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ON A DISTRIBUTION PROBLEM IN FINITE SETS

by

H. G. Meijer and H. Niederreiter

1.

In [2] the following problem emerged which deserves some interest of its own. Let $X = \{x_1, \cdots, x_k\}$ be a nonvoid finite set and let μ be a measure on X with $\mu(x_i) = \lambda_i > 0$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k} \lambda_i = 1$.

Without loss of generality we may suppose that the x_i are arranged in such a way that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. For an infinite sequence ω in X, let $A(i; N; \omega)$ denote the number of occurrences of the element x_i among the first N terms of ω and let $D(\omega) = \sup_{i, N} |A(i; N; \omega) - \lambda_i N|$ (the supremum is taken over $i = 1, 2, \cdots, k; N = 1, 2, \cdots$). We pose the problem: *how small can $D(\omega)$ be?*

Similarly, define $A(M; N; \omega)$ for a subset M of X to be the number of occurrences of elements from M among the first N terms of ω and put $C(\omega) = \sup_{M, N} |A(M; N; \omega) - \mu(M)N|$ (the supremum is taken over all subsets $M \subset X$ and $N = 1, 2, \cdots$). Then we may ask: *how small can $C(\omega)$ be?*

These problems are similar to the well-known problem of constructing a sequence with small discrepancy in the unit interval [0,1] (see e.g. v.d. Corput [1]).

It was shown in [2] that a 'very well' distributed sequence ω in X can be found with

$$D(\omega) \leq k-1, \qquad C(\omega) \leq (k-1) \left\lceil \frac{k}{2} \right\rceil .$$

Those values, however, are far from being optimal. In section 2 of this paper we shall construct a sequence ω in X with

(1) $\qquad D(\omega) \leq \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{k-2} \frac{1}{n}$ and $C(\omega) \leq \frac{1}{2}(k-1)$ for $k \geq 2$.

If $k = 1$ then, trivially, $C(\omega) = D(\omega) = 0$.

For some special measures μ on X better results can be obtained. If e.g. $\lambda_1 = \cdots = \lambda_k = 1/k$ then one easily verifies that the sequence $\omega = (y_n)_{n=1}^{\infty}$ defined by $y_n = x_i$ if $n \equiv i \pmod{k}$ satisfies $D(\omega) = 1 - 1/k$.

In section 3 we construct a sequence η in X which gives a better result than (1) if λ_k is sufficiently small and $k \geq 3$. In fact we prove

$$D(\eta) \leq \begin{cases} \frac{1}{2}+\frac{1}{2}\lambda_k\,(k-2) & \text{if } k \text{ is even} \\ \frac{1}{2}+\frac{1}{2}\lambda_k\,(k-1) & \text{if } k \text{ is odd} \end{cases}$$

$$C(\eta) \begin{cases} = D(\eta) & \text{for } k = 2, 3 \\ \leq \max\left(D(\eta), \frac{5}{4}\right) & \text{for } k = 4 \\ \leq \max\left(D(\eta), \frac{25}{16}\right) & \text{for } k = 5 \\ \leq \max\left(D(\eta), \frac{1}{2}(k-2)\right) & \text{for } k \geq 6. \end{cases}$$

We remark that always $\lambda_k \geq 1/k$.

Added in proof: Recently Tijdeman [3] found by an entirely different method: if $D_k = \sup_\mu \inf_\omega D(\omega)$, then it holds

$$1 - \frac{1}{2(k-1)} \leq D_k \leq 1.$$

Moreover he generalized the results to countable sets.
  A refinement of this method gives

$$D_k = 1 - \frac{1}{2(k-1)}$$

(see [4]).

2.

By using some refinements of the method employed in [2], we can prove the following result.

THEOREM 1. *For any nonvoid finite set* $X = \{x_1, \cdots, x_k\}$ *and every measure* μ *on* X *with* $\mu(x_i) = \lambda_i > 0$ $(i = 1, \cdots, k)$, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ *and* $\sum_{i=1}^{k} \lambda_i = 1$, *there is a sequence* ω *in* X *such that*

(2)
$$|A(i; N; \omega) - \lambda_i N| \leq \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{k-i}\frac{1}{n} \quad \text{if} \quad 2 \leq i \leq k$$

$$|A(1; N; \omega) - \lambda_1 N| \leq \frac{1}{2}\sum_{n=1}^{k-1}\frac{1}{n},$$

therefore

$$D(\omega) = 0 \quad \text{if } k = 1,$$

$$D(\omega) \leq \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{k-2}\frac{1}{n} \quad \text{if} \quad k \geq 2;$$

moreover

$$C(\omega) \leq \frac{1}{2}(k-1).$$

PROOF. We proceed by induction on k. Obviously the case $k = 1$ is trivial. Assuming the proposition to be true for an integer $k \geq 1$, we shall prove that it also holds for $k+1$.

