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On a Deformation of Riemannian Structures on Compact Manifolds

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1. The purpose of this paper is to prove that every compact C^∞-Riemannian manifold with at least 3 dimensions can be deformed conformally to a C^∞-Riemannian structure of constant scalar curvature.

Let S be a d-dimensional C^∞-Riemannian manifold with $d \geqq 3$, and denote its fundamental positive definite tensor by g_{ij}. Throughout this paper we will use the definitions and notations of the book "Curvature and Betti numbers" by K. Yano and S. Bochner, unless otherwise stated. The volume element is written as dV. The total volume is assumed to be 1.

Here we are going to present the outline of the proof. Consider a conformal transformation of a Riemannian structure

$$\bar{g}_{ij} = e^{2\rho} g_{ij}. \tag{1.1}$$

Then the connection coefficients $\bar{\Gamma}^i_{jk}$ corresponding to $\bar{g}_{ij}$ are expressed as[1)]

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \rho_k \delta^i_j + \rho_j \delta^i_k - \rho^i g_{jk}, \tag{1.2}$$

where

$$\rho_i = \frac{\partial \rho}{\partial x^i}. \tag{1.3}$$

From (1.2)

$$\bar{R}^i_{jkl} = R^i_{jkl} - \rho_{jk}\delta^i_l + \rho_{jl}\delta^i_k - g_{jk}\rho^i_l + g_{jl}\rho^i_k \tag{1.4}$$

where

$$\rho_{jk} = \rho_{j,k} - \rho_j \rho_k + \frac{1}{2} g^{\alpha\beta}\rho_\alpha \rho_\beta g_{jk}. \tag{1.5}$$

Hence

$$\bar{R}_{jk} = R_{jk} - (d-2)\rho_{jk} - g_{jk}\rho^\alpha_\alpha \tag{1.6}$$

and

1) see [5] page 78.

$$\bar{R} = e^{-2\rho}(R-2(d-1)\rho_\alpha^\alpha)\,. \tag{1.7}$$

Here $\bar{R}^i_{jkl}$, $\bar{R}_{jk}$ and $\bar{R}$ denote the curvature tensor, the Ricci tensor and the scalar curvature, respectively, of the new structure.

Now let Δ denote the Laplace-Beltrami operator corresponding to g_{ij}. Then (1.7) can be written as

$$\begin{aligned}\bar{R} &= e^{-2\rho}(R-2(d-1)\rho_\alpha^\alpha) \\ &= e^{-2\rho}\left(R-2(d-1)\left(\Delta\rho+\left(\frac{d}{2}-1\right)g^{\alpha\beta}\rho_\alpha\rho_\beta\right)\right. \\ &= e^{-2\rho}\left(R-\frac{4(d-1)}{d-2}\,e^{(-d/2+1)\rho}\Delta(e^{(d/2-1)\rho})\right).\end{aligned} \tag{1.8}$$

Set

$$\bar{u} = e^{(d/2-1)\rho} \tag{1.9}$$

or

$$(\bar{u})^{4/(d-2)} = e^{2\rho}\,, \tag{1.9'}$$

and then

$$-\bar{R}(\bar{u})^{(d+2)/(d-2)} = -R\bar{u}+\frac{4(d-1)}{d-2}\,\Delta\bar{u}\,. \tag{1.10}$$

Conversely, we are going to prove

Theorem A. *There exists a positive C^∞-function $\bar{u}$ satisfying*

$$-(\bar{u})^{(d+2)/(d-2)}C_0 = -R\bar{u}+\frac{4(d-1)}{d-2}\,\Delta\bar{u} \tag{1.10'}$$

where C_0 is a constant.

If such a function $\bar{u}$ be found, we have only to set $\bar{g}_{ij}=(\bar{u})^{4/(d-2)}g_{ij}$ to obtain the desired structure.

On the other hand, if there exists a positive extremal $v^{(q)}$ minimizing a variational function ($q\geqq 2$)

$$F^{(q)}(u) = \frac{\int\left(\frac{4(d-1)}{d-2}\,|\nabla u|^2+Ru^2\right)dV}{\left(\int|u|^q dV\right)^{2/q}} \tag{1.11}$$

to a value $\mu_{(q)}$, then this function satisfies the corresponding Euler's equation

$$\frac{4(d-1)}{d-2}\,\Delta v^{(q)}-Rv^{(q)} = -\mu_{(q)}(v^{(q)})^{q-1}\,. \tag{1.12}$$

Here

$$(1.13)\qquad |\nabla u|^2 = g^{\alpha\beta}\frac{\partial u}{\partial x^\alpha}\frac{du}{dx^\beta}.$$

In order to prove Theorem A, we shall prove the following two theorems.

Theorem B. *For any* $q<2d/(d-2)$, *there exists a positive function* $v^{(q)}$ *satisfying* (1.12).

Theorem C. *As* q *tends to* $2d/(d-2)$, *a uniform limit* $\bar{u}$ *of such* $v^{(q)}$*'s exists, is positive and satisfies* (1.10′) *with* $C_0=\mu_{(2d/(d-2))}$.

Theorem A is an immediate consequence of Theorem C.

