

On a diophantine inequality involving prime numbers

by

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In 1952 Piatetski-Shapiro [4] considered the following analogue of the Goldbach–Waring problem.

Assume that $c > 1$ is not integer and let ε be a positive number. If r is a sufficiently large integer (depending only on c) then the inequality

$$(1) \quad |p_1^c + p_2^c + \dots + p_r^c - N| < \varepsilon$$

has a solution in prime numbers p_1, \dots, p_r for sufficiently large N . More precisely, if the least r such that (1) has a solution in prime numbers for every $\varepsilon > 0$ and $N > N_0(c, \varepsilon)$ is denoted by $H(c)$ then it is proved in [4] that

$$\limsup_{c \rightarrow \infty} \frac{H(c)}{c \log c} \leq 4.$$

Piatetski-Shapiro also proved that if $1 < c < 3/2$ then $H(c) \leq 5$. One can conjecture that if c is near to unity then $H(c) \leq 3$. On the other hand, instead of fixed ε , we may consider ε depending on N and tending to zero as N tends to infinity.

This conjecture is proved in [7] for $1 < c < 27/26$ and

$$\varepsilon = N^{-(1/c)(27/26-c)} \log^{13} N.$$

In this paper we sharpen the last result and prove the following theorem:

THEOREM. *Let $1 < c < 15/14$. There exists a number $N_0(c) > 0$ such that for each real number $N > N_0(c)$ the inequality*

$$|p_1^c + p_2^c + p_3^c - N| < N^{-(1/c)(15/14-c)} \log^9 N$$

has a solution in prime numbers p_1, p_2, p_3 .

Notation

c — fixed real number such that $1 < c < 15/14$,

N — sufficiently large number,

$$\begin{aligned}
(2) \quad & X = (N/2)^{1/c}, \\
(3) \quad & \tau = X^{13/28-c/2}, \\
(4) \quad & T = X^{41/56+c/4}, \\
(5) \quad & \varepsilon = X^{-(15/14-c)} \log^8 X, \\
(6) \quad & \Delta = \frac{1}{10} \varepsilon, \\
(7) \quad & r = [\log X],
\end{aligned}$$

where $[\alpha]$ denotes the integer part of the real number α ;

$$(8) \quad K = X^{15/14-c} (\log X)^{-6},$$

m, n, k, l, d, r (with or without subscripts) — integers,

p, p_1, p_2, \dots — prime numbers,

$\tau(n)$ — the number of positive divisors of n ,

$\Lambda(n)$ — von Mangoldt's function,

$\Psi(x) = \sum_{n \leq x} \Lambda(n)$ — Chebyshev's function,

$\rho = \beta + i\gamma$ — non-trivial zero of the Riemann zeta function $\zeta(s)$,

$\sum_{a < \gamma < b}$ — sum over the non-trivial zeroes of $\zeta(s)$ such that $a < \gamma < b$,

$e(x) = e^{2\pi i x}$,

$A \asymp B$ means $A \ll B \ll A$,

$$(9) \quad S(x) = \sum_{X/2 < p \leq X} \log p \cdot e(p^c x),$$

$$(10) \quad I(x) = \int_{X/2}^x e(t^c x) dt,$$

$$(11) \quad I_\rho(x) = \int_{X/2}^x e(t^c x) t^{\rho-1} dt,$$

$$(12) \quad J(x) = \sum_{|\gamma| \leq T} I_\rho(x).$$

The constants in O -terms and \ll -symbols are absolute or depend only on c .

To prove the theorem we need some lemmas.

LEMMA 1. *Let a, δ be real numbers, $0 < \delta < a/4$, and let k be an integer. There exists a function $\varphi(y)$ which is k times continuously differentiable and such that*

$$\begin{aligned}
\varphi(y) &= 1 && \text{for } |y| \leq a - \delta, \\
0 < \varphi(y) &< 1 && \text{for } a - \delta < |y| < a + \delta, \\
\varphi(y) &= 0 && \text{for } |y| \geq a + \delta,
\end{aligned}$$

and its Fourier transform

$$\Phi(x) = \int_{-\infty}^{\infty} e(-xy)\varphi(y) dy$$

satisfies the inequality

$$|\Phi(x)| \leq \min \left(2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left(\frac{k}{2\pi|x|\delta} \right)^k \right).$$

Proof. See [4] or [5].

Throughout this paper we denote by $\varphi(y)$ the function from Lemma 1 with parameters $a = \frac{9}{10}\varepsilon$, $\delta = \Delta$, $k = r$, and by $\Phi(x)$ the Fourier transform of $\varphi(y)$.

LEMMA 2. Assume that $G(x)$, $F(x)$ are real functions defined in $[a, b]$, $|G(x)| \leq H$ for $a \leq x \leq b$ and $G(x)/F'(x)$ is a monotonous function. Set

$$I = \int_a^b G(x)e(F(x)) dx.$$

If $F'(x) \geq h > 0$ for all $x \in [a, b]$ or if $F'(x) \leq -h < 0$ for all $x \in [a, b]$ then

$$|I| \ll H/h.$$

If $F''(x) \geq h > 0$ for all $x \in [a, b]$ then

$$|I| \ll H/\sqrt{h}.$$

Proof. See [6, p. 71].