We consider the set $X = \{x_1, \cdots, x_{k+1}\}$ and a measure μ on X with

$$\mu(x_i) = \lambda_i > 0, \qquad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k+1}, \sum_{i=1}^{k+1} \lambda_i = 1.$$

On the subset $Y = \{x_1, \cdots, x_k\}$ of X, introduce a measure ν by

$$\nu(x_i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_k} = \alpha_i.$$

Since $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ it follows that

(3) $$\alpha_i \leq \frac{1}{k-i+1} \quad \text{for} \quad 1 \leq i \leq k.$$

By induction hypothesis, there exists a sequence $\tau = (y_n)_{n=1}^{\infty}$ in Y with

$$|A(i; N; \tau) - \alpha_i N| \leq \tfrac{1}{2} + \tfrac{1}{2} \sum_{n=1}^{k-i} \frac{1}{n} \quad \text{if} \quad 2 \leq i \leq k,$$

(4)

$$|A(1; N; \tau) - \alpha_1 N| \leq \tfrac{1}{2} \sum_{n=1}^{k-1} \frac{1}{n}$$

for all $N \geq 1$, and with

(5) $$C(\tau) \leq \tfrac{1}{2}(k-1).$$

We introduce the following notation: for a real number a let $\|a\| = [a + \tfrac{1}{2}]$, i.e. the integer nearest to a. For $n \geq 1$, put $R(n) = n - \|\lambda_{k+1} n\|$. We define a sequence $\omega = (z_n)_{n=1}^{\infty}$ in X by setting

$$\begin{aligned} z_n = x_{k+1} \quad &\text{if} \quad \|\lambda_{k+1} n\| > \|\lambda_{k+1}(n-1)\|, \\ z_n = y_{R(n)} \quad &\text{if} \quad \|\lambda_{k+1} n\| = \|\lambda_{k+1}(n-1)\|. \end{aligned} \quad (n = 1, 2, \cdots)$$

We get then

$$A(k+1; N; \omega) = \|\lambda_{k+1} N\| = \lambda_{k+1} N + \varepsilon$$

with $|\varepsilon| \leq \tfrac{1}{2}$, and therefore

(6) $$|A(k+1; N; \omega) - \lambda_{k+1} N| \leq \tfrac{1}{2}.$$

For $1 \leq i \leq k$, we have $A(i; N; \omega) = A(i; R(N); \tau)$ for all $N \geq 1$ (if $R(N) = 0$, we had to read $A(i; R(N); \tau) = 0$). Now we write

(7) $$|A(i; N; \omega) - \lambda_i N| \leq |A(i; R(N); \tau) - \alpha_i R(N)| + |\alpha_i R(N) - \lambda_i N|.$$

Using the definitions of $R(N)$ of α_i and (3), we obtain

(8) $$|\alpha_i R(N) - \lambda_i N| = |\alpha_i(N - \lambda_{k+1} N - \varepsilon) - \lambda_i N| = |\alpha_i \varepsilon| \leq \frac{1}{2(k-i+1)}.$$

Hence by (7), (4) and (8) we get

$$|A(i; N; \omega) - \lambda_i N| \leqq \tfrac{1}{2} + \tfrac{1}{2} \sum_{n=1}^{k-i+1} \frac{1}{n} \quad \text{if } 2 \leqq i \leqq k,$$

$$|A(1; N; \omega) - \lambda_1 N| \leqq \tfrac{1}{2} \sum_{n=1}^{k} \frac{1}{n}.$$

Moreover (6) implies that the first inequality also holds for $i = k+1$. Therefore the relations (2) have been proved for $k+1$.