2. Let ε be a positive number less than $4/(d-2)$. Set

$$(2.1)\qquad p_\varepsilon = (2d/(d-2)-\varepsilon),$$

$$(2.1')\qquad p'_\varepsilon = p_\varepsilon/(p_\varepsilon-1),$$

and

$$(2.2)\qquad F^{(p_\varepsilon)}(u) = F_\varepsilon(u) = \frac{\int\left(\frac{4(d-1)}{d-2}|\nabla u|^2+Ru^2\right)dV}{\left(\int|u|^{p_\varepsilon}dV\right)^{2/p_\varepsilon}}$$

$$=\int\left(4(d-1)/(d-2)|\nabla u|^2+Ru^2\right)dV/\|u\|^2_{p_\varepsilon},$$

$$(2.3)\qquad \|u\|_q = \left(\int|u|^q dV\right)^{1/q}.$$

By L_p we denote the Banach space of real functions with the norm $\|\ \|_p$.

Lemma 1. *Let* $\{u_i\}$ *be a sequence of* C^∞*-functions with* $\|u_i\|_{p_\varepsilon}=1$ *such that*

$$(2.4)\qquad \lim_i F_\varepsilon(u_i) = \mu_{(p_\varepsilon)} = C_0(\varepsilon) = \mathrm{Min}_u F_\varepsilon(u).$$

Then the sequences $\{|u_i|\}$ *possess a similar property except that* $|u_i|$ *might not be differentiable at the zero points of* u_i.

Proof is almost evident if one notices that

$$(2.5)\qquad |\nabla u|^2 = |\nabla(|u|)|^2,$$

except at zero point of u with non-vanishing Δu. The measure of the set of such points is zero. By the measure we understand the measure with respect to dV.

Lemma 2.[2] *There exists a positive constant C_1 such that for $p_\varepsilon \leqq 2d/(d-2)$*

$$\inf_C \|u-c\|_{p_\varepsilon} \leqq C_1 \| \, |\nabla u| \, \|_2 , \tag{2.6}$$

where c is an arbitrary constant and u is assumed to be a smooth function.

This lemma is similar to Sobolev's lemma. The proof is omitted because a minor modification of the proof of Lemma 4 is sufficient for this lemma. However, it should be noted that even when $\varepsilon=0$, the lemma is valid but this is not necessary for the present paper.

Corollary.

$$F(u) \geqq \frac{4(d-1)}{d-2} \frac{1}{C_1} - \sup_{P\in S} |R(P)| - 1 . \tag{2.8}$$

Let $\psi(x)$ be a function over the unit square $E^d=\{x\,;\, -1 \leqq x^m \leqq 1, m=1, 2, \cdots, d\}$ in a d-dimensional Euclidean space with the property:

$$|||\psi|||_q = \left(\int_{E^d} |\psi|^q dx\right)^{1/q} < \infty \tag{2.9}$$

where $2<q<\infty$.

Consider the multiple Fourier (trigonometrical) series of $\psi(x)$;

$$\begin{aligned} \psi(x) &= \sum (a_{i_1} \cdots {}_{i_d} \cos \pi(i_1 x^1 + \cdots + i_d x^d) \\ &\quad + b_{i_1} \cdots {}_{i_d} \sin \pi(i_1 x^1 + \cdots + i_d x^d) \\ &= \sum (a_I \cos \pi \langle I, x \rangle + b_I \sin \pi \langle I, x \rangle) . \end{aligned} \tag{2.10}$$

Here I denote a "vector" $(i_1, \cdots, i_d)$ with integer components and $\langle I, x \rangle$ an inner product of I and another "vector" $(x^1, \cdots, x^d)$. Define $|I|$ by

$$|I| = (i_1^2 + \cdots + i_d^2)^{1/2} . \tag{2.11}$$

Lemma 3.

$$|||\psi|||_q \leqq (|a_0|^{q'} + \sum_{|I|\geqq 1} (|a_I|^{q'} + |b_I|^{q'}))^{1/q'} \tag{2.12}$$

where $q'=q/(q-1)$.

REMARK. This is the Hausdorff-Young inequality for multiple Fourier series.

2) see [4]

Proof. Let E' be the discrete space of lattice points in a d-dimensional space. Define a one to one operator T onto a functional space over E from another functional space over $E'\times E'$ by

$$(2.13)\qquad (T^{-1}\psi)(I, J) = (a_I, b_J)$$

at a point (I, J) in $E'\times E'$. The space E is given the ordinary Lebesgue measure while in $E'\times E'$ a weight 1 is assigned to each point. The norm is defined by

$$(2.14)\qquad \|T^{-1}\psi\|_{q'} = (|a_0|^{q'}+\textstyle\sum_{|I|\geqq 1}(|a_I|^{q'}+|b_I|^{q'}))^{1/q'}.$$

Then T becomes simultaneously of the type (2.2) and $(1, \infty)$, to which we can apply the Calderon-Zygmund's generalization[3] of M. Riesz's convexity theorem. Hence for any q between 2 and infinity,

$$(2.15)\qquad \||\psi|\|_q \leqq \|T^{-1}\psi\|_{q'}$$

where $q'=q/(q-1)$. This proves the lemma.