LEMMA 3. Suppose that $f''(t)$ exists, is continuous on $[a, b]$ and satisfies

$$f''(t) \asymp \lambda \quad (\lambda > 0) \quad \text{for } t \in [a, b].$$

Then

$$\left| \sum_{a < n \leq b} e(f(n)) \right| \ll (b-a)\lambda^{1/2} + \lambda^{-1/2}.$$

Proof. See [6, p. 104].

LEMMA 4. If $2 \leq t \leq x$ then

$$\Psi(x) = x - \sum_{|\gamma| \leq t} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{t}\right).$$

Proof. See [3, p. 80].

LEMMA 5. Assume that $1 \leq R \leq Y$. There exists a number $\gamma_1 > 0$ such that if

$$\theta(R) = \frac{\gamma_1}{\log^{2/3}(R + 10) \log \log(R + 10)}$$

then the following estimate holds:

$$\sum_{0 < \gamma \leq R} Y^\beta \ll \begin{cases} Y^{1/2} R \log^6 Y, & Y^{3/4} \leq R \leq Y, \\ e^{2 \log Y + 3 \log R - 2 \sqrt{3 \log Y \log R}} \log^6 Y, & Y^{1/3} \leq R \leq Y^{3/4}, \\ Y^{1 - \theta(R)} R^{2.4\theta(R)} \log^{45} Y, & 1 \leq R \leq Y^{1/3}. \end{cases}$$

Proof. This lemma can be deduced from the density theorems of Ingham (see [6, p. 236]) and Huxley (see [2]) combined with the estimate of I. M. Vinogradov for the zero-free region of $\zeta(s)$ (see [3, p. 100]). Detailed computations may be found in [1].

LEMMA 6. We have

$$\int_{-\infty}^{\infty} I^3(x) e(-Nx) \Phi(x) dx \gg \varepsilon X^{3-c}.$$

Proof. Denote the above integral by H . We have

$$H = \int_{X/2}^X \int_{X/2}^X \int_{-\infty}^{\infty} e((t_1^c + t_2^c + t_3^c - N)x) \Phi(x) dx dt_1 dt_2 dt_3$$

(the change of the order of integration is legitimate because of the absolute convergence of the integral). Using the Fourier inversion formula we get

$$H = \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \varphi(t_1^c + t_2^c + t_3^c - N) dt_1 dt_2 dt_3$$

and by the definition of $\varphi(y)$ we get

$$H \geq \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X dt_1 dt_2 dt_3 \geq \int_{\lambda X}^{\mu X} \int_{\lambda X}^{\mu X} \left(\int_{\mathfrak{M}} dt_3 \right) dt_1 dt_2,$$

$|t_1^c + t_2^c + t_3^c - N| < 4\varepsilon/5$

where λ and μ are real numbers such that

$$(13) \quad \frac{1}{2} < \left(\frac{1}{2}\right)^{1/c} < \lambda < \mu < \left(\frac{1}{2}\left(2 - \frac{1}{2^c}\right)\right)^{1/c} < 1$$

and

$$\mathfrak{M} = [X/2, X] \cap [(N - \frac{4}{5}\varepsilon - t_1^c - t_2^c)^{1/c}, (N + \frac{4}{5}\varepsilon - t_1^c - t_2^c)^{1/c}].$$

Because of (13) and the choice of N, X, ε it is easy to show that in fact \mathfrak{M} is the right interval of the above intersection. Thus by the mean-value

theorem

$$H \gg \varepsilon \int_{\lambda X}^{\mu X} \int_{\lambda X}^{\mu X} (\xi_{t_1, t_2})^{1/c-1} dt_1 dt_2,$$

where $\xi_{t_1, t_2} \asymp X^c$. Therefore, $H \gg \varepsilon X^{3-c}$, which proves the lemma.

LEMMA 7. *We have*

- (i) $\int_{-\tau}^{\tau} |S^2(x)| dx \ll X^{2-c} \log^3 X,$
- (ii) $\int_{-\tau}^{\tau} |I^2(x)| dx \ll X^{2-c} \log X,$
- (iii) $\int_n^{n+1} |S^2(x)| dx \ll X \log^3 X$

uniformly with respect to n .

Proof. We only prove (i). Inequalities (ii) and (iii) can be proved likewise.

We have

$$\begin{aligned} (14) \quad \int_{-\tau}^{\tau} |S^2(x)| dx &= \sum_{X/2 < p_1, p_2 \leq X} \log p_1 \log p_2 \int_{-\tau}^{\tau} e((p_1^c - p_2^c)x) dx \\ &\ll \sum_{X/2 < p_1, p_2 \leq X} \log p_1 \log p_2 \min\left(\tau, \frac{1}{|p_1^c - p_2^c|}\right) \\ &\ll U\tau \log^2 X + V \log^2 X, \end{aligned}$$

where

$$U = \sum_{\substack{X/2 < n_1, n_2 \leq X \\ |n_1^c - n_2^c| \leq 1/\tau}} 1, \quad V = \sum_{\substack{X/2 < n_1, n_2 \leq X \\ 1/\tau < |n_1^c - n_2^c|}} \frac{1}{|n_1^c - n_2^c|}.$$

