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ON (α, δ)-SKEW ARMENDARIZ RINGS

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ABSTRACT. For a ring endomorphism α and an α-derivation δ, we introduce (α, δ)-skew Armendariz rings which are a generalization of α-rigid rings and Armendariz rings, and investigate their properties. A semiprime left Goldie ring is α-weak Armendariz if and only if it is α-rigid. Moreover, we study on the relationship between the Baerness and p.p. property of a ring R and these of the skew polynomial ring $R[x; \alpha, \delta]$ in case R is (α, δ)-skew Armendariz. As a consequence we obtain a generalization of [11], [14] and [16].

Throughout this paper R denotes an associative ring with unity, $\alpha : R \longrightarrow R$ is an endomorphism and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. We denote $S = R[x; \alpha, \delta]$ the Öre extension whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. Recall that a ring R is *reduced* if R has no nonzero nilpotent elements. Observe that reduced rings are *abelian* (i.e., all idempotents are central).

According to Krempa[15], an endomorphism α of a ring R is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We call a ring R α-rigid if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced rings by Hong et al.[10]. Properties of α-rigid rings have been studied in Krempa[15], Hong[10], and Hirano[8].

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In [9] Hong et al. defines a ring R with an endomorphism α to be *α-skew Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) = 0$ for each i, j. Motivated by results in Armendariz[2], Anderson and Camillo[1], Tsiu-Kwen Lee and Tsai-Lien Wong[16], and Hong et al.[9], we investigate a generalization of α-rigid rings and Armendariz rings which we call an (α, δ)-skew Armendariz ring.

Definition 1. Let α be an endomorphism and δ be an α-derivation of a ring R. We say that R is an *(α, δ)-skew Armendariz* (or simply, (α, δ)-Armendariz) ring, if for polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[x;\alpha,\delta]$, $f(x)g(x) = 0$ implies $a_ix^ib_jx^j = 0$ for each i, j.

Note that each α-skew Armendariz ring is (α, δ)-skew Armendariz, where δ is the zero mapping. As it is mentioned in Hong et al. [9], if R is an Armendariz ring, then it is an I_R-Armendariz ring, where I_R is the identity endomorphism of R and thus every reduced ring R is I_R-Armendariz. However, there exists an I_R-Armendariz ring R which is not reduced. For example, $R = \mathbb{Z}_{n^2}$, where $\mathbb{Z}_{n^2}$ is the ring of integers modulo n^2 and $n \geq 2$ is a positive integer, is a commutative Armendariz ring which is not reduced. Thus R is an I_R-Armendariz ring, but it is not I_R-rigid.

Clearly every subring of an (α, δ)-Armendariz ring is (α, δ)-Armendari z. Also, every α-rigid ring is (α, δ)-Armendariz, but the converse dose not hold [9, Example 1].

Definition 2. Let α be an endomorphism and δ be an α-derivation of R. We say that R is an *(α, δ)-skew weak Armendariz* (or simply (α, δ)-weak Armendariz) ring, if for linear polynomials $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x$ in $R[x;\alpha,\delta]$, $f(x)g(x) = 0$ implies $a_ix^ib_jx^j = 0$ for $i, j \in \{0, 1\}$.

It is clear that every (α, δ)-Armendariz ring is (α, δ)-weak Armendariz and that every subring of an (α, δ)-weak Armendariz ring is still (α, δ)-weak Armendariz. However, an (α, δ)-weak Armendariz ring is not necessarily (α, δ)-Armendariz in general [16, Example 3.2]. But, weak Armendariz von Neumann regular rings are reduced [1, Theorem 6]. In this paper we give an affirmative answer to a question of Hong et al. [9, page 115] and show that, a ring R with a monomorphism α and α-derivation δ, is α-rigid if and only if it is reduced and (α, δ)-weak Armendariz. Moreover we provide several examples of non semiprime