Furthermore we have to show that ω satisfies $C(\omega) \leqq k/2$. If M is a subset of X and M^c denotes its complement in X, then

(9) $|A(M^c; N; \omega) - \mu(M^c)N| = |A(M; N; \omega) - \mu(M)N|.$

Consequently, it suffices to consider subsets M of Y. Using (5) and the same type of arguments as above, we arrive at

$$|A(M; N; \omega) - \mu(M)N| \leqq |A(M; R(N); \tau) - \nu(M)R(N)|$$
$$+ |\nu(M)R(N) - \mu(M)N| \leqq \tfrac{1}{2}(k-1) + |\nu(M)\varepsilon| \leqq \tfrac{1}{2}k.$$

3.

In this section we exhibit another construction principle which gives better results than the sequence of section 2 if $\lambda_k = \max \lambda_i$ is small and $k \geqq 3$. Since the case $k = 1$ is trivial we restrict ourselves to $k \geqq 2$. For a real number a we denote as above $||a|| = [a + \tfrac{1}{2}]$; moreover we define

(10) $\{\{a\}\} = a - ||a||.$

Hence

(11) $-\tfrac{1}{2} \leqq \{\{a\}\} < \tfrac{1}{2}.$

We consider the following scheme consisting of an infinite number of rows and k columns.

x_1	x_2	$\cdots$	x_k	
λ_1	λ_2	$\cdots$	λ_k	
$||\lambda_1||$	$||\lambda_2||$	$\cdots$	$||\lambda_k||$	1st row
$||2\lambda_1||$	$||2\lambda_2||$	$\cdots$	$||2\lambda_k||$	2nd row
$||n\lambda_1||$	$||n\lambda_2||$	$\cdots$	$||n\lambda_k||$	nth row

The ith column consists of $||\lambda_i|| \leqq ||2\lambda_i|| \leqq \cdots \leqq ||n\lambda_i|| \leqq \cdots$, where $||(n+1)\lambda_i|| = ||n\lambda_i||$ or $||(n+1)\lambda_i|| = ||n\lambda_i|| + 1$. Now we change this column in the following way.

If $||(n+1)\lambda_i|| = ||n\lambda_i||$ $(n = 0, 1, 2, \cdots)$ we omit $||(n+1)\lambda_i||$ such that we get a void place in the scheme.

If on the other hand $||(n+1)\lambda_i|| = ||n\lambda_i||+1$ $(n = 0, 1, 2, \cdots)$ we replace $||(n+1)\lambda_i||$ by x_i. We remark that in the last case

$$(12) \qquad\qquad \{\{n\lambda_i\}\} \geqq \tfrac{1}{2}-\lambda_i,$$

$$(13) \qquad\qquad \{\{(n+1)\lambda_i\}\} < -\tfrac{1}{2}+\lambda_i.$$

The ith column now consists of places with x_i and void places. Up till the nth row there are exactly $||n\lambda_i||$ places with x_i. We do so for $i = 1, 2, \cdots, k$. The sequence $\eta = (\eta_n)_{n=1}^\infty$ is the sequence which we get if we read the consecutive rows from the left to the right. After we have passed through the nth row we have had $||n\lambda_i||$ times the element x_i and altogether $T(n) = \sum_{i=1}^k ||n\lambda_i||$ elements of η. For this sequence η we will prove the following result.

THEOREM 2. *For the sequence η we have*

$$(14) \qquad\qquad |A(i; N; \eta)-\lambda_i N| \leqq \tfrac{1}{2}+\tfrac{1}{2}\lambda_i(k-d),$$

where $d = 1$ if k is odd and $d = 2$ if k is even. Therefore

$$D(\eta) \leqq \tfrac{1}{2}+\tfrac{1}{2}\lambda_k(k-d).$$

Moreover

$$C(\eta) \begin{cases} = D(\eta) & \text{for } k = 2, 3 \\ \leqq \max\left(D(\eta), \tfrac{5}{4}\right) & \text{for } k = 4 \\ \leqq \max\left(D(\eta), \tfrac{25}{16}\right) & \text{for } k = 5 \\ \leqq \max\left(D(\eta), (k-2)/2\right) & \text{for } k \geqq 6. \end{cases}$$

PROOF. Since there is no risk of ambiguity we omit the η in $A(i; N; \eta)$ and $A(M, N; \eta)$.