3. It is well known that there are countably many non-positive eigenvalues of the elliptic operator $\frac{4(d-1)}{d-2}\Delta$ diverging to $-\infty$. We write them in non-increasing order, $-\lambda_1, -\lambda_2, \cdots, -\lambda_m, \cdots$. To each λ_m is attached an eigenfunction ϕ_m with $\|\phi_m\|_2=1$. These ϕ_m's are mutually orthogonal in the sense of L_2. The first eigenvalue $\lambda_1=0$ and the corresponding function $\phi_1\equiv 1$. Then every square integrable function $u(P)$ can be expanded into the Fourier series with respect to $\{\phi_m\}$. In particular

$$(3.1)\qquad u_i = \textstyle\sum_{i=1}^{\infty} a_{ij}\phi_j.$$

Lemma 4. *Suppose that ψ_N's are smooth functions on S with $\|\psi_N\|_{p_g}<\infty$, such that for integer j between* 1 *and N,*

$$(3.2)\qquad \int_S \psi_N\phi_j dV = 0.$$

More generally, if $\psi_N/\|\psi_N\|_{p_g}$ is weakly close to 0, *then for any given small $\delta<0$, there exists an integer N_0 such that if $N\geqq N_0$, then*

$$(3.3)\qquad \inf_c \|\psi_N - c\|_{p_g} \leqq \delta\| \,|\Delta\psi_N|\, \|_2.$$

Proof. Without loss of generality we may assume that $\|\psi_N\|_{p_g}$ is uniformly bounded as N tends to infinity.

3) See Theorem D in page 117 in "On the theorem of Hausdorff and Young" in [2].

Firstly, we consider the case when the carrier of ψ_N is contained in a coordinate neighborhood E. In this case we can consider the trigonometrical expansion of ψ_N. Take a sufficiently large integer M, so that for a preassigned small $\delta'>0$,

$$(3.3)\qquad (\textstyle\sum_{|I|\geqq M}|I|^{-2p_\varepsilon/(p_\varepsilon-2)})^{(p_\varepsilon-2)/2p_\varepsilon}\leqq\delta'\,.$$

This is possible whenever ε is positive because

$$(3.5)\qquad 2p_\varepsilon/(p_\varepsilon-2)=\frac{d\left(\frac{4}{d-2}-\varepsilon\right)+(d-1)\varepsilon}{\frac{4}{d-2}-\varepsilon}>d\,.$$

We set $c=a_0$ (and b_0 is always assumed to be zero). By virtue of Lemma 3,

$$(3.6)\qquad \begin{aligned}|||\psi_N|||_{p_\varepsilon}&\leqq(\textstyle\sum_I(|a_I|^{p_\varepsilon'}+|b_I|^{p_\varepsilon'})^{1/p_\varepsilon'}\\ &=(\textstyle\sum_{|I|<M}(|a_I|^{p_\varepsilon'}+|b_I|^{p_\varepsilon'}))^{1/p_\varepsilon'}\\ &\quad+(\textstyle\sum_{|I|\geqq M}(|a_I|^{p_\varepsilon'}+|b_I|^{p_\varepsilon'}))^{1/p_\varepsilon'}\end{aligned}$$

where

$$(3.7)\qquad p_\varepsilon'=\frac{p_\varepsilon}{p_\varepsilon-1}<2\,.$$

Set

$$(3.8)\qquad q_\varepsilon=2/p_\varepsilon'$$

and

$$(3.9)\qquad q_\varepsilon'=q_\varepsilon/(q_\varepsilon-1)\,.$$

By a simple calculation

$$(3.10)\qquad p_\varepsilon'q_\varepsilon'=2p_\varepsilon/(p_\varepsilon-2)\,.$$

The second term of the right hand side of (3.6) will be dominated by

$$(3.11)\qquad \begin{aligned}&(\textstyle\sum_{|I|\geqq M}(|a_I|^{p_\varepsilon'}+|b_I|^{p_\varepsilon'}))^{1/p_\varepsilon'}\\ &=\left(\textstyle\sum_{|I|\geqq M}\left(\frac{|a_I|\ |I|}{|I|}\right)^{p_\varepsilon'}+\left(\frac{|b_I|\ |I|}{|I|}\right)^{p_\varepsilon'}\right)^{1/p_\varepsilon'}\\ &\leqq(\textstyle\sum_{|I|\geqq M}(|a_I|\ |I|)^2+(|b_I|\ |I|)^2)^{1/2}\cdot\\ &\quad\cdot(\textstyle\sum_{|I|\geqq M}|I|^{-p_\varepsilon'q_\varepsilon'})^{1/p_\varepsilon'q_\varepsilon'}\leqq\delta'\left(\int|\operatorname{grad}\psi_N|^2dx\right)^{1/2}\end{aligned}$$

where

$$(3.13)\qquad |\operatorname{grad}\psi_N|=\left(\textstyle\sum_{m=1}^d\left(\frac{\partial\psi_N}{\partial x^m}\right)^2\right)^{1/2}.$$

On account of the uniform ellipticity of g^{ij}, there exists a constant C_2 such that

$$C_2^{-1}|||\psi|||_{p_g} \leqq ||\psi||_{p_g} \leqq C_2|||\psi|||_{p_g} \tag{3.14}$$

and

$$C_2^{-1}|||\,|\mathrm{grad}\,\psi|\,|||_2 \leqq ||\,|\nabla\psi|\,||_2 \leqq C_2|||\,|\mathrm{grad}\,\psi|\,|||_2\,. \tag{3.15}$$

