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ON A DECOMPOSITION OF AN EXTENDED CONTRAVARIANT ALMOST ANALYTIC VECTOR IN A COMPACT *K*-SPACE WITH CONSTANT SCALAR CURVATURE

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1. Introduction.

We have defined an another kind of an almost analytic vector in [5], which is called an extended contravariant almost analytic vector, that is, in an almost complex manifold we have called v^i an extended contravariant almost analytic vector if it satisfies

(1. 1) $$\underset{v}{\pounds} F_j{}^i+\lambda F_j{}^r N_{rl}{}^i v^l=0$$

where $\underset{v}{\pounds}$ is the operator of Lie derivation with respect to v^i, $F_j{}^i$ the almost complex structure tensor, λ a scalar function and $N_{ji}{}^h$ the Nijenhuis tensor:

$$N_{ji}{}^h=F_j{}^r(\partial_r F_i{}^h-\partial_i F_r{}^h)-F_i{}^r(\partial_r F_j{}^h-\partial_j F_r{}^h).$$

When $\lambda=0$, (1. 1) is the defining equation of usual contravariant almost analytic vector [6] and when $\lambda=-1/2$, (1. 1) is Satô's contravariant almost (φ, Φ)-analytic vector obtained by the cross-section of a tangent bundle [3].

On the other hand, we have proved that a contravariant almost analytic vector v^i in a compact K-space with constant scalar curvature can be decomposed into the form

(1. 2) $$v^i=p^i+F_s{}^i q^s$$

where p^i and q^i are both Killing vectors [9]. This generalizes the well known Matsushima's theorem [2] and also results of Lichnerowicz [1] and Sawaki [4].

The purpose of the present paper is to prove that an extended contravariant almost analytic vector for a constant λ such that $-3/4\leqq\lambda\leqq 0$ in a compact K-space with constant scalar curvature can be decomposed into the form (1. 2).

In §2 we shall give some definitions and identities. In §3 we shall give a characterization of the extended contravariant almost analytic vector. In §4 we shall prepare some lemmas on the extended contravariant almost analytic vector in a K-space. The last section will be devoted to the proof of the main theorem. Throughout this paper, indices run over the range $1, 2, \cdots, 2n$.

2. Preliminaries.

Let M be a $2n$-dimensional almost-Hermitian manifold which admits an almost

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complex structure tensor $F_j{}^i$ and a positive definite Riemannian metric tensor g_{ji} satisfying

$$F_l{}^i F_j{}^l = -\delta_j^i, \tag{2.1}$$

$$g_{lt} F_j{}^l F_i{}^t = g_{ji}. \tag{2.2}$$

Then from (2.1) and (2.2), we have

$$F_{ji} = -F_{ij} \tag{2.3}$$

where $F_{ji} = F_j{}^l g_{li}$.

In an almost Hermitian manifold, if it satisfies

$$\nabla_j F_{ih} + \nabla_i F_{jh} = 0, \tag{2.4}$$

where ∇_j denotes the operator of covariant derivative with respect to the Riemannian connection, the manifold is called a K-space or Tachibana space.

From (2.4) we have easily

$$\nabla_j F_i{}^j = 0. \tag{2.5}$$

Generally, in an almost complex manifold, a tensor $T_{ji}(T_j{}^i)$ is called pure in j, i, if it satisfies

$$*O_{ji}^{ab} T_{ab} = 0 \qquad (*O_{jb}^{ai} T_a{}^b = 0)$$

and $T_{ji}(T_j{}^i)$ is called hybrid in j, i, if it satisfies

$$O_{ji}^{ab} T_{ab} = 0 \qquad (O_{jb}^{ai} T_a{}^b = 0)$$

where

$$*O_{ji}^{ab} = \frac{1}{2}(\delta_j^a \delta_i^b + F_j{}^a F_i{}^b) \qquad \text{and} \qquad O_{ji}^{ab} = \frac{1}{2}(\delta_j^a \delta_i^b - F_j{}^a F_i{}^b).$$

For instance in an almost-Hermitian manifold, $\nabla_j F_{ih}$ is pure in j, i and g_{ji} is hybrid in j, i.