First we remark that by (10)

$$\sum_{h=1}^k \{\{n\lambda_h\}\} = n - \sum_{h=1}^k ||n\lambda_h||,$$

which implies that $\Sigma\{\{n\lambda_h\}\}$ has to be an integer. If we exclude the case k even, $(\{\{n\lambda_1\}\}, \cdots, \{\{n\lambda_k\}\}) = (-\tfrac{1}{2}, \cdots, -\tfrac{1}{2})$ we may conclude from (11)

$$(15) \qquad\qquad |\sum_{h=1}^k \{\{n\lambda_h\}\}| \leqq \tfrac{1}{2}(k-d),$$

where $d = 1$ if k is odd, $d = 2$ if k is even. Using again (10) we get

$$(16) \qquad A(i; T(n))-\lambda_i T(n) = ||n\lambda_i||-\lambda_i \sum_{h=1}^k ||n\lambda_h||$$

$$= -\{\{n\lambda_i\}\}+\lambda_i \sum_{h=1}^k \{\{n\lambda_h\}\}.$$

Let N be an integer with $T(n) \leq N \leq T(n+1)$. Then $A(i; N) = A(i; T(n))$ or $A(i; N) = A(i; T(n)) + 1$. In the first case we have by (16), (11) and (15)

$$A(i; N) - \lambda_i N \leq A(i; T(n)) - \lambda_i T(n) = -\{\{n\lambda_i\}\} + \lambda_i \sum_{h=1}^{k} \{\{n\lambda_h\}\}$$

$$\leq \tfrac{1}{2} + \tfrac{1}{2}\lambda_i(k-d).$$

In the second case x_i is an element of the $(n+1)^{\text{th}}$ row. Then by (12)

(17) $$\{\{n\lambda_i\}\} \geq \tfrac{1}{2} - \lambda_i.$$

Moreover $N \geq T(n) + 1$. Therefore using (16), (17) and (15) we arrive at

$$A(i; N) - \lambda_i N \leq A(i; T(n)) - \lambda_i T(n) + 1 - \lambda_i$$

$$= -\{\{n\lambda_i\}\} + \lambda_i \sum_{h=1}^{k} \{\{n\lambda_h\}\} + 1 - \lambda_i \leq -\tfrac{1}{2} + \lambda_i + \tfrac{1}{2}\lambda_i(k-d) + 1 - \lambda_i$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\lambda_i(k-d).$$

This upper bound trivially holds as well with $d = 2$ in the exceptional case excluded above.

In order to get a lower bound we proceed in a similar way. We have $A(i; N) = A(i; T(n+1))$ or $A(i; N) = A(i; T(n+1)) - 1$.

For the calculations we first exclude the case k even,

$$(\{\{(n+1)\lambda_1\}\}, \cdots, \{\{(n+1)\lambda_k\}\}) = (-\tfrac{1}{2}, \cdots, -\tfrac{1}{2}).$$

Then we obtain in the first case

$$A(i; N) - \lambda_i N \geq A(i; T(n+1)) - \lambda_i T(n+1)$$

$$= -\{\{(n+1)\lambda_i\}\} + \lambda_i \sum_{h=1}^{k} \{\{(n+1)\lambda_h\}\} \geq -\tfrac{1}{2} - \tfrac{1}{2}\lambda_i(k-d).$$

In the second case we have $N \leq T(n+1) - 1$. Moreover x_i occurs in the $(n+1)^{\text{th}}$ row and (13) gives

$$\{\{(n+1)\lambda_i\}\} < -\tfrac{1}{2} + \lambda_i.$$

Therefore

$$A(i; N) - \lambda_i N \geq A(i; T(n+1)) - \lambda_i T(n+1) - 1 + \lambda_i$$

$$= -\{\{(n+1)\lambda_i\}\} + \lambda_i \sum_{h=1}^{k} \{\{(n+1)\lambda_h\}\} - 1 + \lambda_i$$

$$\geq \tfrac{1}{2} - \lambda_i - \tfrac{1}{2}\lambda_i(k-d) - 1 + \lambda_i = -\tfrac{1}{2} - \tfrac{1}{2}\lambda_i(k-d).$$

One easily verifies that these lower bounds also hold with $d = 2$ for the case k even,

$$(\{\{(n+1)\lambda_1\}\}, \cdots, \{\{(n+1)\lambda_k\}\}) = (-\tfrac{1}{2}, \cdots, -\tfrac{1}{2}).$$

Hence (14) has been proved.