Combining (3.11) and (3.15), we have

$$(\textstyle\sum_{|I|\geqq M}(|a_I|^{p_g}+|b_I|^{p_g}))^{1/p_g} \leqq \delta' C_2||\,|\nabla\psi_N|\,||_2\,. \tag{3.16}$$

REMARK. If we set $M=1$, we have Lemma 2 for the case when the carrier of ψ_N is within a coordinate neighborhood. Namely if $a_0=0$,

$$||\psi||_{p_g} \leqq C_1||\,|\nabla\psi|\,||_2\,.$$

As for the first term of (3.6), $a_I/||\psi_N||_{p_g}$'s and $b_I/||\psi_N||_{p_g}$'s can be taken arbitrarily close to zero if N_0 is sufficiently large, because these coefficients are linear functionals over L_{p_g}. By virtue of (3.14), $a_I/||\psi_N||_p$ and $b_I/||\psi_N||_{p_g}$'s are also small, say less than $\frac{1}{2}M^{-d}\delta'$. Hence

$$(\textstyle\sum_{|I|<M}(|a_I|^{p_g'}+|b_I|^{p_g'}))^{1/p_g'} \leqq \delta'||\psi_N||_p\ . \tag{3.17}$$

However, by virtue of Lemma 2 (see the remark above),

$$(\textstyle\sum_{|I|<M}(|a_I|^{p_g'}+|b_I|^{p_g'}))^{1/p_g'} \leqq \delta' C_1||\,|\nabla\psi_N|\,||_2\,. \tag{3.18}$$

Combining (3.6), (3.16), and (3.18),

$$||\psi_N||_{p_g} \leqq \delta(||\,|\nabla\psi_N|\,||_2) \tag{3.19}$$

if $a_0=0$. Here

$$\delta = \delta'(c_1+c_2). \tag{3.20}$$

This concludes the proof for the case when the carrier of ψ_N is in a coordinate neighborhood.

As for the general case, we decompose the manifold S into a union of finitely many, say l_0's closures of coordinate neighborhood $U_1, \cdots, U_{l_0}$ such that U_l is a cubic neighborhood of a point in U_l, and any two of these U_l's intersect only at the boundary. The restriction of ψ_N over U_l is denoted by $\psi_{N,l}$. The mean of $\psi_{N,l}$ over U_l with respect to dx will be denoted by

$$c_{N,l} = \int_{U_l}\psi_{N,l}dx\,. \tag{3.21}$$

Then

(3.22) $$\|\psi_N-\sum_{l=1}^{l_0}c_{N,l}\|_{p_\varepsilon}\leqq\sum_{l=1}^{l_0}\|\psi_{N,l}-c_{N,l}\|_{p_\varepsilon}.$$

However, the trigonometrical Fourier coefficients of a function ψ in L_p over the coordinate systems of U_l's are also linear functionals over L_{p_ε}. Hence, if N_0 is sufficiently large, we can apply the result of the previous case so that for each l and a preassigned $\delta''=l_0^{-1}\delta$

(3.23) $$\|\psi_{N,l}-c_{N,l}\|\leqq\delta''(\|\,|\nabla\psi_{N,l}|\,\|_2).$$

From this it follows that

(3.24) $$\|\psi_{N,l}-\sum_{l=1}^{l_0}c_{N,l}\|\leqq l_0\delta''(\sup_l\|\,|\nabla\psi_{N,l}|\,\|_2)\leqq\delta(\|\,|\nabla\psi_N|\,\|_2)$$

This completes the proof of Lemma 4.

REMARK (I). If $M=1$, Lemma 2 follows.

REMARK (II). This $\delta=\delta(N_0)$ depends upon N_0 and goes to zero as N_0 tends to infinity.

4. Consider the Fourier expansion (3.1) of u_i's in (2.4) with $\|u_i\|_{p_\varepsilon}=1$. It is easily seen that all $F_\varepsilon(u_i)$'s are bounded by a positive constant C_3. From this fact, it follows that the convex closure of the set $\{u_i\}$ in L_2, and more generally in L_{p_ε}, is compact strongly. This will be formulated in

Lemma 5. *The convex closure of $\{u_i\}$ compact; and a limit $v^{(p_\varepsilon)}$ of a convergent subsequence is not zero.*

Proof. The latter half is an immediate consequence of the former half because for each i, $\|u_i\|_{p_\varepsilon}=1$.

Now without loss of generality we may assume that $\{u_i\}$ converges weakly to $v^{(p_\varepsilon)}$. Consider the Fourier expansion with respect to ϕ_j's.

(4.1) $$v^{(p_\varepsilon)}=\sum_{j=1}^{\infty}b_j\phi_j.$$

Then

(4.2) $$\|u_i-v^{(p_\varepsilon)}\|_{p_\varepsilon}\leqq\sum_{j=1}^{N-1}\|(a_{ij}-b_j)\phi_j\|_{p_\varepsilon}+\|\sum_{j=N}^{\infty}(a_{ij}-b_j)\phi_j\|_{p_\varepsilon}.$$

By virtue of the previous lemma, there exists a sequence of constants $c_i(N)$ such that for $N\geqq N_0$,

(4.3) $$\|\sum_{j=N}^{\infty}(a_{ij}-b_j)\phi_j-c_i(N)\|_{p_\varepsilon}\leqq\delta(N_0)2\sum_N^{\infty}\lambda_j(a_{ij}-b_j)^2\leqq\delta(N_0)2C_3.$$