We have easily the following

PROPOSITION 1. *If $T_j{}^i$ is pure (hybrid) in j, i, then we have*

$$F_t{}^i T_j{}^t = F_j{}^t T_t{}^i \qquad (F_t{}^i T_j{}^t = -F_j{}^t T_t{}^i).$$

PROPOSITION 2. *If S^{ji} is pure (hybrid) in j, i, then we have*

$$F_t{}^j S^{ti} = F_t{}^i S^{jt} \qquad (F_t{}^j S^{ti} = -F_t{}^i S^{jt}).$$

PROPOSITION 3. *If T_{ji} is pure in j, i and $S_j{}^i$ is pure (hybrid) in j, i, then $T_{jr} S_i{}^r$ is pure (hybrid) in j, i.*

PROPOSITION 4. *If T_{ji} is pure in j, i and S^{ji} is hybrid in j, i, then we have $T_{ji} S^{ji} = 0$.*

PROPOSITION 5.[1] *$N_{ji}{}^h$ is pure in j, i and hybrid in i, h.*

1) See Yano [10].

Now in a K-space, let $R_{kji}{}^{h}$ and $R_{ji}=R_{tji}{}^{t}$ be Riemannian curvature tensor and Ricci tensor respectively. Then we have the following identities:[2)]

$$*O_{ji}^{ab}\nabla_a F_{bh}=0, \tag{2.6}$$

$$F_{hk}\nabla^t\nabla_j F_t{}^h=R^*{}_{kj}-R_{jk} \tag{2.7}$$

where $\nabla^t=g^{ta}\nabla_a$ and $R^*{}_{ji}=(1/2)F^{ab}R_{abti}F_j{}^t$.

$$O_{ji}^{ab}R_{ab}=0, \qquad O_{ji}^{ab}R^*{}_{ab}=0, \tag{2.8}$$

$$R^*{}_{ji}=R^*{}_{ij}, \tag{2.9}$$

$$\nabla_j F_{tl}(\nabla_i F^{tl})=R_{ji}-R^*{}_{ji} \tag{2.10}$$

where $F^{ji}=F_t{}^i g^{tj}$,

$$R-R^*=\text{constant} \tag{2.11}$$

where $R=g^{ji}R_{ji}$ and $R^*=g^{ji}R^*{}_{ji}$.

In a Riemannian manifold, we have

$$\frac{1}{2}\nabla_i R=\nabla^j R_{ji} \tag{2.12}$$

and in a K-space

$$\frac{1}{2}\nabla_i R^*=\nabla^j R^*{}_{ji}.\text{[3)]} \tag{2.13}$$

Therefore from (2.11), (2.12) and (2.13), we have

$$\nabla^k(R_{ik}-R^*{}_{ik})=\frac{1}{2}\nabla_i(R-R^*)=0. \tag{2.14}$$

Moreover, for any vector v^i, we have

$$\nabla_k(N_{tl}{}^k\nabla^t v^l)=0 \tag{2.15}$$

and

$$N_{jl}{}^k=4F_j{}^s\nabla_s F_l{}^k. \tag{2.16}$$

3. A characterization of an extended contravariant almost analytic vector.

Let M and $T(M)$ be a $2n$-dimensional almost complex manifold with structure tensor F and a tangent bundle of M respectively. We denote the natural projection $T(M)\to M$ by π. It is well known that a differentiable cross-section f defines a contravariant almost analytic vector if it satisfies that

$$df_p\circ F_p=\Phi_{f(p)}\circ df_p \quad \text{for} \quad p\in M \tag{3.1}$$

where Φ is an almost complex structure on $T(M)$.

2) See Tachibana [7], [8].

3) See Sawaki [4].

Let x^i be local coordinates in a neighborhood U of a fixed point p of M and y^i be the components of a tangent vector v with respect to the natural frame $\partial/\partial x^i$. Then (x^i, y^i) is a local coordinate in a neighborhood $\pi^{-1}(U)$ of $T(M)$.

If we put

$$(3.2)\qquad \begin{cases} \Phi_j{}^i = F_j{}^i, & \Phi_{\bar{j}}{}^i = 0, \\ \Phi_j{}^{\bar{i}} = (\partial_r F_j{}^i) y^r + \lambda F_j{}^s N_{sr}{}^i y^r, & \Phi_{\bar{j}}{}^{\bar{i}} = F_j{}^i, \end{cases}$$

where $\bar{j} = 2n + j$ and λ is a scalar function, then we have a tensor field Φ of type (1, 1) on $T(M)$ whose component are $\Phi_J{}^I$ with respect to the coordinate neighborhood $\pi^{-1}(U)(x^i, y^i)$, and it is easily verified that Φ is an almost complex structure on $T(M)$ by virtue of Proposition 5 where $I, J = 1, 2, \cdots, 4n$.