In order to get an estimate for $C(\eta)$ we consider a nonvoid subset M of X. Put

$$M = \{x_{i_1}, \cdots, x_{i_j}\}, \quad \mu M = \lambda_{i_1} + \cdots + \lambda_{i_j} = \Lambda,$$
$$X \backslash M = \{x_{i_{j+1}}, \cdots, x_{i_k}\}.$$

Then

$$A(M; T(n)) - \Lambda T(n) = \sum_{v=1}^{j} \|n\lambda_{i_v}\| - \Lambda \sum_{h=1}^{k} \|n\lambda_h\|$$

$$= -\sum_{v=1}^{j} \{\{n\lambda_{i_v}\}\} + \Lambda \sum_{h=1}^{k} \{\{n\lambda_h\}\}$$

$$= -(1-\Lambda) \sum_{v=1}^{j} \{\{n\lambda_{i_v}\}\} + \Lambda \sum_{v=j+1}^{k} \{\{n\lambda_{i_v}\}\}.$$

Let N be an integer with $T(n) \leqq N \leqq T(n+1)$ and suppose

$$A(M; N) = A(M; T(n)) + t \quad \text{with } 0 \leqq t \leqq j.$$

Then $N \geqq T(n) + t$ and

$$A(M; N) - \Lambda N \leqq A(M; T(n)) - \Lambda T(N) + t - \Lambda t$$

$$= -(1-\Lambda) \sum_{v=1}^{j} \{\{n\lambda_{i_v}\}\} + \Lambda \sum_{v=j+1}^{k} \{\{n\lambda_{i_v}\}\} + t - \Lambda t.$$

Suppose that $x_{u_1}, \cdots x_{u_t}$ are the elements of the $(n+1)^{\text{th}}$ row which are counted in $A(M; N)$ and not in $A(M; T(n))$.

Then by (12)

$$\{\{n\lambda_{u_\tau}\}\} \geqq \tfrac{1}{2} - \lambda_{u_\tau}. \quad (\tau = 1, \cdots, t)$$

Therefore

$$\sum_{v=1}^{j} \{\{n\lambda_{i_v}\}\} \geqq \tfrac{1}{2}t - (\lambda_{u_1} + \cdots + \lambda_{u_t}) - \tfrac{1}{2}(j-t) \geqq t - \tfrac{1}{2}j - \Lambda.$$

Hence

$$A(M; N) - \Lambda N \leqq -(1-\Lambda)(t - \tfrac{1}{2}j - \Lambda) + \tfrac{1}{2}\Lambda(k-j) + t - \Lambda t$$

$$= \frac{j}{2} + \Lambda \left(\frac{k}{2} - j + 1\right) - \Lambda^2.$$

In a similar way we find a lower bound for $A(M; N) - \Lambda N$ which has the same absolute value.

Hence

$$|A(M; N) - \Lambda N| \leqq \frac{j}{2} + \Lambda \left(\frac{k}{2} - j + 1\right) - \Lambda^2.$$

Since for $k = 2$, trivially, $C(\eta) = D(\eta)$, we suppose $k \geqq 3$. We observe

that we can restrict ourselves to $\frac{1}{2}k \leqq j \leqq k-1$ (compare (9)). If $j = k-1$, the complement of M is a singleton which was dealt with in $D(\eta)$. In particular this implies $C(\eta) = D(\eta)$ for $k = 3$. If $\frac{1}{2}k+1 \leqq j \leqq k-2$, then clearly

$$\frac{j}{2} + \Lambda \left(\frac{k}{2} - j + 1 \right) - \Lambda^2 \leqq \tfrac{1}{2}(k-2).$$

If $\frac{1}{2}k \leqq j \leqq \frac{1}{2}k+\frac{1}{2}$, then

$$\frac{j}{2} + \Lambda \left(\frac{k}{2} - j + 1 \right) - \Lambda^2 \leqq \tfrac{1}{2} \left(\frac{k}{2} + \tfrac{1}{2} \right) + \Lambda - \Lambda^2 \leqq \frac{k}{4} + \tfrac{1}{2}.$$

For $k \geqq 6$, we have $\frac{1}{4}k+\frac{1}{2} \leqq (k-2)/2$ and so $C(\eta) \leqq \max (D(\eta), (k-2)/2)$. For $k = 4, 5$ one finds by separate discussion of the permissible values for $j : C(\eta) \leqq \max (D(\eta), \frac{5}{4})$ for $k = 4$, $C(\eta) \leqq \max (D(\eta), \frac{25}{16})$ for $k = 5$. This completes the proof.

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