It is easily seen that these $c_i(N)$'s tend uniformly to zero as N goes to infinity. Now, i tends to infinity. Then the first term of (4.2) vanishes and

$$\overline{\lim}_i \, ||u_i - c_i(N) - v^{(p_\varepsilon)}||_{p_\varepsilon} \leqq 2C_3\delta(N_0). \tag{4.4}$$

However this $\delta(N_0)$ can be made arbitrarily small. Hence

$$\overline{\lim}_N \, \overline{\lim}_i \, ||u_i - c_i(N) - v^{(p_\varepsilon)}||_{p_\varepsilon} = 0\,. \tag{4.5}$$

By making use of the fact that $\overline{\lim} \, ||u_i - v^{(p_\varepsilon)}||_{p_\varepsilon}$ is independent of N, we can easily obtain the relation that

$$\overline{\lim} \, ||u_i - v^{(p_\varepsilon)}||_{p_\varepsilon} \leqq \overline{\lim}_N \, \overline{\lim}_i \, c_i(N) = 0\,.$$

Thus the lemma has been proved.

NOTICE. The function $v^{(p_\varepsilon)}$, being a limit of non-negative u_i's, is non-negative.

Lemma 6. *If $v^{(p_\varepsilon)}$, a non-negative C^2 function satisfies the equation* (1.12) *for $q=p_\varepsilon$, $\varepsilon \geqq 0$, then $v^{(n_\varepsilon)} = v^{(q)}$ is positive.*

Proof. For simplicity, we shall use $v^{(q)}$ instead of $v^{(p_\varepsilon)}$. Suppose that $v^{(q)}$ vanishes at a point P. Take the polar coordinates $r, \theta^m, m=1, 2, \cdots, d-1$ of a normal geodesic coordinates around P. The volume element and the Laplace-Beltrami operator with respect to the induced and normalized (total volume 1) Riemannian structure on the concentric sphere $\Omega(r)$ of the radius r around P will be denoted by $\sqrt{\sigma(r)}$ and $\Delta_{\theta(r)}$ respectively. Then

$$\Delta v^{(q)} = ((\partial/\partial r)^2 + (\partial/\partial r)(\log\sqrt{\sigma})\partial/\partial r + r^{-2}\Delta_{\theta(r)})v^{(q)}\,. \tag{4.6}$$

By integrating (4.6) over $\Omega(r)$ with the volume element $\sqrt{\sigma}\,d\theta$

$$r^{-d+1}(\partial/\partial r)r^{d-1}\int_{\Omega(r)}(\partial v^{(q)}/\partial r)\sqrt{\sigma}\,d\theta + r^{-2}\int_{\Omega(r)}\Delta v^{(q)}\sqrt{\sigma}\,d\theta$$
$$= \int (Rv^{(q)} - \mu_{(q)}(v^{(q)})^{q-1})\sqrt{\sigma}\,d\theta\,.$$

When r ranges over a small interval $(0, r_0)$, there exists a positive constant K_1 such that

$$\left|\int (Rv^{(q)} - \mu^{(q)}(v^{(q)})^{q-1})\sqrt{\sigma}\,d\theta\right| \leqq K_1\int v^{(q)}\sqrt{\sigma}\,d\theta$$

4) see [1]

because $q>2$. Hence

$$r^{d-1}\int_{\Omega(r)}(\partial v^{(q)}/\partial r)\sqrt{\sigma}\,d\theta \leqq \int_0^r \rho^{d-1}K_1\int_{\Omega(r)} v^{(q)}\sqrt{\sigma}\,d\theta d\rho$$

or

$$\int_{\Omega(r)}(\partial v^{(q)}/\partial r)\sqrt{\sigma}\,d\theta \leqq r^{-d+1}\int_0^r K_1\rho^{d-1}\int v^{(q)}\sqrt{\sigma}\,d\theta d\rho\,.$$

Integrating both sides from 0 to s,

$$\begin{aligned}(4.7)\qquad &\int_{\Omega(s)} v^{(q)}\sqrt{\sigma}(s)d\theta - \int_0^s\int_{\Omega(r)}(v^{(q)}\partial\log\sqrt{\sigma}/\partial r)(r)\sqrt{\sigma}\,\sigma(r)d\theta dr \\ &\leqq \int_0^s r^{-d+1}\int_0^r K_1\rho^{d-1}\int_{\Omega(\rho)} v^{(q)}\sqrt{\sigma}(\rho)d\theta d\rho dr\,.\end{aligned}$$

Now set

$$X(r) = \int_{\Omega(r)} v^{(q)}\sqrt{\sigma}(r)d\theta\,,$$

and take positive constants K_2 and K_3 such that both

$$|(\partial\log\sqrt{\sigma}/\partial r)(r)| \leqq K_2 r$$
$$|v^{(q)}(r)| \leqq K_3$$

hold for $0\leqq r\leqq r_0$.