We have

$$U \ll \sum_{\substack{X/2 < n_1 \leq X \\ (n_1^c - 1/\tau)^{1/c} \leq n_2 \leq (n_1^c + 1/\tau)^{1/c}}} \sum_{X/2 < n_2 \leq X} 1 \ll \sum_{X/2 < n_1 \leq X} (1 + (n_1^c + 1/\tau)^{1/c} - (n_1^c - 1/\tau)^{1/c})$$

and by the mean-value theorem

$$(15) \quad U \ll X + \frac{1}{\tau} X^{2-c}.$$

Obviously $V \leq \sum_l V_l$ where

$$(16) \quad V_l = \sum_{\substack{X/2 < n_1, n_2 \leq X \\ l < |n_1^c - n_2^c| \leq 2l}} \frac{1}{|n_1^c - n_2^c|}$$

and l takes the values $2^k/\tau$, $k = 0, 1, 2, \dots$, with $l \leq X^c$. We have

$$V_l \ll \frac{1}{l} \sum_{\substack{X/2 < n_1 \leq X \\ (n_1^c + l)^{1/c} \leq n_2 \leq (n_1^c + 2l)^{1/c}}} \sum_{X/2 < n_2 \leq X} 1.$$

For $l \geq 1/\tau$ and $X/2 < n_1 \leq X$ it is easy to see that

$$(n_1^c + 2l)^{1/c} - (n_1^c + l)^{1/c} > 1.$$

Hence

$$(17) \quad V_l \ll \frac{1}{l} \sum_{X/2 < n_1 \leq X} ((n_1^c + 2l)^{1/c} - (n_1^c + l)^{1/c}) \ll X^{2-c}$$

by the mean-value theorem.

The conclusion follows from formulas (3) and (14)–(17).

LEMMA 8. Let a_m, b_n be arbitrary complex numbers and let

$$\begin{aligned} \tau \leq |x| \leq K, \quad X^{1/4} < R \leq X^{1/2}, \\ X^{1/4} < L < L_1 \leq 2L, \quad LR \leq X. \end{aligned}$$

Define

$$W = \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq R} a_m b_n e((mn)^c x).$$

Then

$$|W| \ll (\mathcal{A}\mathcal{B})^{1/2} X^{3/7} (\log X)^{-1},$$

where

$$\mathcal{A} = \sum_{X^{1/4} < m \leq R} |a_m|^2, \quad \mathcal{B} = \sum_{L < n \leq L_1} |b_n|^2.$$

Proof. Obviously, we may suppose that $\tau \leq x \leq K$. Take $s \in [1, R]$ whose exact value will be determined later. We define the numbers R_i , $0 \leq i \leq Q$, in the following way:

$$R_0 = X^{1/4}, \quad R_{i+1} = \min(R_i + s, R), \quad R_Q = R.$$

Obviously $Q \ll R/s$. We have

$$W = \sum_{L < n \leq L_1} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i} a_m b_n e((mn)^c x),$$

hence

$$|W| \leq \sum_{L < n \leq L_1} |b_n| \left| \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i} a_m e((mn)^c x) \right|$$

and Cauchy's inequality gives

$$\begin{aligned} |W|^2 &\leq \mathcal{B} \sum_{L < n \leq L_1} \left| \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i} a_m e((mn)^c x) \right|^2 \\ &\leq \mathcal{B}Q \sum_{L < n \leq L_1} \sum_{1 \leq i \leq Q} \left| \sum_{R_{i-1} < m \leq R_i} a_m e((mn)^c x) \right|^2 \\ &= \mathcal{B}Q \sum_{L < n \leq L_1} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m_1, m_2 \leq R_i} a_{m_1} \bar{a}_{m_2} e((m_1^c - m_2^c)n^c x). \end{aligned}$$

After some rearrangements we obtain

$$\begin{aligned} (18) \quad |W|^2 &\ll \mathcal{B}Q \left(\sum_{L < n \leq L_1} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i} |a_m|^2 \right. \\ &\quad \left. + \sum_{1 \leq i \leq Q} \sum_{\substack{R_{i-1} < m_1, m_2 \leq R_i \\ m_1 \neq m_2}} |a_{m_1}| |a_{m_2}| \right) \\ &\quad \times \left| \sum_{L < n \leq L_1} e((m_1^c - m_2^c)n^c x) \right| \\ &\ll \mathcal{B}Q \left(\mathcal{A}L + \sum_{1 \leq h \leq s} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i - h} |a_m| |a_{m+h}| \right) \\ &\quad \times \left| \sum_{L < n \leq L_1} e(f(n)) \right|, \end{aligned}$$

where

$$f(n) = ((m + h)^c - m^c)n^c x.$$

It is easy to see that if $L < n \leq L_1$ then

$$f''(n) \asymp ((m + h)^c - m^c)L^{c-2}x$$

and by Lemma 3 we obtain

$$\begin{aligned} \left| \sum_{L < n \leq L_1} e(f(n)) \right| &\ll ((m + h)^c - m^c)^{1/2} L^{c/2} x^{1/2} \\ &\quad + ((m + h)^c - m^c)^{-1/2} L^{1-c/2} x^{-1/2} \\ &\ll ((m + h)^c - m^c)^{1/2} L^{c/2} K^{1/2} \\ &\quad + ((m + h)^c - m^c)^{-1/2} L^{1-c/2} \tau^{-1/2} \\ &\ll ((m + h)^c - m^c)^{1/2} L^{c/2} K^{1/2} \\ &\ll s^{1/2} R^{(c-1)/2} L^{c/2} K^{1/2}. \end{aligned}$$