(hence non α-rigid) rings which are (α, δ)-weak Armendariz. Furthermore we prove that, for a semiprime left Goldie ring R with a monomorphism α, R is an α-rigid ring if and only if R is an α-weak Armendariz ring if and only if its classical left quotient ring $Q(R)$ is $\widetilde{\alpha}$-weak Armendariz. Finally, we show that, if R is an (α, δ)-Armendariz ring, then (1) $R[x; \alpha, \delta]$ is an abelian ring, (2) R is a Baer (resp. p.p.-) ring if and only if $R[x; \alpha, \delta]$ is a Baer (resp. p.p.-) ring.
The following results provide some properties of (α, δ)-Armendariz rings, which is needed in the sequel:

LEMMA 3. *Let R be an (α, δ)-weak Armendariz ring and $ab = 0$. Then $\alpha(a)\delta(b) = \delta(a)b = 0$.*

Proof. Since $ab = 0$, $\delta(a)b + \alpha(a)\delta(b) = 0$. Put $f(x) = \delta(a) + \alpha(a)x$ and $g(x) = b + bx$ in $R[x; \alpha, \delta]$. Since $f(x)g(x) = 0$, we have $\delta(a)b = \alpha(a)\delta(b) = 0$. □

LEMMA 4. *Let R be an (α, δ)-weak Armendariz ring. Then for each idempotent element $e \in R$, we have $\alpha(e) = e$ and $\delta(e) = 0$.*

Proof. Since $e = e^2$, we have $\delta(e) = \delta(e)e + \alpha(e)\delta(e)$. Let $f(x) = \delta(e) + \alpha(e)x$ and $g(x) = (e-1) + (e-1)x$. Then $f(x)g(x) = \delta(e)e - \delta(e) + \alpha(e)\delta(e) + (\delta(e)e - \delta(e) + \alpha(e)\delta(e))x = 0$. Since R is (α, δ)-Armendariz, $\delta(e)e = \delta(e)$ and hence $\alpha(e)\delta(e) = 0$. Now suppose that, $h(x) = \delta(e) - (1 - \alpha(e))x$ and $k(x) = e + ex$. Then $h(x)k(x) = (\delta(e)e - \delta(e) + \alpha(e)\delta(e)) + (\delta(e)e - (1 - \alpha(e))\delta(e))x = 0$. Hence $\delta(e)e = 0$ and so $\delta(e) = \delta(e)e = 0$.
Now take $p(x) = (1 - e) + (1 - e)\alpha(e)x$ and $q(x) = e + (e - 1)\alpha(e)x$. Then $p(x)q(x) = 0$, since $\delta(e) = \delta(\alpha(e)) = 0$. Hence $\alpha(e) = e\alpha(e)$. Now suppose that $t(x) = e + e(1 - \alpha(e))x$ and $s(x) = (1 - e) - e(1 - \alpha(e))x$. Then $t(x)s(x) = -e(1 - \alpha(e))(\delta(e) - \delta(e\alpha(e)))x = 0$, since $\delta(e) = \delta(e\alpha(e)) = 0$. Hence $e(1-\alpha(e)) = 0$, since R is (α, δ)-Armendariz. Therefore $e = e\alpha(e) = \alpha(e)$. □

In the following lemma we employ the same method in the proof of [9, Lemma 19]:

LEMMA 5. *Let R be an (α, δ)-Armendariz ring. If $e^2 = e \in R[x; \alpha, \delta]$, where $e = e_0 + e_1x + \cdots + e_nx^n$, then $e = e_0$.*

Proof. Since $e(1-e) = 0 = (1-e)e$, we have $(e_0+e_1x+\cdots+e_nx^n)((1-e_0) - e_1x - \cdots - e_nx^n) = 0$ and $((1-e_0) - e_1x - \cdots - e_nx^n)(e_0 + e_1x + \cdots + e_nx^n) = 0$. Hence, $e_0(1 - e_0) = 0$, $e_0e_i = 0$ and $(1 - e_0)e_i = 0$ for $1 \le i \le n$, since R is (α, δ)-Armendariz. Thus $e_i = 0$ for $1 \le i \le n$, and so $e = e_0 = e_0^2$. □