Then, from (4.7) it follows that

$$\begin{aligned}(4.8)\qquad X(s) &\leqq K_2\int_0^s X(r)r dr + K_1\int_0^s r^{-d+1}\int_0^r \rho^{d-1}X(\rho)d\rho dr \\ &\leqq K_2K_3(s^2/2)+K_1K_3(s^2/2d).\end{aligned}$$

In general, it can be shown that

$$(4.9)\qquad X(s)\leqq K_3(K_1+K_2)^n 2^{-n}s^{2n}/n!\,.$$

The proof can be given by induction on n. If (4.9) holds up to $n=N-1$. Then by subtituting $X(r)$ in the right hand side of (4.8) by (4.9),

$$\begin{aligned}X(s) &\leqq K_3[(K_1+K_2)^{N-1}2^{-N+1}/(N-1)!][(K_2s^{2N}/2N)+ \\ &\quad + K_1s^{2N}/(2(N-1)+d)2N] \\ &\leqq K_3(K_1+K_2)^N 2^{-N}s^{2N}/N!\end{aligned}$$

Since N can be taken arbitrarily large, $X(s)=0$ for all s. From this we can conclude that $v^{(q)}=0$ around P. This means the zero points of $v^{(q)}$ is open. Therefore $v^{(q)}$ must be identically zero. This contradiction proves that $v^{(q)}$ is positive everywhere.

Lemma 7. *The $v^{(q)}$ is a weak solution of* (4.8). *Here Δ is understood as the extension of Laplace-Beltrami operator over L_2.*

Proof. Take a C^∞ function v on S with

$$\sup_{P\in S}|v(P)|\leqq 1\,, \tag{4.10}$$

and a small positive real η. Define a subset S_1 of S by

$$S_1=\{P\,;\,v^{(q)}(P)>\eta\} \tag{4.11}$$

and

$$S_2=S'-S_1\,,\ \text{where}\ S'=\{P\,;\,v^{(q)}(P)>0\}\,. \tag{4.12}$$

Set

$$\eta'=\int_{S_2}dV\,. \tag{4.13}$$

It is easily seen that η' goes to zero as η goes to zero. Take a function

$$w_\eta=v^{(q)}+\eta v\,. \tag{4.14}$$

Then

$$\begin{aligned}&\left|\,\|w_\eta\|_q-\left(\int_{S_1}|w_\eta|^q dV\right)^{1/q}\right|\\ &\quad\leqq\frac{1}{q}\left(\int_{S_2}(2\eta)^q dV\right)+\frac{1}{q}\eta^q+0(\eta^2)\\ &\quad=\frac{1}{q}2^q\eta^q\eta'+\frac{1}{q}\eta^q+0(\eta^2).\end{aligned} \tag{4.15}$$

However the quantity $\left(\int_{S_1}|w_\eta|^q dV\right)^{1/q}$ is a C^2 function in η and

$$\begin{aligned}&\left(\int_{S_1}|w_\eta|^q dV\right)^{1/q}-\left(\int_{S_1}(v^{(q)})^q dV\right)^{1/q}-\left(\int_{S_1}(v^{(q)})^q dV\right)^{(1/q)-1}\cdot\\ &\quad\cdot\int_{S_1}(v^{(q)})^{q-1}v\,dV\leqq C_4\eta^2\,,\end{aligned} \tag{4.16}$$

where C_4 is a positive constant.

On the other hand

$$\left|\left(\int_S(v^{(q)})^q dV\right)^{1/q}-\left(\int_{S_1}(v^{(q)})^q dV\right)^{1/q}\right|\leqq\eta(\eta')^{1/q}\,, \tag{4.17}$$

$$\left|\int_S(v^{(q)})^q dV-\int_{S_1}(v^{(q)})^q dV\right|=\eta^q\eta' \tag{4.18}$$

and

$$\left|\int_S (v^{(q)})^{q-1} v\,dV - \int_{S_1} (v^{(q)})^{q-1} v\,dV\right| = \eta^{q-1}\eta'. \tag{4.19}$$

These are all obtained in the same manner as (4.15). Combining (4.15), (4.16), (4.17) (4.18) and (4.19) we have

$$\begin{aligned}&\lim_{\eta\to 0}\frac{1}{\eta}(\|w_\eta\|_q - \|v^{(q)}\|_q)\\ &= \lim_{\eta\to 0}\frac{1}{\eta}(\|w_\eta\|_q - 1) = \int_S (v^{(q)})^{q-1} v\,dV.\end{aligned} \tag{4.20}$$

Now set

$$H_\varepsilon(u) = F_\varepsilon(u)\|u\|_q^2 \tag{4.21}$$

and let η be a real number with small absolute value. Then

$$F_\varepsilon(v^{(q)}) = H_\varepsilon(v^{(q)}) = \lim_i F_\varepsilon(u_i) = \lim_i H_\varepsilon(u_i) = \min_u F_\varepsilon(u) = \mu_{(q)}. \tag{4.22}$$

Hence

$$\begin{aligned}0 &\leq \frac{1}{|\eta|}(F_\varepsilon(w_\eta) - F_\varepsilon(v^{(q)})) = \frac{1}{|\eta|}(F_\varepsilon(w_\eta) - \mu_{(q)})\\ &= \frac{1}{|\eta|}\frac{1}{\|w_\eta\|_q}(H_\varepsilon(w_\eta) - \mu_{(q)}\|w_\eta\|_q^2)\\ &= \frac{\operatorname{sgn}(\eta)}{\|w_\eta\|_q^2}\left(-2\int_S\left(\frac{4(d-1)}{d-2}\Delta v^{(q)} - Rv^{(q)} + \mu_{(q)}(v^{(q)})^{q-1}\right)v\,dV\right) + \Phi_\eta,\end{aligned} \tag{4.23}$$

where Φ_η tends to zero as η goes to zero.