We substitute this estimate in (18) to get

$$(19) \quad |W|^2 \ll \mathcal{B}Q(\mathcal{A}L + s^{1/2}R^{(c-1)/2}L^{c/2}K^{1/2}\Sigma_0),$$

where

$$\Sigma_0 = \sum_{1 \leq h \leq s} \sum_{1 \leq i \leq Q} \sum_{R_{i-1} < m \leq R_i - h} |a_m| |a_{m+h}|.$$

Clearly

$$\begin{aligned} \Sigma_0 &\leq \sum_{1 \leq h \leq s} \sum_{X^{1/4} < m \leq R-h} |a_m| |a_{m+h}| \\ &\leq \sum_{1 \leq h \leq s} \left(\sum_{X^{1/4} < m \leq R-h} |a_m|^2 \right)^{1/2} \left(\sum_{X^{1/4} < m \leq R-h} |a_{m+h}|^2 \right)^{1/2} \leq s\mathcal{A}. \end{aligned}$$

The last estimate, (19) and the inequality $Q \ll R/s$ give us

$$(20) \quad |W|^2 \ll \mathcal{A}BLR(1/s + s^{1/2}R^{(c-1)/2}L^{(c-2)/2}K^{1/2}).$$

We now determine s in such a way that the two summands in brackets are equal. This gives the value

$$s = K^{-1/3}R^{(1-c)/3}L^{(2-c)/3}.$$

The assumption of our lemma and (8) imply easily $1 \leq s \leq R$. Substituting s in (20) we get after some calculations

$$|W|^2 \ll \mathcal{A}BX^{6/7}(\log X)^{-2},$$

i.e.

$$|W| \ll (\mathcal{A}B)^{1/2}X^{3/7}(\log X)^{-1},$$

which is the desired result.

LEMMA 9. Suppose that a_m, b_n are arbitrary complex numbers and that $L < L_1 \leq 2L, L \leq X$. Set

$$V = \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/n} a_m b_n e((mn)^c x).$$

Then

$$|V| \ll \log X \cdot \left| \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/L} a'_m b'_n e((mn)^c x) \right|$$

for some complex numbers a'_m, b'_n such that $|a'_m| \leq |a_m|, |b'_n| \leq |b_n|$.

Proof. Let q be an odd integer such that $2X/L \leq q \leq 4X/L$. We have

$$(21) \quad \begin{aligned} V &= \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/L} a_m b_n e((mn)^c x) \\ &\quad \times \sum_{X^{1/4} < m_1 \leq X/n} \frac{1}{q} \sum_{|k| \leq (q-1)/2} e\left(\frac{k(m-m_1)}{q}\right) \end{aligned}$$

$$= \sum_{|k| \leq (q-1)/2} \frac{1}{\max(1, |k|)} \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/L} a_m^{(k)} b_n^{(k)} e((mn)^c x),$$

where

$$a_m^{(k)} = a_m e\left(\frac{km}{q}\right), \quad b_n^{(k)} = b_n \frac{\max(1, |k|)}{q} \sum_{X^{1/4} < m_1 \leq X/n} e\left(-\frac{km_1}{q}\right).$$

Obviously $|a_m^{(k)}| = |a_m|$ and because of the well-known estimate

$$\left| \sum_{a < k \leq b} e(\alpha k) \right| \ll \min(b - a, 1/|\alpha|)$$

we get

$$|b_n^{(k)}| \ll |b_n|.$$

The absolute value of the double sum over m and n on the right-hand side of (21) takes its maximum for some value $k = k_0$.

Denote by a'_m, b'_n the numbers $a_m^{(k_0)}, b_n^{(k_0)}$, multiplied by a sufficiently small positive constant.

Obviously this proves the lemma.

LEMMA 10. Assume that $\tau \leq |x| \leq K$. Then

$$|S(x)| \ll X^{13/14} \log^3 X.$$

Proof. Without loss of generality we may assume that $\tau \leq x \leq K$. Clearly

$$S(x) = V_0(x) - V_1(x) + O(X^{1/2}),$$

where

$$V_i(x) = \sum_{X^{1/4} < n \leq 2^{-i} X} \Lambda(n) e(n^c x), \quad i = 0, 1.$$

Hence it is sufficient to prove that

$$(22) \quad |V_0(x)|, |V_1(x)| \ll X^{13/14} \log^3 X.$$

Consider, for instance, $V_0(x)$. We use Vaughan's identity (see [8]) to get

$$(23) \quad V_0(x) = S_1 - S_2 - S_3,$$

where

$$\begin{aligned} S_1 &= \sum_{d \leq X^{1/4}} \mu(d) \sum_{l \leq X/d} \log l \cdot e((ld)^c x), \\ S_2 &= \sum_{k \leq X^{1/2}} \sum_{r \leq X/k} c_k e((kr)^c x), \\ S_3 &= \sum_{X^{1/4} < m \leq X^{3/4}} \sum_{X^{1/4} < n \leq X/m} a_m \Lambda(n) e((mn)^c x), \end{aligned}$$

where $|c_k| \leq \log k, |a_m| \leq \tau(m)$.