Now we give an affirmative answer to a question of Hong et al. [9, Page 115]:

THEOREM 6. *A ring R with a monomorphism α, is an α-rigid ring if and only if it is a reduced (α,δ)-weak Armendariz ring.*

Proof. It is clear that an α-rigid ring is reduced and by [10, Proposition 6] R is (α,δ)-weak Armendariz. Now, suppose that R is a reduced ring and $a\alpha(a)=0$ for $a\in R$. By Lemma 3, $\delta(a)\alpha(a)=\alpha(a)\delta(\alpha(a))=0$. Now, let $h(x)=\alpha(a)-\alpha(a)x$ and $k(x)=a+\alpha(a)x$. Then $h(x)k(x)=0$. Since R is (α,δ)-weak Armendariz, we have $\alpha(a)\alpha(a)=0$. Hence $a=0$, since R is reduced and α is a monomorphism. □

Next we provide several examples of (α,δ)-weak Armendariz rings:

Let R be a ring and let

$$T_n(R) := \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \end{pmatrix} \mid a_i\in R \right\},$$

with $n\geq 2$. We can denote elements of $T_n(R)$ by $(a_1,a_2,\dots,a_n)$. Then $T_n(R)$ is a ring with addition point-wise and multiplication given by $(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n)=(a_1b_1,a_1b_2+a_2b_1,\dots,a_1b_n+a_2b_{n-1}+\dots+a_nb_1)$, for each $a_i,b_j\in R$. For an endomorphism α and an α-derivation δ of R, the natural extension $\overline{\alpha}:T_n(R)\to T_n(R)$ defined by $\overline{\alpha}((a_i))=(\alpha(a_i))$ is an endomorphism of $T_n(R)$ and $\overline{\delta}:T_n(R)\to T_n(R)$ defined by $\overline{\delta}((a_i))=(\delta(a_i))$, is an $\overline{\alpha}$-derivation of $T_n(R)$.

LEMMA 7. *Let α be an endomorphism and δ an α-derivation of a ring R. Let R be an α-rigid ring, $A=(a_1,a_2,a_3,\dots,a_{n+1})$ and $B=(b_1,b_2,b_3,\dots,b_{n+1})\in T_{n+1}(R)$. If $AB=0$, then $A\overline{\delta}(B)=0$. Also $AB=0$ if and only if $A\overline{\alpha}(B)=0$.*

Proof. Let $A=(a_1,a_2)$ and $B=(b_1,b_2)\in T_2(R)$ with $AB=0$. Then $a_1b_1=a_1b_2+a_2b_1=0$. Since R is reduced, $a_1b_2=a_2b_1=0$. Now suppose that $A=(a_1,a_2,a_3,\dots,a_{n+1})$ and $B=(b_1,b_2,b_3,\dots,b_{n+1})\in T_{n+1}(R)$ with $AB=0$. Then we have the following system of equations:

$a_1b_1=0$;
$a_1b_2+a_2b_1=0$;
$a_1b_3+a_2b_2+a_3b_1=0$;

$\vdots$
$a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 = 0;$
$a_1b_{n+1} + a_2b_n + \cdots + a_{n+1}b_1 = 0.$
Hence $a_1b_j = 0$ for all $1 \leq j \leq n+1$, since R is reduced. Thus
$a_2b_1 = 0;$
$a_2b_2 + a_3b_1 = 0;$
$\vdots$
$a_2b_{n-1} + \cdots + a_nb_1 = 0;$
$a_2b_n + \cdots + a_{n+1}b_1 = 0,$
and hence $(a_2, a_3, \ldots, a_{n+1})(b_2, b_3, \ldots, b_{n+1}) = 0$. By induction hypothesis, we have $a_ib_j = 0$ for all $2 \leq i \leq n+1$ and $2 \leq j \leq n+1-i+1$. Thus $a_ib_j = 0$ for all $1 \leq i \leq n+1$ and $1 \leq j \leq n-i+1$. Hence $a_i\alpha(b_j) = a_i\delta(b_j) = 0$ for all $1 \leq i \leq n+1$ and $1 \leq j \leq n-i+1$, since R is α-rigid. Therefore $A\overline{\alpha}(B) = A\overline{\delta}(B) = 0$. By a similar way one can show that, $A\overline{\alpha}(B) = 0$ implies $AB = 0$. □