Now, since cross-section f can be locally expressed by

$$(3.3)\qquad \begin{cases} 'x^i = x^i, \\ 'x^{\bar{i}} = y^i(x^1, x^2, \cdots, x^{2n}) \end{cases}$$

in terms of the local coordinate system (x^i, y^i) on $T(M)$, (3.1) can be written

$$(3.4)\qquad \begin{cases} F_j{}^r \partial_r {}'x^i = \Phi_r{}^i \partial_j {}'x^r + \Phi_{\bar{r}}{}^i \partial_j {}'x^{\bar{r}}, \\ F_j{}^r \partial_r {}'x^{\bar{i}} = \Phi_r{}^{\bar{i}} \partial_j {}'x^r + \Phi_{\bar{r}}{}^{\bar{i}} \partial_j {}'x^{\bar{r}}. \end{cases}$$

The first equation in (3.4) is an identity and from the second equation in (3.4) we have

$$(3.5)\qquad F_j{}^r \partial_r y^i = y^r \partial_r F_j{}^i + \lambda F_j{}^r N_{rl}{}^i y^l + F_r{}^i \partial_j y^r.$$

If we denote the components of vector field v by v^i, (3.5) can be written as

$$\underset{v}{\pounds}\, F_j{}^i + \lambda F_j{}^r N_{rl}{}^i v^l = 0$$

which is nothing but the formula which defines our extended contravariant almost analytic vector.

4. Some lemmas.

In this section, we assume that we are in a K-space. In a K-space, by (2.16), (1.1) turns to

$$(4.1)\qquad \sigma v^r \nabla_r F_j{}^i - F_j{}^r \nabla_r v^i + F_r{}^i \nabla_j v^r = 0$$

or

$$(4.2)\qquad \sigma v^t \nabla_t F_{ji} - F_j{}^t \nabla_t v_i + F_{ti} \nabla_j v^t = 0$$

where $\sigma = 1 + 4\lambda$.

Now, we need following lemmas to prove the main theorem, LEMMA 4. 1.[4] *In an almost-Hermitian space, if tensor S_{jti} is skew-symmetric, then we have*

$$\nabla^i\nabla^t S_{jti}=0.$$

LEMMA 4. 2.[5] *In a compact K-space with constant scalar curvature, if $\nabla_j p_i+\nabla_i p_j$ is pure in j, i and r_i is a vector such that $r_i=\nabla_i r$ for a certain scalar r, then we have*

$$\int_M p^i r^j R_{ji} dV=0$$

where dV means the volume element of the space M.

LEMMA 4. 3. *In a K-space, if v^i is an extended contravariant almost analytic vector for a constant λ, then following relation holds good:*

$$\sigma(R_{ri}-R^*{}_{ri})v^r+\frac{1}{2}N_{jri}\nabla^j v^r=0. \tag{4. 3}$$

Proof. Operating ∇^j to (4. 1) and taking account of (2. 5), we have

$$\sigma\nabla^j v^t(\nabla_t F_j{}^i)+\sigma v^t\nabla^j\nabla_t F_j{}^i-F_j{}^t\nabla^j\nabla_t v^i+\nabla^j F_t{}^i(\nabla_j v^t)+F_t{}^i\nabla^j\nabla_j v^t=0. \tag{4. 4}$$

In this place, for the second term of the left hand side of (4. 4), by (2. 7) and (2. 9), we have

$$\sigma v^t\nabla^j\nabla_t F_j{}^i=\sigma v^t(-R^*{}_t{}^a F_a{}^i+R_t{}^s F_s{}^i)$$

where $R^*{}_j{}^i=g^{ti}R^*{}_{jt}$, and for the third term, we have

$$F_j{}^t\nabla^j\nabla_t v^i=\frac{1}{2}F^{jt}(\nabla_j\nabla_t v^i-\nabla_t\nabla_j v^i)$$

$$=\frac{1}{2}F^{jt}R_{jts}{}^i v^s.$$

Thus (4. 4) turns to

$$(\sigma-1)\nabla_r F_j{}^i(\nabla^j v^r)+\sigma F_a{}^i R_r{}^a v^r-\sigma F_a{}^i R^*{}_r{}^a v^r-\frac{1}{2}F^{jr}R_{jrs}{}^i v^s+F_r{}^i\nabla^j\nabla_j v^r=0.$$

Transvecting this equation with F_{ik}, and using (2. 16), we have

$$\nabla^j\nabla_j v_k+\sigma R_{rk}v^r-(\sigma-1)R^*{}_{rk}v^r-\frac{(\sigma-1)}{4}N_{kjr}\nabla^j v^r=0. \tag{4. 5}$$

On the other hand, operating $F^{kj}\nabla_k$ to (4. 1), we have

$$\begin{aligned}\sigma F^{kj}(\nabla_k v^r)\nabla_r F_j{}^i+\sigma v^r F^{kj}\nabla_k\nabla_r F_j{}^i-F^{kj}(\nabla_k F_j{}^r)\nabla_r v^i-F^{kj}F_j{}^r\nabla_k\nabla_r v^i\\+F^{kj}(\nabla_k F_r{}^i)\nabla_j v^r+F^{kj}F_r{}^i\nabla_k\nabla_j v^r=0.\end{aligned} \tag{4. 6}$$

4), 5) See Takamatsu [9].