We split the sum S_2 in the following way:

$$(24) \quad S_2 = S_2^{(1)} + S_2^{(2)} + O(X^{3/4} \log X),$$

where

$$S_2^{(1)} = \sum_{k \leq X^{1/4}} \sum_{r \leq X/k} c_k e((kr)^c x),$$

$$S_2^{(2)} = \sum_{X^{1/4} < k \leq X^{1/2}} \sum_{X^{1/4} < r \leq X/k} c_k e((kr)^c x).$$

Let us treat S_1 . We have

$$(25) \quad |S_1| \leq \sum_{d \leq X^{1/4}} \left| \sum_{l \leq X/d} \log l \cdot e((ld)^c x) \right|.$$

We break the sum over l into sums of the type

$$T_L = \sum_{L < l \leq L_1} \log l \cdot e((ld)^c x)$$

whose number is $O(\log X)$ and $L < L_1 \leq 2L$, $L_1 \leq X/d$. Abel's transformation formula gives us

$$(26) \quad |T_L| \ll \log X \max_{L_2 \in [L, L_1]} \left| \sum_{L < l \leq L_2} e(g(l)) \right|,$$

where $g(l) = (ld)^c x$. Obviously $g''(l) \asymp L^{c-2} d^c x$ for $L \leq l \leq L_2$ and by Lemma 3

$$\begin{aligned} \left| \sum_{L < l \leq L_2} e(g(l)) \right| &\ll L^{c/2} d^{c/2} x^{1/2} + L^{1-c/2} d^{-c/2} x^{-1/2} \\ &\leq L^{c/2} d^{c/2} K^{1/2} + L^{1-c/2} d^{-c/2} \tau^{-1/2} \\ &\leq X^{c/2} K^{1/2} + X^{1-c/2} d^{-1} \tau^{-1/2} \ll X^{c/2} K^{1/2}. \end{aligned}$$

The last estimate, together with (25) and (26), implies

$$(27) \quad |S_1| \ll \log^2 X \cdot X^{1/4+c/2} K^{1/2} \ll X^{11/14}.$$

We estimate $S_2^{(1)}$ in the same way to get

$$(28) \quad |S_2^{(1)}| \ll X^{11/14}.$$

To estimate $S_2^{(2)}$ and S_3 we need Lemmas 8 and 9. Consider S_3 . We split it in the following way:

$$(29) \quad S_3 = W_1 + W_2 + W_3,$$

where

$$\begin{aligned} W_1 &= \sum_{X^{1/2} < n \leq X^{3/4}} \sum_{X^{1/4} < m \leq X/n} a_m \Lambda(n) e((mn)^c x), \\ W_2 &= \sum_{X^{1/2} < n \leq X^{3/4}} \sum_{X^{1/4} < m \leq X/n} \Lambda(m) a_n e((mn)^c x), \\ W_3 &= \sum_{X^{1/4} < n \leq X^{1/2}} \sum_{X^{1/4} < m \leq X^{1/2}} a_m \Lambda(n) e((mn)^c x). \end{aligned}$$

We break W_1 into sums of the type

$$W_1(L) = \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/n} a_m \Lambda(n) e((mn)^c x)$$

whose number is $O(\log X)$ and $X^{1/2} \leq L < L_1 \leq X^{3/4}$, $L_1 \leq 2L$. According to Lemma 9 there exist complex numbers a'_m, b'_n such that $|a'_m| \leq |a_m| \leq \tau(m)$, $|b'_n| \leq \Lambda(n)$ and

$$|W_1(L)| \ll (\log X) \cdot |W'_1(L)|,$$

where

$$W'_1(L) = \sum_{L < n \leq L_1} \sum_{X^{1/4} < m \leq X/L} a'_m b'_n e((mn)^c x).$$

Using the well-known inequalities

$$\sum_{m \leq y} \tau^2(m) \ll y \log^3 y, \quad \sum_{n \leq y} \Lambda^2(n) \ll y \log y$$

and Lemma 8 we obtain

$$|W'_1(L)| \ll \left(\frac{X}{L} \log^3 X \cdot L \log X \right)^{1/2} X^{3/7} (\log X)^{-1} = X^{13/14} \log X,$$

hence

$$(30) \quad |W_1| \ll X^{13/14} \log^3 X.$$

We estimate W_2 and W_3 in the same way (for W_3 , of course, we do not use Lemma 9) to get

$$(31) \quad |W_2|, |W_3| \ll X^{13/14} \log^3 X.$$

We treat $S_2^{(2)}$ analogously:

$$(32) \quad S_2^{(2)} = U_1 + U_2,$$

where

$$U_1 = \sum_{X^{1/4} < r \leq X^{1/2}} \sum_{X^{1/4} < k \leq X^{1/2}} c_k e((kr)^c x),$$

$$U_2 = \sum_{X^{1/2} < r \leq X^{3/4}} \sum_{X^{1/4} < k \leq X/r} c_k e((kr)^c x).$$

We estimate these sums just as W_1, W_2, W_3 , with the help of Lemmas 8 and 9, using the inequality

$$\sum_{k \leq y} \log^2 k \ll y \log^2 y.$$

Thus we get

$$(33) \quad |U_1|, |U_2| \ll X^{13/14} \log^3 X.$$

The inequality (22) follows from (23), (24), (27)–(33). The lemma is proved.