THEOREM 8. *Let α be a monomorphism and δ an α-derivation of a ring R. Then R is an α-rigid ring if and only if $T_n(R)$ is an $(\overline{\alpha}, \overline{\delta})$-weak Armendariz ring, for each $n \geq 2$.*

Proof. One can show that the map $\varphi : T_n(R)[x; \overline{\alpha}, \overline{\delta}] \to T_n(R[x; \alpha, \delta])$, given by $\varphi(A_0 + A_1x + \cdots + A_kx^k) = (g_1, g_2, g_3, \ldots, g_n)$ with $A_i = (a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in})$ and $g_j = a_{0j} + a_{1j}x + \cdots + a_{kj}x^k$, for all $0 \leq i \leq k$ and $1 \leq j \leq n$, is an isomorphism. Now suppose that $f = A_0 + A_1x + \cdots + A_tx^t$ and $g = B_0 + B_1x + \cdots + B_mx^m$ are polynomials in $T_n(R)[x; \overline{\alpha}, \overline{\delta}]$ with $fg = 0$, where $A_i = (a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in})$ and $B_j = (b_{i1}, b_{i2}, b_{i3}, \ldots, b_{in})$ for all $0 \leq i \leq t$ and $0 \leq j \leq m$. By Lemma 7, $f_ig_j = 0$ where $f_i = a_{0i} + a_{1i}x + \cdots + a_{ti}x^t$ and $g_j = b_{0j} + b_{1j}x + \cdots + b_{mj}x^m$ for $1 \leq i \leq n$ and $1 \leq j \leq n-i+1$. Since R is α-rigid, we have $a_{ri}b_{sj} = 0$ for $1 \leq i \leq n$, $1 \leq j \leq n-i+1$, $0 \leq r \leq t$ and $0 \leq s \leq m$. Hence $A_rB_s = 0$ for $0 \leq r \leq t$ and $0 \leq s \leq m$. Thus $A_r\overline{\alpha}(B_s) = A_r\overline{\delta}(B_s) = 0$, by Lemma 7. Therefore $T_n(R)$ is $(\overline{\alpha}, \overline{\delta})$-Armendariz.

Conversely, suppose that $T_n(R)$ is $(\overline{\alpha}, \overline{\delta})$-weak Armendariz. Hence R is (α, δ)-weak Armendariz as a subring. Let $r\alpha(r) = 0$, for $r \in R$. Then $\delta(r)\alpha(r) = \alpha(r)\delta\alpha(r) = 0$, by Lemma 3. Take $h(x) = (0, 0, 1, \ldots, 0) - (0, \alpha(r), 0, \ldots, 0)x$ and $k(x) = (0, 0, \ldots, 1, 0) + (0, \ldots, \alpha(r), 0, 0)x \in T_n(R)$ $[x; \overline{\alpha}, \overline{\delta}]$. Then $h(x)k(x) = 0$. Since $T_n(R)[x; \overline{\alpha}, \overline{\delta}]$ is $(\overline{\alpha}, \overline{\delta})$-weak Armendariz, $(0, 0, 1, \ldots, 0)(0, \ldots, \alpha(r), 0, 0) = 0$ and hence $\alpha(r) = 0$. Therefore $r = 0$, since α is monomorphism.

