}{t}\,dt \le c_1 b(c_1 s) \quad \textit{for all } s \ge s_0 > 1;$$

(2) *suppose $f \in L^1(\mathbb{T})$; then $Mf \in L^\Phi$ if and only if $f \in L^\Psi$.*

Proof. This is an easy consequence of Theorem 1.2 and Corollary 2.2.

3. Examples. In this section, some examples of functions $\Phi(t), \Psi(t), a(t)$ and $b(t)$ will be given. Let $\varphi(t)$ and $\psi(t)$ be functions defined on $[0,\infty)$. We write $\varphi(t) \sim \psi(t)$ if there exist positive constants c_1, c_2 and t_0 such that $c_1\psi(t) \le \varphi(t) \le c_2\psi(t)$ for all $t \ge t_0$.

EXAMPLE 1. Let $1 < p < \infty$ and

$$\Phi(t) = \frac{1}{p}t^p,\quad a(t) = t^{p-1}\quad \text{for } t \ge 0;$$
$$\Psi(t) = \frac{1}{p}t^p,\quad b(t) = t^{p-1}\quad \text{for } t \ge 0.$$

EXAMPLE 2. Let $0 < \alpha \le 1$ and

$$\Phi(t) \sim \frac{t}{(\log t)^{1-\alpha}},\quad a(t) \sim \frac{1}{(\log t)^{1-\alpha}};$$
$$\Psi(t) \sim t(\log t)^{\alpha},\quad b(t) \sim (\log t)^{\alpha}.$$

EXAMPLE 3. Let

$$\Phi(t) \sim \frac{t}{\log t},\quad a(t) \sim \frac{1}{\log t};$$
$$\Psi(t) \sim t(\log\log t),\quad b(t) \sim \log\log t.$$

EXAMPLE 4. Put $L_1(t) = \log^+ t$, $L_n(t) = \log^+ L_{n-1}(t)$ for $n \ge 2$, and

$$\Phi(t) \sim \frac{t}{L_1(t)L_2(t)\ldots L_{n-1}(t)},\quad a(t) \sim \frac{1}{L_1(t)L_2(t)\ldots L_{n-1}(t)};$$
$$\Psi(t) \sim tL_n(t),\quad b(t) \sim L_n(t).$$

References

[1] M. de Guzmán, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Math. 481, Springer, Berlin, 1975.

[2] H. Kita, *On maximal functions in Orlicz spaces*, submitted.

[3] H. Kita and K. Yoneda, *A treatment of Orlicz spaces as a ranked space*, Math. Japon. 37 (1992), 775–802.

[4] V. Kokilashvili and M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Sci., 1991.
[5] G. Moscariello, *On the integrability of the Jacobian in Orlicz spaces*, Math. Japon. 40 (1994), 323–329.
[6] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, 1991.
[7] E. M. Stein, *Note on the class* $L \log L$, Studia Math. 31 (1969), 305–310.
[8] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, New York, 1985.
[9] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1959.

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RNP and KMP are equivalent for some Banach spaces with shrinking basis

by

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Abstract. We get a characterization of PCP in Banach spaces with shrinking basis. Also, we prove that the Radon–Nikodym and Krein–Milman properties are equivalent for closed, convex and bounded subsets of some Banach spaces with shrinking basis.

Introduction. We begin by recalling some geometrical properties in Banach spaces (see [3]–[5]).

Let X be a Banach space and let C be a closed, bounded, convex and nonempty subset of X.

C is said to have the *point of continuity property* (PCP) if for every closed, bounded and nonempty subset F of C, the identity map from (F, weak) into $(F, \|\ \|)$ has some point of continuity.

C is said to have the *convex point of continuity property* (CPCP) if for every closed, bounded, convex and nonempty subset F of C, the identity map from (F, weak) into $(F, \|\ \|)$ has some point of continuity.

C is said to have the *Radon–Nikodym property* (RNP) if for every measure space (Ω, Σ, μ) and for every μ-continuous vector measure $F : \Sigma \to X$ such that

$$F(A)/\mu(A) \in C \quad \forall A \in \Sigma,\ \mu(A) > 0,$$

there is $f : \Omega \to X$ Bochner integrable with

$$F(A) = \int_A f\, d\mu \quad \forall A \in \Sigma.$$

C is said to have the *Krein–Milman property* (KMP) if for every closed, bounded, convex and nonempty subset F of C, we have

$$F = \overline{\text{co}}(\text{Ext}\, F),$$

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