In order for this inequality to hold for a positive η as well as a negative η,

$$\int_S\left(\frac{4(d-1)}{d-2}\Delta v^{(q)} - Rv^{(q)} + \mu_{(q)}(v^{(q)})^{q-1}\right)v\,dV = 0. \tag{4.24}$$

Since v can range over all C^∞ functions with $\sup_P|v(P)| \leq 1$, $v^{(q)}$ must satisfy

$$\frac{4(d-1)}{d-2}\Delta v^{(q)} - Rv^{(q)} = -\mu_{(q)}(v^{(q)})^{q-1} \tag{4.25}$$

in the sense of weak solution. This is the same equation as (4.8) and (1.12).

This completes the proof.

REMARK. q has only to be larger than 2.

5. In this paragraph we shall prove that a weak solution $v^{(q)}$ of (4.25) which gives the minimal value of $F_\varepsilon(u)$ for $q=p_\varepsilon$, is actually a C^∞ function solution, and thus the gap between Lemma 6 and Lemma 7 will be closed.

Lemma 8. *The non-negative function $v^{(q)}$, $q\geqq 2$, satisfying* (4.25) *in the sense of a weak solution, is C^2 everywhere and C^∞ except at zero points of $v^{(q)}$.*

REMARK. By virtue of Lemma 6, there is no zero point of $v^{(q)}$.

Proof. Firstly, the boundedness of $v^{(q)}$ will be proved.

By $G(P, Q)$ we denote the Green's function[5] for $\frac{4(d-1)}{d-2}\Delta$. The Sobolev's lemma will be formulated in the following form. If

$$u_1(P) = \int G(P, Q)u(Q)dV(Q)\,, \tag{5.1}$$

where u belongs to $L_{q'}$, then u_1 belongs to $L_{p'}$ where

$$(p')^{-1}\geqq (q')^{-1}-(2/d), \tag{5.2}$$

and

$$||u_1||_{p'} = C_5||u||_{q'}\,. \tag{5.3}$$

Here C_5 is an absolute constant if $(p')^{-1}-(q')^{-1}+(2/d)$ is larger than a fixed constant. Applying this to (4.25),

$$\begin{aligned} v^{(q)} &= -\int G(P, Q)(-\mu_{(q)}(v^{(q)})^{q-1}+Rv^{(q)})dV(Q)+\int v^{(q)}dV(Q) \\ &\equiv -\int G(P, Q)A(Q)dV(Q)+\int v^{(q)}dV(Q) \end{aligned} \tag{5.4}$$

where the function $A(Q)$ belongs to L_{m_1} with

$$m_1 = \left(\frac{2d}{d-2}-\varepsilon\right)\left(\frac{d+2}{d-2}-\varepsilon\right)^{-1}.$$

Hence $v^{(q)}$ belongs to $L_{q_1'}$ where

$$\begin{aligned} (q_1')^{-1} &= (m_1)^{-1}-(2/d) \\ &\leqq \frac{d-2}{2d}\left(1-\frac{d-2}{d}\,\varepsilon\right)\left(1+\frac{d-2}{2d}\,\varepsilon\right)+O(\varepsilon^2) \\ &= \frac{d-2}{2d}\left(1-\frac{d-2}{2d}\,\varepsilon\right)+O(\varepsilon^2) \end{aligned} \tag{5.5}$$

5) see [3]

where ε is small. Hence we can find a q_1 such that

$$q_1 = \frac{2d}{d-2} + \zeta > \frac{2d}{d-2} \tag{5.6}$$

where ζ is a positive real number.

Now, notice that the quantity $A(Q)$ belongs to L_{m_2} where

$$m_2 = \left(\frac{2d}{d-2} + \zeta\right)\left(\frac{d+2}{d-2} - \varepsilon\right)^{-1}. \tag{5.7}$$

Then, again by virtue of Sobolev's lemma, $v^{(q)}$ belongs to $L_{q'_2}$ with

$$\begin{aligned}(q'_2)^{-1} &= (m_2)^{-1} - (2/d) \leqq \frac{d+2}{2d+(d-2)\zeta} - \frac{2}{d} \\ &= \frac{d(d+2)-4d-2(d-2)\zeta}{(2d+(d-2)\zeta)d} = \frac{d-2}{2d}\left(1-\frac{2}{d}\zeta\right)\left(1+\frac{d-2}{2d}\zeta\right)^{-1}.\end{aligned} \tag{5.8}$$

Therefore q_2 can be taken as

$$q_2 \geqq \frac{2d}{d-2}\left(1-\frac{2}{d}\zeta\right)^{-1}\left(1+\frac{d-2}{2d}\zeta\right) \tag{5.9}$$

or in particular

$$q_2 = \frac{2d}{d-2} + \frac{d+2}{d-2}\zeta.$$

By repeating these procedures, we can easily show that $v^{(q)}$ belongs to L_{q_n}, with

$$q_n = \frac{2d}{d-2} + \left(\frac{d+2}{d-2}\right)^{n-1}\zeta. \tag{5.10}$$

Take an integer n large enough so that

$$q_n = \left(\frac{d}{2} + \zeta_1\right)\frac{d+2}{d-2} > \frac{d}{2}\,\frac{d+2}{d-2}. \tag{5.11}$$

Here ζ_1 is a positive real number. Then,

$$\sup_P v^{(q)}(P) \leqq \sup_P \|(P, Q)\|_{q_n/(q_n-1)} \|v^{(q)}\|_{q_n} + \text{finite number.} \tag{5.12}$$

The right hand side is bounded because the part involving the Green's function is finite.