By [10, Proposition 6], every α-rigid ring is (α, δ)-Armendariz. The following is an example of a non semiprime (hence non α-rigid) ring which is (α, δ)-Armendariz:

EXAMPLE 9. Let α be an endomorphism and δ be an α-derivation of R. Let R be an α-rigid ring and

$$R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

be a subring of the upper triangular matrix ring over a ring. The endomorphism α of R is extended to the endomorphism $\overline{\alpha} : R_3 \to R_3$ defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ and the α-derivation δ of R is also extended to $\overline{\delta} : R_3 \to R_3$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. We can easily see that $\overline{\delta}$ is an $\overline{\alpha}$-derivation of R_3. We show that, (i) for A, $B \in R_3$, $AB = 0$ if and only if $A\overline{\alpha}(B) = 0$, also $AB = 0$ implies that $A\overline{\delta}(B) = 0$, (ii) R_3 is not $\overline{\alpha}$-rigid, (iii) R_3 is $(\overline{\alpha}, \overline{\delta})$-Armendariz.

(i) Suppose that $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \overline{\alpha} \left(\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \right) = 0$. Then we have the following equations:

$$a_1\alpha(a_2) = 0 \tag{1}$$

$$a_1\alpha(b_2) + b_1\alpha(a_2) = 0 \tag{2}$$

$$a_1\alpha(c_2) + b_1\alpha(d_2) + c_1\alpha(a_2) = 0 \tag{3}$$

$$a_1\alpha(d_2) + d_1\alpha(a_2) = 0. \tag{4}$$

Since R is reduced, from Eq.(1), we have $\alpha(a_2)a_1 = 0$. Multiplying $\alpha(a_2)$ to Eq.(2) from the left-hand side, then we have $b_1\alpha(a_2) = \alpha(a_2)b_1 = 0$, since R is reduced. Hence $a_1\alpha(b_2) = \alpha(b_2)a_1 = 0$. Multiplying $\alpha(a_2)$ to Eq.(3) from the left-hand side, then we have $c_1\alpha(a_2) = \alpha(a_2)c_1 = 0$, since R is reduced. Hence Eq.(3) becomes

$$a_1\alpha(c_2) + b_1\alpha(d_2) = 0. \tag{5}$$

Multiplying $\alpha(a_2)$ to Eq.(4) from left-hand side, then $d_1\alpha(a_2) = \alpha(a_2)d_1 = 0$ and $a_1\alpha(d_2) = \alpha(d_2)a_1 = 0$, since R is reduced. Multiplying a_1 to Eq.(5) from the right-hand side, then we have $a_1\alpha(c_2) = \alpha(c_2)a_1 = 0$ and $b_1\alpha(d_2) = \alpha(d_2)b_1 = 0$. Thus $a_1a_2 = a_1b_2 = a_1c_2 = a_1d_2 = b_1a_2 = b_1d_2 = c_1a_2 = d_1a_2 = 0$, since R is α-rigid.

Hence $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} = 0.$

Now assume that $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} = 0.$

Then by a similar argument, we have $a_1\alpha(a_2) = a_1\alpha(b_2) = b_1\alpha(a_2) = c_1\alpha(a_2) = d_1\alpha(a_2) = a_1\alpha(d_2) = a_1\alpha(c_2) = b_1\alpha(d_2) = 0$.

Hence $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \overline{\alpha} \left(\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \right) = 0.$

Assume that $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} = 0.$

Then $a_1a_2 = a_1b_2 = a_1c_2 = a_1d_2 = b_1a_2 = b_1d_2 = c_1a_2 = d_1a_2 = 0$. Hence $a_1\delta(a_2) = a_1\delta(b_2) = a_1\delta(c_2) = a_1\delta(d_2) = b_1\delta(a_2) = b_1\delta(d_2) = c_1\delta(a_2) = d_1\delta(a_2) = 0$, since R is α-rigid.

Thus $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \overline{\delta} \left(\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \right) = 0.$

(ii) Since R_3 is not reduced, R_3 is not $\overline{\alpha}$-rigid.