LEMMA 11. *If $1 \leq T_1 \leq T$ then*

$$\frac{1}{\sqrt{T_1}} \sum_{0 < \gamma \leq T_1} X^\beta \ll X e^{-(\log X)^{1/4}}.$$

Proof. Assume that $T_1 \leq X^{1/3}$. Then using Lemma 5 we have

$$\begin{aligned} \frac{1}{\sqrt{T_1}} \sum_{0 < \gamma \leq T_1} X^\beta &\leq \sum_{0 < \gamma \leq T_1} X^\beta \leq \sum_{0 < \gamma \leq X^{1/3}} X^\beta \\ &\ll \log^{45} X \cdot X^{1-0.2\theta(X^{1/3})} \ll X e^{-(\log X)^{1/4}}. \end{aligned}$$

If $X^{1/3} \leq T_1 \leq X^{3/4}$ then using Lemma 5 we get

$$\frac{1}{\sqrt{T_1}} \sum_{0 < \gamma \leq T_1} X^\beta \ll \log^6 X \cdot e^\omega,$$

where

$$\omega = 2 \log X + \frac{5}{2} \log T_1 - 2\sqrt{3 \log X \log T_1}.$$

Obviously ω is a convex function of $\sqrt{\log T_1}$. Thus

$$\max_{X^{1/3} \leq T_1 \leq X^{3/4}} \omega = \max(\omega|_{T_1=X^{1/3}}, \omega|_{T_1=X^{3/4}}) = \frac{7}{8} \log X.$$

Hence

$$\log^6 X \cdot e^\omega \ll X^{7/8} \log^6 X \ll X e^{-(\log X)^{1/4}}.$$

If $X^{3/4} \leq T_1 \leq T$ then Lemma 5 gives us

$$\begin{aligned} \frac{1}{\sqrt{T_1}} \sum_{0 < \gamma \leq T_1} X^\beta &\ll \log^6 X \cdot \frac{1}{\sqrt{T_1}} X^{1/2} T_1 \\ &\leq \log^6 X \cdot X^{1/2} T^{1/2} \ll X e^{-(\log X)^{1/4}}. \end{aligned}$$

Lemma 11 is proved.

LEMMA 12. *If $1 \leq T_1 \leq T$ then*

$$\frac{1}{T_1} \sum_{0 < \gamma \leq T_1} X^\beta \ll X e^{-(\log X)^{1/4}}.$$

PROOF. This lemma follows immediately from Lemma 11.

LEMMA 13. *If $|x| \leq \tau$ then*

$$|J(x)| \ll X e^{-(\log X)^{1/5}},$$

where $J(x)$ is defined in (12).

PROOF. Without loss of generality we may assume that $x \geq 0$. By the definition we have

$$I_\varrho(x) = \int_{X/2}^X t^{\beta-1} e(F(t)) dt$$

where

$$F(t) = t^c x + \frac{\gamma}{2\pi} \log t.$$

Define the following sets of non-trivial zeroes of $\zeta(s)$:

$$\begin{aligned} M_1 &= \{\varrho \mid |\gamma| \leq T, -\gamma/2\pi > \frac{3}{2}cX^c x\}, \\ M_2 &= \{\varrho \mid |\gamma| \leq T, \frac{1}{2}c(X/2)^c x \leq -\gamma/2\pi \leq \frac{3}{2}cX^c x\}, \\ M_3 &= \{\varrho \mid |\gamma| \leq T, -\gamma/2\pi < \frac{1}{2}c(X/2)^c x\}. \end{aligned}$$

(The set M_2 may be empty.)

Let $X^{-c} \leq x \leq \tau$. If $\varrho \in M_3$ then

$$F'(t) \gg \frac{1}{X} \left(c \left(\frac{X}{2} \right)^c x - \frac{-\gamma}{2\pi} \right) > 0$$

and according to Lemma 2

$$|I_\varrho(x)| \ll \frac{X^\beta}{c(X/2)^c x + \gamma/2\pi}.$$

Hence

$$\begin{aligned} \sum_{\varrho \in M_3} |I_\varrho(x)| &\ll \sum_{-\pi c(X/2)^c x \leq \gamma \leq T} \frac{X^\beta}{c(X/2)^c x + \gamma/2\pi} \\ &\ll \sum_{-\pi c(X/2)^c x \leq \gamma \leq X^c x} \frac{X^\beta}{X^c x} + \sum_{X^c x < \gamma \leq T} \frac{X^\beta}{\gamma} \\ &\ll \log X \cdot \max_{1 \leq T_1 \leq T} \left(\frac{1}{T_1} \sum_{0 < \gamma \leq T_1} X^\beta \right). \end{aligned}$$