In the left hand side of this equation, for the first term and the fifth term, by (2.16), we have

$$\sigma F^{kj}(\nabla_k v^r)\nabla_r F_j{}^i + F^{kj}(\nabla_k F_r{}^i)\nabla_j v^r = -(\sigma+1)F_j{}^k(\nabla_k F_r{}^i)\nabla^j v^r$$
$$= -\frac{(\sigma+1)}{4} N_{jr}{}^i \nabla^j v^r,$$

for the second term, by (2.9), we have

$$F^{kj}\nabla_k \nabla_r F_j{}^i = -\frac{1}{2} F^{kj}(\nabla_k \nabla_j F_r{}^i - \nabla_j \nabla_k F_r{}^i)$$
$$= -\frac{1}{2} F^{kj}(R_{kjs}{}^i F_r{}^s - R_{kjr}{}^s F_s{}^i)$$
$$= -R^*{}_r{}^i + R^{*i}{}_r = 0.$$

For the third term $F^{kj}\nabla_k F_j{}^r$, F^{kj} being hybrid in k, j and $\nabla_k F_j{}^r$ pure in k, j, then this term vanishes by virtue of Proposition 4. For the last term we have

$$F^{kj}F_r{}^i \nabla_k \nabla_j v^r = \frac{1}{2} F_r{}^i F^{kj}(\nabla_k \nabla_j v^r - \nabla_j \nabla_k v^r)$$
$$= \frac{1}{2} F_r{}^i F^{kj} R_{kjs}{}^r v^s$$
$$= R^*{}_s{}^i v^s.$$

Hence, (4.6) becomes

$$\nabla^r \nabla_r v_k + R^*{}_{rk} v^r - \frac{(\sigma+1)}{4} N_{jrk} \nabla^j v^r = 0. \tag{4.7}$$

Thus, subtracting (4.7) from (4.5), we get (4.3).

LEMMA 4.4. *In a compact K-space, if v^i is an extended contravariant almost analytic vector for a constant $\lambda \neq -1/4$ and r^i is a vector such that $r^i = \nabla^i r$ for a certain scalar r, then we have*

$$\int_M r^j v^h (R_{hj} - R^*{}_{hj}) dV = 0. \tag{4.8}$$

Proof. From

$$\nabla^j\{r v^h(R_{hj} - R^*{}_{hj})\} = r^j v^h(R_{hj} - R^*{}_{hj}) + r\nabla^j v^h(R_{hj} - R^*{}_{hj}) + r v^h \nabla^j(R_{hj} - R^*{}_{hj}),$$

by Green's theorem, we have

$$\int_M [r^j v^h(R_{hj} - R^*{}_{hj}) + r\nabla^j v^h(R_{hj} - R^*{}_{hj}) + r v^h \nabla^j(R_{hj} - R^*{}_{hj})] dV = 0. \tag{4.9}$$

On the other hand, operating ∇^i to (4.3), we have

$$\sigma \nabla^i(R_{ri} - R^*{}_{ri})v^r + \sigma(R_{ri} - R^*{}_{ri})\nabla^i v^r + \frac{1}{2}\nabla^i(N_{jri}\nabla^j v^r) = 0.$$

In this place, since $1+4\lambda=\sigma\neq 0$, taking account of (2.14) and (2.15), we have

$$\nabla^i v^r(R_{ri}-R^*_{ri})=0. \tag{4.10}$$

Consequently, from (4.9), we have (4.8).