(iii) Suppose that $p = \sum_{i=0}^{m} \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} x^i$ and

$q = \sum_{j=0}^{n} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} x^j$ are polynomials in $R_3[x; \overline{\alpha}, \overline{\delta}]$ such that $pq = 0$. By Hong et al. [10, Proposition 6], $R[x; \alpha, \delta]$ is reduced. So that by Hong et al. [9, Proposition 17],

$\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \overline{\alpha}^i \left(\begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} \right) = 0$, for all i, j. Hence,

$\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \begin{pmatrix} a_j' & b_j' & c_j' \\ 0 & a_j' & d_j' \\ 0 & 0 & a_j' \end{pmatrix} = 0$ for all i, j. Therefore by (i) R_3 is $(\overline{\alpha}, \overline{\delta})$-Armendariz.

Suppose that R is a semiprime left Goldie ring and α is a monomorphism of R. Let $\mathcal{C} = \mathcal{C}_R(0)$ be the regular elements of R. By Goldie's Theorem [5, Theorem 5.10], R has a semisimple Artinian quotient ring $Q = \mathcal{C}^{-1}R$. By [12, Proposition 2.4], we have $\alpha^{-1}(\mathcal{C}_R(0)) = \mathcal{C}_R(0)$. We extend α to Q, with $\widetilde{\alpha}(c^{-1}r) = \alpha(c)^{-1}\alpha(r)$, for $c \in \mathcal{C}_R(0)$ and $r \in R$. Now we consider the classical left quotient ring $Q = \mathcal{C}^{-1}R$ of an α-Armendariz semiprime left Goldie ring R, in the following:

THEOREM 10. *Let R be a semiprime left Goldie ring and α be a monomorphism of R such that $\alpha(1) = 1$. Then R is an α-Armendariz (resp. α-weak Armendariz) ring if and only if $Q = \mathcal{C}^{-1}R$ is an $\widetilde{\alpha}$-Armendariz (resp. $\widetilde{\alpha}$-weak Armendariz) ring.*

Proof. Let $f(x) = c_0^{-1}a_0 + c_1^{-1}a_1x + \cdots + c_n^{-1}a_nx^n$ and $g(x) = s_0^{-1}b_0 + s_1^{-1}b_1x + \cdots + s_m^{-1}b_mx^m$ be polynomials in $Q[x;\widetilde{\alpha}]$ such that $f(x)g(x) = 0$. Since $\mathcal{C}$ is a left denominator set, there exist $c, s \in \mathcal{C}$ and $a_i', b_j' \in R$ such that $c_i^{-1}a_i = c^{-1}a_i'$ and $s_j^{-1}b_j = s^{-1}b_j'$ for $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$. Then $(a_0' + a_1'x + \cdots + a_n'x^n)s^{-1}(b_0' + b_1'x + \cdots + b_m'x^m) = 0$. Thus $(a_0's^{-1} + a_1'\alpha(s)^{-1}x + \cdots + a_n'\alpha^n(s)^{-1}x^n)(b_0' + b_1'x + \cdots + b_m'x^m) = 0$. There exist $s' \in \mathcal{C}$ and $a_i'' \in R$ such that $a_i'\alpha^i(s)^{-1} = s'^{-1}a_i''$ for $i = 0, 1, \ldots, n$. Hence $(a_0'' + a_1''x + \cdots + a_n''x^n)(b_0' + b_1'x + \cdots + b_m'x^m) = 0$. Since R is α-Armendariz, $a_i''\alpha^i(b_j') = 0$ for $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$. Hence $s'^{-1}a_i''\alpha^i(b_j') = 0$ for $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$. Thus $a_i'\alpha^i(s^{-1}b_j') = a_i'\alpha^i(s)^{-1}\alpha^i(b_j') = 0$ for $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$ and hence $c_i^{-1}a_i\alpha^i(s_j^{-1}b_j) = c^{-1}a_i'\alpha^i(s^{-1}b_j) = 0$, for $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$. Therefore Q is $\widetilde{\alpha}$-Armendariz. The converse is clear. In the above argument by replacing α-Armendariz with α-weak Armendariz, the result follows. □