Once the essential boundedness is established, apply $G(P, Q)$, and we have the proof immediately[6].

6. Proofs of Theorems B and C.

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6) See Appendix.

Theorem B follows immediately from Lemmas 6, 7, and 8. We shall proceed to prove Theorem C.

Lemma 9. *The family of functions* $\{v^{(q)}\}$ *are uniformly bounded for* $2<q<2d/(d-2)$.

Proof. Take a positive fixed $\zeta_2>1$. Using the procedure in the proof of Lemma 8, starting at $q_1=2d/(d-2)+\zeta_2$, we can see that at each i

$$(6.1) \qquad (q_{i+1})^{-1}-(q_i)^{-1}\left(\frac{d+2}{d-2}\right)+\frac{2}{d}=\frac{1}{q_{i+1}q_i}\frac{\zeta_2^2}{\left(\frac{d+2}{d-2}\right)}>\frac{8}{d^3}\zeta_2^2$$

and if

$$q_n=\left(\frac{d}{2}+\zeta_1\right)\frac{d+2}{d-2},$$

then

$$(6.2) \qquad \|v^{(q)}\|_{q_n}\leqq C_5^{n-1}\|v^{(q)}\|_{q_1}.$$

Here C_5 is defined in (5.3). From this it follows that

$$(6.3) \qquad \begin{aligned}\Lambda(q)&\leqq \sup_P\|G(P,Q)\|_{q_n/(q_n-1)}C_5^{n-1}\|v^{(q)}\|_{q_1}\\ &\leqq C_6\|v^{(q)}\|_{q_1}\end{aligned}$$

where

$$(6.4) \qquad \Lambda(q)=\sup_P v^{(q)}(P)$$

and C_6 is an absolute constant.

However

$$(6.5) \qquad \begin{aligned}(\|v^{(q)}\|_{q_1})^{q_1}&\leqq \Lambda(q)^{q_1-q}(\|v^{(q)}\|_q)^q\\ &=\Lambda(q)^{q_1-q}=\Lambda(q)^{\zeta_2+\varepsilon}.\end{aligned}$$

Hence for small ε,

$$(6.6) \qquad \begin{aligned}\|v^{(q)}\|_{q_1}&\leqq \Lambda(q)^{(\zeta_2+\varepsilon)/q_1}\\ &\leqq \Lambda(q)^{\zeta_2}.\end{aligned}$$

Notice that $\Lambda(q)$ may be assumed to be $\leqq 1$.

Combining this with (6.3), we can see that

$$(6.7) \qquad \Lambda(q)\leqq C_6\Lambda(q)^{\zeta_2},$$

or

$$\Lambda(q)^{1-\zeta_2}\leqq C_6.$$

This proves the uniformly boundedness of $v^{(q)}$'s.

Proof of Theorem C. Since $q^{(q)}$'s are uniformly bounded,

$$\lim_{q\to 2d/(d-2)}\int_S G(P,Q)(Rv^{(q)}-\mu_{(q)}(v^{(q)})^{q-1})\,dV(Q) + \int_S v^{(q)}(Q)dV(Q) = \bar{u} \tag{6.11}$$

converges uniformly to a C^1 function when we take a suitable sequence of q's. This limit $\bar{u}$ must satisfy

$$\frac{4(d-1)}{(d-2)}\Delta\bar{u}-R\bar{u} = -\mu_{(2d/(d-2))}(\bar{u})^{(d+2)/(d-2)} \tag{6.14}$$

weakly. From this we can easily obtain the C^2 property for $\bar{u}$ because of the boundedness of $\bar{u}$. Again, Lemma 6 is available and $\bar{u}$ can be proved to be C^∞ because it is bounded C^2 function without zero points, satisfying (6.14). This is nothing but the equation (1.10). Thus Theorem C has been proved.

A direct consequnce of Therem A is that if R is everywhere non negative, then $\bar{R}$, the scalar curvature of the new structure, is a non-negative constant and is zero just in case R is everywhere zero. If R is everywhere non-positive and not identically zero, then $\bar{R}$ is negative because it is less than $F^{(2d/(d-2))}(1)<0$.

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Appendix

Supplement to the proof of Lemma 8.

Once the essential boundedness of $v^{(q)}$ is established, it immediately follows that $v^{(q)}$ is C^1. Hence $v^{(q)}$, being a solution of an equation

$$\Delta v^{(q)} = Rv^{(q)}-\mu_{(q)}(v^{(q)})^{q-1},$$

is a C^2 solution except at zero point of $v^{(q)}$. Repeating this kind of procedures, we can see that $v^{(q)}$ is C^∞ except at zero point of $v^{(q)}$.

REMARK. As is seen very easily, if the original structure is C^k, $k \geqq 3$, and ω, then $\bar{u}$ itself is also C^k, $k \geqq 1$ and ω.

REMARK. Prof. J. Serrin notified the author that Lemma 6 can be proved by using E. Hopf's maximum principle (cf. [5]).

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