LEMMA 4.5. *In a compact K-space, an extended contravariant almost analytic vector v^i for a constant λ such that $-3/4\leqq\lambda\leqq 0$, $\lambda\neq -1/4$, can be decomposed into the form*

$$v^i=p^i+r^i \tag{4.11}$$

where $\nabla_i p^i=0$ and r^i is a vector such that $r^i=\nabla^i r$ for a certain scalar r and

$$*O^{ji}_{ab}(\nabla^a p^b+\nabla^b p^a)=0, \tag{4.12}$$

$$r^t\nabla_t F_{ji}=0. \tag{4.13}$$

Proof. By the theory of harmonic integrals, (4.11) is the result that holds good for any vector v^i in a compact orientable Riemannian space. Next putting

$$T_{ji}=\nabla_j p_i+\nabla_i p_j+F_j{}^a F_i{}^b(\nabla_a p_b+\nabla_b p_a)$$

and writing out the square of T_{ji}, we get

$$\frac{1}{4}T_{ji}T^{ji}=(\nabla_j p_i+\nabla_i p_j)\nabla^j p^i+F_j{}^a F_i{}^b\nabla^j p^i(\nabla_a p_b+\nabla_b p_a).$$

Now, operating ∇^i to $p^j T_{ji}$, we have

$$\begin{aligned}\nabla^i(p^j T_{ji})&=\frac{1}{4}T_{ji}T^{ji}+p^j\nabla^i T_{ji}\\ &=\frac{1}{4}T_{ji}T^{ji}+p^j\{\nabla^i(\nabla_j p_i+\nabla_i p_j)+F_j{}^a(\nabla^i F_i{}^b)(\nabla_a p_b+\nabla_b p_a)\\ &\qquad+(\nabla^i F_j{}^a)F_i{}^b(\nabla_a p_b+\nabla_b p_a)+F_j{}^a F_i{}^b\nabla^i(\nabla_a p_b+\nabla_b p_a)\}\\ &=\frac{1}{4}T_{ji}T^{ji}+p^j\{\nabla^i(\nabla_j p_i+\nabla_i p_j)+F_j{}^a F_i{}^b\nabla^i(\nabla_a p_b+\nabla_b p_a)\},\end{aligned} \tag{4.14}$$

because $\nabla^i F_i{}^b=0$ and since $(\nabla^i F_j{}^a)F_i{}^b=(\nabla_i F^{ba})F_j{}^i$ is skew-symmetric with respect to a and b, $(\nabla^i F_j{}^a)F_i{}^b(\nabla_a p_b+\nabla_b p_a)$ vanishes.

On the other hand, interchanging j and i in (4.2) and subtracting the equation thus obtained from (4.2), we get

$$2\sigma v^t\nabla_t F_{ji}-F_j{}^t(\nabla_t v_i-\nabla_i v_t)+F_{ti}(\nabla_j v^t-\nabla^t v_j)=0. \tag{4.15}$$

Substituting (4.11) into (4.15) and taking account of $\nabla_j r_i=\nabla_i r_j$, we have

$$2\sigma v^t\nabla_t F_{ji}-F_j{}^t(\nabla_t p_i-\nabla_i p_t)+F_{ti}(\nabla_j p^t-\nabla^t p_j)=0.$$

Since $\nabla_i F_j{}^i=0$ and $\nabla_i p^i=0$, this equation can be easily written as

$$\begin{aligned}&F_j{}^t(\nabla_i p_t+\nabla_t p_i)-F_i{}^t(\nabla_j p_t+\nabla_t p_j)\\ &=-2\sigma v^t\nabla_t F_{ji}+2p^t\nabla_t F_{ji}+2\nabla^t(F_{jt}p_i+F_{ti}p_j+F_{ij}p_t).\end{aligned} \tag{4.16}$$

Operating ∇^i to (4.16) and using (4.11) and $\nabla^i r^t(\nabla_t F_{ji})=0$, we have

$$\begin{aligned}(4.17)\quad &\nabla^i F_j{}^t(\nabla_i p_t+\nabla_t p_i)+F_j{}^t\nabla^i(\nabla_i p_t+\nabla_t p_i)-F_i{}^t\nabla^i(\nabla_j p_t+\nabla_t p_j)\\ &=-2(\sigma-1)(\nabla^i v^t)\nabla_t F_{ji}-2(\sigma-1)v^t\nabla^i\nabla_t F_{ji}-2r^t\nabla^i\nabla_t F_{ji}+2\nabla^i\nabla^t S_{jti}\end{aligned}$$

where $S_{jti}=F_{jt}p_i+F_{ti}p_j+F_{ij}p_t$.

In (4.17), since $\nabla^i F_j{}^t$ is skew-symmetric with respect to i and t, $\nabla^i F_j{}^t(\nabla_i p_t+\nabla_t p_i)=0$ and by Lemma 4.1, $\nabla^i\nabla^t S_{jti}=0$.