Now we extend [14, Proposition 18] and [16, Theorem 3.3], to the more general case as in the following:

THEOREM 11. *Let R be a semiprime left Goldie ring and α be a monomorphism of R such that $\alpha(1) = 1$. Then R is an α-weak Armendariz ring if and only if R is an α-rigid ring.*

Proof. By [9, Corollary 4], every α-rigid ring is α-weak Armendariz. Conversely, suppose that R is a semiprime left Goldie α-weak Armendariz ring. By Theorem 6, it is enough to show that R is reduced. Let $Q = \mathcal{C}^{-1}R$ be the left Goldie quotient ring of R. Then $\widetilde{\alpha} : Q \to Q$ defined by $\widetilde{\alpha}(s^{-1}r) = \alpha(s)^{-1}\alpha(r)$ is a monomorphism of Q. By Theorem 10, Q is $\widetilde{\alpha}$-weak Armendariz. Since R is semiprime left Goldie, Q is semisimple Artinian by [5, Theorem 5.10], and hence $Q \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where D_i are division rings and n_i are positive integers for $1 \leq i \leq k$. We claim that $n_i = 1$ for $1 \leq i \leq k$. First we show that $n_1 = 1$. Since Q ia $\widetilde{\alpha}$-weak Armendariz, by Lemma 4, $\widetilde{\alpha}(e) = e$ for each $e^2 = e \in Q$. Let E_{ij} be the matrix units in M_{n_1} for $1 \leq i \leq n$, $1 \leq j \leq n$. Hence $\widetilde{\alpha}((E_{ii}, 0, \ldots, 0)) = (E_{ii}, 0, \ldots, 0)$ for $1 \leq i \leq n$. Since $E_{1n}E_{nn} = E_{1n}$ and $E_{11}E_{1n} = E_{1n}$, we have $\widetilde{\alpha}((E_{1n}, 0, \ldots, 0))(E_{nn}, 0, \ldots, 0) = \widetilde{\alpha}((E_{1n}, 0, \ldots, 0))$ and $(E_{11}, 0, \ldots, 0)\widetilde{\alpha}((E_{1n}, 0, \ldots, 0)) = \widetilde{\alpha}((E_{1n}, 0, \ldots, 0))$. Thus there exists $a_{1n} \in D_1$ such that $\widetilde{\alpha}((E_{1n}, 0, \ldots, 0)) = (a_{1n}E_{1n}, 0, \ldots, 0)$. Since $E_{n1}E_{11} = E_{n1}$ and $E_{nn}E_{n1} = E_{n1}$, we have $\widetilde{\alpha}((E_{n1}, 0, \ldots, 0))(E_{11}, 0, \ldots, 0) = \widetilde{\alpha}((E_{n1}, 0, \ldots, 0))$ and $(E_{nn}, 0, \ldots, 0)\widetilde{\alpha}((E_{n1}, 0, \ldots, 0)) = \widetilde{\alpha}((E_{n1}, 0, \ldots, 0))$. Thus there exists b_{n1} in D_1 such that $\widetilde{\alpha}((E_{n1}, 0, \ldots, 0)) = (b_{n1}E_{n1}, 0, \ldots, 0)$. Since $E_{1n}E_{n1} = E_{11}$ and $E_{n1}E_{1n} = E_{nn}$, we have $\widetilde{\alpha}((E_{n1}, 0, \ldots, 0))\widetilde{\alpha}((E_{1n}, 0, \ldots, 0)) = (E_{nn}, 0, \ldots, 0)$ and $\widetilde{\alpha}((E_{1n}, 0, \ldots, 0))\widetilde{\alpha}((E_{n1}, 0, \ldots, 0)) = (E_{11}, 0, \ldots, 0)$. Thus $a_{1n}b_{n1} = 1 = b_{n1}a_{1n}$. Now suppose that $f(x) = \widetilde{\alpha}((E_{1n}, 0, \ldots, 0)) - (E_{11}, 0, \ldots, 0)x$ and $g(x) = (E_{1n}, 0, \ldots, 0) + (E_{nn}, 0, \ldots, 0)x$. Then $f(x)g(x) = 0$. Since Q is $\widetilde{\alpha}$-weak Armendariz, $\widetilde{\alpha}((E_{1n}, 0, \ldots, 0))(E_{nn}, 0, \ldots, 0) = 0$. Hence $a_{1n} = 0$. This is a contradiction. Thus $n_1 = 1$. By a similar argument one can show that $n_i = 1$ for $2 \leq i \leq k$. Hence $Q \simeq D_1 \times \cdots \times D_k$ and so Q is reduced. Therefore R is reduced. □