Using Lemma 12 we obtain

$$(34) \quad \sum_{\varrho \in M_3} |I_\varrho(x)| \ll Xe^{-(\log X)^{1/5}}.$$

If $\varrho \in M_1$ then

$$F'(t) \leq \frac{1}{t} \left(cX^c x + \frac{\gamma}{2\pi} \right) \leq \frac{1}{X} \left(cX^c x + \frac{\gamma}{2\pi} \right) < 0$$

and by Lemma 2 we get

$$|I_\varrho(x)| \ll \frac{X^\beta}{-\gamma/2\pi - cX^c x} \ll \frac{X^\beta}{|\gamma|}.$$

Therefore

$$\begin{aligned} \sum_{\varrho \in M_1} |I_\varrho(x)| &\ll \sum_{0 < \gamma \leq T_1} \frac{X^\beta}{\gamma} \\ &\ll \log X \cdot \max_{1 \leq T_1 \leq T} \left(\frac{1}{T_1} \sum_{0 < \gamma \leq T_1} X^\beta \right) \end{aligned}$$

and Lemma 12 gives

$$(35) \quad \sum_{\varrho \in M_1} |I_\varrho(x)| \ll Xe^{-(\log X)^{1/5}}.$$

If $\varrho \in M_2$ then $-\gamma/2\pi \asymp X^c x$ and $F''(t) \asymp X^{c-2}x$. According to Lemma 2

$$|I_\varrho(x)| \ll \frac{X^\beta}{\sqrt{X^c x}}.$$

Hence, using Lemma 11 we get

$$(36) \quad \sum_{\varrho \in M_2} |I_\varrho(x)| \ll \frac{1}{\sqrt{X^c x}} \sum_{0 < \gamma \leq X^c x} X^\beta \ll Xe^{-(\log X)^{1/4}}.$$

(If the set M_2 is empty then the last estimate is also true.)

Inequalities (34)–(36) give us

$$(37) \quad |J(x)| \ll Xe^{-(\log X)^{1/5}}.$$

If $0 \leq x \leq X^{-c}$ then estimating $|I_\varrho(x)|$ trivially we have

$$\sum_{\varrho \in M_2} |I_\varrho(x)| \ll \sum_{\varrho \in M_2} X^\beta \ll X^{\beta_0},$$

where $\beta_0 = \max_{\varrho \in M_2} \beta < 1$. The terms $\sum_{\varrho \in M_1} |I_\varrho(x)|$ and $\sum_{\varrho \in M_3} |I_\varrho(x)|$ can be estimated as in the previous case. Thus (37) is always true.

LEMMA 14. *If $|x| \leq \tau$ then*

$$S(x) = I(x) + O(Xe^{-(\log X)^{1/5}}).$$

Proof. Obviously

$$(38) \quad S(x) = U(x) + O(X^{1/2}),$$

where

$$U(x) = \sum_{X/2 < n \leq X} \Lambda(n)e(n^c x).$$

Using Abel's transformation we get

$$U(x) = - \int_{X/2}^X (\Psi(t) - \Psi(X/2)) \frac{d}{dt}(e(t^c x)) dt + (\Psi(X) - \Psi(X/2))e(X^c x).$$

According to Lemma 4 we obtain

$$\begin{aligned} U(x) &= - \int_{X/2}^X \left(t - \frac{X}{2} - \sum_{|\gamma| \leq T} \frac{t^\rho - (X/2)^\rho}{\rho} \right. \\ &\quad \left. + O\left(\frac{X \log^2 X}{T}\right) \right) \frac{d}{dt}(e(t^c x)) dt \\ &\quad + \left(X - \frac{X}{2} - \sum_{|\gamma| \leq T} \frac{X^\rho - (X/2)^\rho}{\rho} + O\left(\frac{X \log^2 X}{T}\right) \right) e(X^c x) \\ &= - \int_{X/2}^X \left(t - \frac{X}{2} - \sum_{|\gamma| \leq T} \frac{t^\rho - (X/2)^\rho}{\rho} \right) \frac{d}{dt}(e(t^c x)) dt \\ &\quad + \left(X - \frac{X}{2} - \sum_{|\gamma| \leq T} \frac{X^\rho - (X/2)^\rho}{\rho} \right) e(X^c x) + O\left(\frac{\tau X^{1+c}}{T} \log^2 X\right). \end{aligned}$$

Using partial integration we get

$$\begin{aligned} U(x) &= \int_{X/2}^X e(t^c x) \left(1 - \sum_{|\gamma| \leq T} t^{\rho-1} \right) dt + O\left(\frac{\tau X^{1+c}}{T} \log^2 X\right) \\ &= I(x) - J(x) + O\left(\frac{\tau X^{1+c}}{T} \log^2 X\right). \end{aligned}$$

The conclusion follows from formulas (3), (4), (38) and Lemma 13.