Hence, (4.17) turns to

$$\begin{aligned}&F_j{}^t\nabla^i(\nabla_i p_t+\nabla_t p_i)-F_i{}^t\nabla^i(\nabla_j p_t+\nabla_t p_j)\\ &=-2r^t\nabla^i\nabla_t F_{ji}-2(\sigma-1)\nabla^i v^t(\nabla_t F_{ji})-2(\sigma-1)v^t\nabla^i\nabla_t F_{ji}.\end{aligned}$$

Transvecting this equation with $p^h F_h{}^j$ and taking account of (2.7) and (2.16), we have

$$\begin{aligned}(4.18)\quad &p^h\{\nabla^i(\nabla_i p_h+\nabla_h p_i)+F_h{}^j F_i{}^t\nabla^i(\nabla_j p_t+\nabla_t p_j)\}\\ &=2p^h r^t(R^*_{th}-R_{th})+\frac{1}{2}(\sigma-1)N_{ith}(\nabla^i v^t)p^h+2(\sigma-1)p^h v^t(R^*_{th}-R_{th}).\end{aligned}$$

Substituting (4.3) into (4.18), we get

$$\begin{aligned}(4.19)\quad &p^h\{\nabla^i(\nabla_i p_h+\nabla_h p_i)+F_h{}^j F_i{}^t\nabla^i(\nabla_j p_t+\nabla_t p_j)\}\\ &=2p^h r^t(R^*_{th}-R_{th})+(\sigma-1)(\sigma+2)p^h v^t(R^*_{th}-R_{th}).\end{aligned}$$

Thus, substituting (4.19) into (4.14) and making use of Green's theorem, we have

$$(4.20)\quad \int_M\left[\frac{1}{4}T_{ji}T^{ji}+2p^h r^t(R^*_{th}-R_{th})+(\sigma-1)(\sigma+2)p^h v^t(R^*_{th}-R_{th})\right]dV=0.$$

Substituting $p^h=v^h-r^h$ into (4.20) and taking account of Lemma 4.4, (4.20) becomes

$$(4.21)\quad \int_M\left[\frac{1}{4}T_{ji}T^{ji}+2r^h r^t(R_{th}-R^*_{th})+(\sigma-1)(\sigma+2)v^h v^t(R^*_{th}-R_{th})\right]dV=0,$$

or by (2.10),

$$(4.22)\quad \int_M\left[\frac{1}{4}T_{ji}T^{ji}+2r^h\nabla_h F_{ji}(r^t\nabla_t F^{ji})-(\sigma-1)(\sigma+2)v^h\nabla_h F_{ji}(v^t\nabla_t F^{ji})\right]dV=0.$$

Thus, if $-2\leqq\sigma\leqq 1$, that is, $-3/4\leqq\lambda\leqq 0$ and $\lambda\neq -1/4$, then we can deduce $T_{ji}=0$ and $r^h\nabla_h F_{ji}=0$.

LEMMA 4.6. *If $-3/4<\lambda<0$, $\lambda\neq -1/4$, we have*

$$(4.23)\quad v^h\nabla_h F_{ji}=0,\qquad r^h\nabla_h F_{ji}=0.$$

Proof. This follows from (4.22).

LEMMA 4.7. *In a compact K-space, if v^i is an extended contravariant almost* *analytic vector for a constant* λ *such that* $-3/4\leqq\lambda\leqq 0$, $\lambda\neq -1/4$, *then it satisfies*

$$\nabla^l\nabla_l v^i + R_t{}^i v^t = 0. \tag{4.24}$$

Proof. When $-3/4<\lambda<0$, $\lambda\neq -1/4$, multiplying (4.23) by $\nabla_k F^{ji}$ and taking account of (2.10), we have

$$(R_{tk}-R^*{}_{tk})v^t=0,$$

and hence, from (4.3), $N_{tli}\nabla^t v^l=0$. Consequently by (4.7), we get (4.24).

When $\lambda=-3/4$ or $\lambda=0$, forming $(4.5)\times(\sigma+1)-(4.7)\times(\sigma-1)$, we have

$$2\nabla^i\nabla_i v_k+\sigma(\sigma+1)R_{rk}v^r-(\sigma-1)(\sigma+2)R^*{}_{rk}v^r=0. \tag{4.25}$$

In this place, if $\lambda=-3/4$ or $\lambda=0$, that is, if $\sigma=1$ or $\sigma=-2$, then from (4.25) we have (4.24).