By [10, Proposition 9], a ring R is α-rigid if and only if $R[x; \alpha, \delta]$ is reduced. Next we extend this result and [9, Proposition 20], to the more general setting, in the following:

THEOREM 12. *Let R be an (α, δ)-Armendariz ring. Then $R[x; \alpha, \delta]$ is an abelian ring.*

Proof. By Lemma 4, for each idempotent $e \in R$, $\alpha(e) = e$ and $\delta(e) = 0$. By Lemma 5, the set of idempotent elements of $R[x; \alpha, \delta]$ is a subset of the set of idempotent elements of R. So it is enough to show that R is abelian. Let A be the set of idempotents of R. For each $e, f \in A$, we claim that $efR \cap (1-f)(1-e)A = 0$. Suppose that $ef(-t) = (1-f)(1-e)s \in efR \cap (1-f)(1-e)A$ for some $t \in R$ and

$s \in A$. Let $g(x) = e + (1-f)x$ and $h(x) = (1-e)s + ftx$. Then $g(x)h(x) = [(1-f)\delta(f)t + (1-f)\alpha(f)\delta(t)]x = 0$, since $\delta(1-e) = \delta(f) = 0$, $\alpha(f) = f$ and $ef(-t) = (1-f)(1-e)s$. Hence $eft = 0$, since R is (α, δ)-Armendariz. Thus $efR \cap (1-f)(1-e)A = 0$. Now suppose that $fe = 0$. So $-ef = (1-f)(1-e)f \in efR \cap (1-f)(1-e)A = 0$. Hence $ef = 0$. Next, take $k = e + er(1-e)$ and $l = (1-e) + (1-e)re$, for $r \in R$. It is clear that $k^2 = k, l^2 = l$ and $(1-e)k = el = 0$. Hence $k(1-e) = le = 0$. Thus $er = ere = re$. Therefore R is an abelian ring. □

Now we turn our attention to the relationship between the Baerness and p.p. property of a ring R and these of the skew polynomial ring $R[x; \alpha, \delta]$ in case R is (α, δ)-Armendariz.

Recall that R is a Baer ring if the right annihilator of every non-empty subset of R is generated by an idempotent of R. These definitions are left-right symmetric. A ring R is called a *right* (resp. *left*) *p.p.-ring* if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of R is generated (as a right (resp. left) ideal) by an idempotent of R). R is called a p.p.-ring if it is both right and left p.p.

Now we extend [9 , Theorem 21] and [14, Theorem 10], in the following:

THEOREM 13. *Let α be a monomorphism and δ be an α-derivation of R. If R is an (α, δ)-Armendariz ring, then R is a Baer ring if and only if $R[x; \alpha, \delta]$ is a Baer ring.*

Proof. Since R is (α, δ)-Armendariz, R is abelian by Theorem 12. But everey abelian Baer ring is reduced, so R is α-rigid by Theorem 6. Therefore by [10, Theorem 11], the result follows. □

By a similar proof as Theorem 13, we can obtain:

THEOREM 14. *Let α be a monomorphism and δ be an α-derivation of R. If R is an (α, δ)-Armendariz ring then, R is a p.p.-ring if and only if $R[x; \alpha, \delta]$ is a p.p.-ring.*

Note that Theorem 14, is a generalization of [9, Theorem 22] and [14, Theorem 9].

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