Proof of the Theorem. It is sufficient to show that the sum

$$B = \sum_{\substack{X/2 < p_1, p_2, p_3 \leq X \\ |p_1^c + p_2^c + p_3^c - N| < \varepsilon}} \log p_1 \log p_2 \log p_3$$

tends to infinity as X tends to infinity. Set

$$B_1 = \sum_{X/2 < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \cdot \varphi(p_1^c + p_2^c + p_3^c - N).$$

By the definition of $\varphi(y)$ we have

$$(39) \quad B \geq B_1.$$

The Fourier transformation formula gives us

$$\begin{aligned} B_1 &= \sum_{X/2 < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \int_{-\infty}^{\infty} e((p_1^c + p_2^c + p_3^c - N)x) \Phi(x) dx \\ &= \int_{-\infty}^{\infty} S^3(x) e(-Nx) \Phi(x) dx. \end{aligned}$$

Let us represent B_1 in the form

$$(40) \quad B_1 = D_1 + D_2 + D_3,$$

where

$$\begin{aligned} D_1 &= \int_{-\tau}^{\tau} S^3(x) e(-Nx) \Phi(x) dx, \\ D_2 &= \int_{\tau < |x| < K} S^3(x) e(-Nx) \Phi(x) dx, \\ D_3 &= \int_{|x| > K} S^3(x) e(-Nx) \Phi(x) dx. \end{aligned}$$

Using (5)–(8) and Lemma 1 we have

$$\begin{aligned} (41) \quad |D_3| &\ll \int_K^{\infty} |S^3(x)| |\Phi(x)| dx \\ &\ll X^3 \int_K^{\infty} \frac{1}{x} \left(\frac{r}{2\pi \Delta x} \right)^r dx \ll X^3 \left(\frac{r}{2\pi \Delta K} \right)^r \ll 1. \end{aligned}$$

We next treat D_2 :

$$\begin{aligned} (42) \quad |D_2| &\ll \int_{\tau}^K |S^3(x)| |\Phi(x)| dx \\ &\ll \left(\max_{\tau \leq x \leq K} |S(x)| \right) \int_{\tau}^K |S^2(x)| |\Phi(x)| dx. \end{aligned}$$

According to Lemma 1 we get

$$\begin{aligned} \int_{\tau}^K |S^2(x)| |\Phi(x)| dx &\ll \varepsilon \int_{\tau}^{1/\varepsilon} |S^2(x)| dx + \int_{1/\varepsilon}^K |S^2(x)| \frac{dx}{x} \\ &\ll \varepsilon \sum_{0 \leq n \leq 1/\varepsilon} \int_n^{n+1} |S^2(x)| dx + \sum_{1/\varepsilon - 1 \leq n \leq K} \frac{1}{n} \int_n^{n+1} |S^2(x)| dx. \end{aligned}$$

The last estimate and Lemma 7(iii) give

$$\int_{\tau}^K |S^2(x)| |\Phi(x)| dx \ll X \log^4 X.$$

Therefore, by (42) and by Lemma 10 we have

$$(43) \quad |D_2| \ll \varepsilon \frac{X^{3-c}}{\log X}.$$

We now find an asymptotic formula for D_1 . Set

$$H_1 = \int_{-\tau}^{\tau} I^3(x) e(-Nx) \Phi(x) dx.$$

Using Lemmas 7(i), 7(ii) and 14 we have

$$\begin{aligned} (44) \quad |D_1 - H_1| &\ll \int_{-\tau}^{\tau} |S^3(x) - I^3(x)| |\Phi(x)| dx \\ &\ll \varepsilon \int_{-\tau}^{\tau} |S(x) - I(x)| (|S^2(x)| + |I^2(x)|) dx \\ &\ll \varepsilon X e^{-(\log X)^{1/5}} \left(\int_{-\tau}^{\tau} |S^2(x)| dx + \int_{-\tau}^{\tau} |I^2(x)| dx \right) \\ &\ll \varepsilon X^{3-c} e^{-(\log X)^{1/6}}. \end{aligned}$$

Define

$$H = \int_{-\infty}^{\infty} I^3(x) e(-Nx) \Phi(x) dx.$$

Then we have

$$(45) \quad |H - H_1| \ll \int_{\tau}^{\infty} |I^3(x)| |\Phi(x)| dx.$$

By Lemma 2 we get $|I(x)| \ll 1/(|x|X^{c-1})$. Therefore, by (45) and Lemma 1

we obtain

$$(46) \quad |H - H_1| \ll \frac{1}{X^{3(c-1)}} \int_{\tau}^{\infty} |\Phi(x)| \frac{dx}{x^3} \ll \frac{\varepsilon}{X^{3(c-1)\tau^2}} \ll \varepsilon \frac{X^{3-c}}{\log X}.$$

It now follows from formulas (40), (41), (43)–(46) that

$$B_1 = H + O\left(\frac{\varepsilon X^{3-c}}{\log X}\right).$$

Hence by Lemma 6 we have $B_1 \gg \varepsilon X^{3-c}$. Together with (40) this implies that $B \gg \varepsilon X^{3-c}$, and so $B \rightarrow \infty$ as $X \rightarrow \infty$.

The Theorem is proved.

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