LEMMA 4.8. *In a compact K-space, an extended contravariant almost analytic vector* v^i *for a constant* λ *such that* $\lambda=-1/4$, *can be decomposed into the form*

$$v^i=p^i+r^i \tag{4.26}$$

where $\nabla_i p^i=0$ *and* r^i *is a vector such that* $r^i=\nabla^i r$ *for a certain scalar* r *and*

$$*O_{ab}^{ji}(\nabla^a p^b+\nabla^b p^a)=0, \tag{4.27}$$

$$p^t\nabla_t F_{ji}=0. \tag{4.28}$$

Proof. In (4.2) if $\lambda=-1/4$, i.e. $\sigma=0$, we have

$$\nabla_j v_i - F_j{}^a F_i{}^b\nabla_a v_b=0. \tag{4.29}$$

Interchanging j and i in (4.29) and subtracting the equation thus obtained from (4.29) and substituting $v^i=p^i+r^i$, we have

$$(\nabla_j p_i-\nabla_i p_j)-F_j{}^aF_i{}^b(\nabla_a p_b-\nabla_b p_a)=0. \tag{4.30}$$

Transvecting (4.30) with $F_k{}^j$ and taking account of $\nabla^j F_{ji}=0$ and $\nabla_a p^a=0$, we have

$$F_k{}^a(\nabla_i p_a+\nabla_a p_i)-F_i{}^b(\nabla_k p_b+\nabla_b p_k)=2p^a\nabla_a F_{ki}+2\nabla^a S_{kai} \tag{4.31}$$

where $S_{kai}=F_{ka}p_i+F_{ai}p_k+F_{ik}p_a$.

Operating ∇^i to (4.31), taking account of that $\nabla^i F_k{}^a$ is skew-symmetric in i,a and $\nabla^i F_i{}^b=0$, by Lemma 4.1, we have

$$F_k{}^a\nabla^i(\nabla_i p_a+\nabla_a p_i)-F_i{}^a\nabla^i(\nabla_k p_a+\nabla_a p_k)=2\nabla^i p_a(\nabla^a F_{ki})+2p^a\nabla^i\nabla_a F_{ki}. \tag{4.32}$$

Transvecting (4.32) with $F_h{}^k$ and making use of (2.7), we have

$$\begin{aligned}\nabla^i(\nabla_i p_h+\nabla_h p_i)+F_h{}^kF_i{}^a\nabla^i(\nabla_k p_a+\nabla_a p_k)&=-2F_h{}^k(\nabla^i p_a)\nabla^a F_{ki}-2p^aF_h{}^k\nabla^i\nabla_a F_{ki}\\ &=-F_h{}^k(\nabla^a F_k{}^i)(\nabla_i p_a-\nabla_a p_i)-2p^a(R^*{}_{ha}-R_{ha})\\ &=2p^a(R_{ha}-R^*{}_{ha})\end{aligned} \tag{4.33}$$

because, since $\nabla^a F_h{}^i$ is pure in a,i and by (4.30), $\nabla_i p_a-\nabla_a p_i$ is hybrid in a,i,

$F_h{}^k(\nabla^a F_k{}^i)(\nabla_i p_a - \nabla_a p_i)$ vanishes by virtue of Proposition 4.

Next substituting (4.33) into (4.14) in which $h=j$, we have

$$\nabla^i(p^h T_{ih}) = \frac{1}{4} T^{ih} T_{ih} + 2p^h p^a (R_{ah} - R^*{}_{ah}) \tag{4.34}$$

and by Green's theorem, we have

$$\int_M \left[\frac{1}{4} T^{ih} T_{ih} + 2p^h p^a (R_{ah} - R^*{}_{ah})\right] dV = 0.$$

Thus, we get $T_{ih}=0$ and $p^h \nabla_h F_{ji} = 0$.

LEMMA 4.9. *In a compact K-space, if v^i is an extended contravariant almost analytic vector for a constant λ such that $\lambda = -1/4$, then it satisfies*

$$\nabla^l \nabla_l v^i + R^*{}_t{}^i v^t = 0. \tag{4.35}$$

Proof. (4.35) follows from (4.25).

5. Proof of the main theorem.

THEOREM. *In a compact K-space with constant scalar curvature, an extended contravariant almost analytic vector v^i for a constant λ such that $-3/4 \leqq \lambda \leqq 0$ is decomposed into the form*

$$v^i = p^i + F_r{}^i q^r$$

where p^i and q^i are both Killing vectors.

Proof. First of all, we shall prove that p^i is a Killing vector. When $-3/4 \leqq \lambda \leqq 0$ and $\lambda \neq -1/4$, we put

$$U_{ji} = \nabla_j p_i + \nabla_i p_j.$$

Operating ∇^i to $p^j U_{ji}$ and making use of $p_i = v_i - r_i$, we have

$$\begin{aligned} \nabla^i(p^j U_{ji}) &= \frac{1}{2} U_{ji} U^{ji} + p^j(\nabla^i \nabla_j v_i + \nabla^i \nabla_i v_j - 2\nabla^i \nabla_j r_i) \\ &= \frac{1}{2} U_{ji} U^{ji} + p^j(\nabla^i \nabla_i v_j + \nabla^i \nabla_j v_i - \nabla_j \nabla^i v_i + \nabla_j \nabla^i v_i - 2\nabla^i \nabla_j r_i + 2\nabla_j \nabla^i r_i - 2\nabla_j \nabla^i r_i). \end{aligned} \tag{5.1}$$

In this place, by the Ricci's identity and (4.24), we have

$$\nabla^i \nabla_i v_j + \nabla^i \nabla_j v_i - \nabla_j \nabla^i v_i = \nabla^i \nabla_i v_j + R_{ji} v^i = 0 \tag{5.2}$$

and

$$\nabla^i \nabla_j r_i - \nabla_j \nabla^i r_i = r^i R_{ji}. \tag{5.3}$$

Hence, making use of (5.2) and (5.3), from (5.1) by Green's theorem we find

$$\int_M \left[\frac{1}{2} U_{ji} U^{ji} - 2p^j r^i R_{ji} + p^j \nabla_j \alpha\right] dV = 0 \tag{5.4}$$

where $\alpha=\nabla^i v_i-2\nabla^i r_i$.

From $\nabla_j(\alpha p^j)=p^j\nabla_j\alpha+\alpha\nabla_j p^j=p^j\nabla_j\alpha$, we have

$$\int_M p^j\nabla_j\alpha dV=0. \tag{5.5}$$

Thus taking account of (4.12) and Lemma 4.2, (5.4) becomes

$$\int_M \frac{1}{2}U_{ji}U^{ji}dV=0$$

from which it follows that

$$U_{ji}=\nabla_j p_i+\nabla_i p_j=0, \tag{5.6}$$

that is, p^i is a Killing vector.

Next, when $\lambda=-1/4$, again we consider (5.1). In this place, by the Ricci's identity and Lemma 4.9, we have

$$\begin{aligned}\nabla^i\nabla_i v_j+\nabla^i\nabla_j v_i-\nabla_j\nabla^i v_i&=\nabla^i\nabla_i v_j+R_{ji}v^i\\ &=-R^*{}_{ji}v^i+R_{ji}v^i.\end{aligned} \tag{5.7}$$

Hence, making use of (5.3) and (5.7), from (5.1) by Green's theorem, we find

$$\int_M\left[\frac{1}{2}U_{ji}U^{ji}+p^jv^i(R_{ji}-R^*{}_{ji})-2p^jr^iR_{ji}+p^j\nabla_j\alpha\right]dV=0 \tag{5.8}$$

where $\alpha=\nabla^i v_i-2\nabla^i r_i$.

Multiplying (4.28) by $\nabla_h F^{ji}$ and using (2.10), we have

$$p^i(R_{ji}-R^*{}_{ji})=0.$$

Thus taking account of (4.27), by Lemma 4.2 and (5.5), (5.8) becomes

$$\int_M \frac{1}{2}U_{ji}U^{ji}dV=0$$

from which it follows that

$$U_{ji}=\nabla_j p_i+\nabla_i p_j=0.$$

If we put

$$q^i=-F_t{}^i r^t, \qquad \text{or} \qquad r^i=F_t{}^i q^t \tag{5.9}$$

then, $v^i=p^i+r^i$ can be written as

$$v^i=p^i+F_t{}^i q^t. \tag{5.10}$$

Lastly, we shall prove that q^i is a Killing vector. From (5.9), we have $q_i=F_i{}^t r_t$ from which it follows

$$\nabla_h q_i+\nabla_i q_h=(\nabla_h F_i{}^t+\nabla_i F_h{}^t)r_t+(F_i{}^t\nabla_h r_t+F_h{}^t\nabla_i r_t). \tag{5.11}$$

Interchanging j and i in (4.2) and adding the equation thus obtained to (4.2), we get

$$F_i^{\ t}(\nabla_j v_t+\nabla_t v_j)+F_j^{\ t}(\nabla_i v_t+\nabla_t v_i)=0.$$

Substituting $v_i=p_i+r_i$ into this equation and using (5.6) and $\nabla_t r_i=\nabla_i r_t$, we have

$$F_i^{\ t}\nabla_j r_t+F_j^{\ t}\nabla_i r_t=0. \tag{5.12}$$

Thus, by (2.4) and (5.12), the right hand side of (5.11) vanishes. Consequently we find q^i is a Killing vector. q.e.d.

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