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On a dual locally uniformly rotund norm on a dual Vašák space

by

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Abstract. We transfer a renorming method of transfer, due to G. Godefroy, from weakly compactly generated Banach spaces to Vašák, i.e., weakly $\mathcal{K}$-countably determined Banach spaces. Thus we obtain a new construction of a locally uniformly rotund norm on a Vašák space. A further cultivation of this method yields the new result that every dual Vašák space admits a dual locally uniformly rotund norm.

0. Introduction. Let V be a (subspace of a) weakly compactly generated Banach space. Then, according to Troyanski [10] modulo Amir and Lindenstrauss [1], V has an equivalent locally uniformly rotund (LUR) norm. If V is moreover a dual space, then it even admits a dual LUR norm [6]. However, the proof of the last fact is quite different; in fact, starting from [1], then a method of transfer due to Godefroy [5] is used.

Let us consider a more general situation when V is a Vašák space, that is, V, provided with the weak topology, is countably $\mathcal{K}$-determined; see below for an exact definition. Then, replacing [1] by a result of Vašák [11], Troyanski's theorem [10] also yields a LUR norm on V. In this paper we show that *a Vašák space which is, moreover, dual admits an equivalent dual LUR norm*; thus a question raised in [4] is settled affirmatively. This assertion really extends the theorem from [6] mentioned above because Mercourakis has constructed a dual Vašák space which is not a subspace of a weakly compactly generated space [8].

Of course, a hopeful candidate for a proof of our result is the method of transfer. Indeed, it does work but we have to refine this approach in accordance with the more complicated structure of the Vašák spaces.

In the paper we consider three stages of complexity: from weakly compactly generated space through Vašák space to dual Vašák space. In the second section we reprove the well known facts that *a (dual) weakly compactly generated Banach space admits a (dual) LUR norm* [2, p. 164] ([6, Corollary 2.2]). We present here the method of transfer but we translate

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the geometrical explanation from [6] to a purely analytical form. By the way, the method of transfer also shows that both the non-parenthetic and the parenthetic assertion can be proved at once (which completely avoids Troyanski's method here).

The third section shows how to use the method of transfer in *LUR renorming of a Vašák space*, thus obtaining an alternate approach to that renorming. The basic material we start from are the investigations of Mercourakis [7] (see also [4]). In particular, we use the fact that the dual to a Vašák space can be nicely embedded into the so-called C_1 space (a substitute of $c_0(\Gamma)$ if the space is weakly compactly generated). Also the fact that the topology of the space of irrational numbers has a countable base is very important here. It enables us to replace a family of $2^{\aleph_0}$ norms which we have to do with by a countable one. A similar idea was used by Mercourakis in strictly rotund renorming of the C_1 space [7].

The fourth section contains finally the new result: *We construct a dual LUR norm on a dual Vašák space.* Here the reasoning from the previous section is "weak* lower semicontinuously cultivated".

1. Notation and definitions. Let V be a Banach space. Its dual is denoted by V^* and the second dual by V^{**}. We always assume that V is a subspace of V^{**}. $\langle v^*, v\rangle$ means the value of $v^* \in V^*$ at $v \in V$. The closed linear span of a set M in V is denoted by $\overline{\text{sp}}M$. Let E be a total subset in V^*. The E-topology on V, denoted by $w(V, E)$, is the topology whose base consists of all sets of the form

$$\{u \in V : |\langle v^*, u - v\rangle| < \varepsilon,\ v^* \in F\},$$

where $v \in V$, $\varepsilon > 0$ and $F \subset E$ is a finite set. In particular, $w(V, V^*)$ is called the weak topology and $w(V^*, V)$ the weak* topology. A norm $\|\cdot\|$ on V is said to be *LUR* if $\|v - v_j\| \to 0$ whenever $v, v_j \in V$ and $2\|v\|^2 + 2\|v_j\|^2 - \|v + v_j\|^2 \to 0$. Let $\mathbf{N}$ denote the natural numbers with the discrete topology and consider $\mathbf{N}^{\mathbf{N}}$ with the product topology. V is called a *Vašák space* [9], [11] if there exist a subset Σ' of $\mathbf{N}^{\mathbf{N}}$ and a multivalued mapping φ from Σ' onto V such that $\varphi(\sigma)$ is a nonempty compact set for each $\sigma \in \Sigma'$ and that the set $\{\sigma \in \Sigma' : \varphi(\sigma) \cap C \neq \emptyset\}$ is closed for every weakly closed subset C in V. The lower semicontinuity of a real-valued function is abbreviated to l.s.c. Recall that an equivalent norm on V is weakly l.s.c., and an equivalent norm on V^* is dual if and only if it is weak* l.s.c. For $x \in l_\infty(\Gamma)$ we define $\operatorname{supp} x = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. If M is a subset of Γ we put $\chi_M(t) = 1$ for $t \in M$ and $\chi_M(t) = 0$ for $t \in \Gamma \backslash M$.

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2. The well known case: (dual) weakly compactly generated Banach space

THEOREM 1. *Let V be a (dual) weakly compactly generated Banach space. Then it admits an equivalent (dual) LUR norm.*

Proof. By Amir and Lindenstrauss [2, Theorem 2, p. 147] there is a set Γ and a linear bounded one-to-one and weak*-to-weak continuous mapping $S : V^* \to c_0(\Gamma)$. Consider the adjoint mapping $S^* : l_1(\Gamma) \to V^{**}$.

We claim that $S^*(l_1(\Gamma))$ *lies in V and is in fact dense in V*. So take any y in $l_1(\Gamma)$. According to [3, Ch. 5, §3, Theorem 9] we are to show that S^*y is weak* continuous. Let $\{v^*_\tau\}$ be a net in V^* weak* converging to $v^* \in V^*$. Then

$$\langle S^*y, v^*_\tau\rangle = \langle y, Sv^*_\tau\rangle \to \langle y, Sv^*\rangle = \langle S^*y, v^*\rangle$$

since S is weak*-to-weak continuous. Hence S^*y belongs to V. Further, choose any $v^* \in V^*$ and assume that for each $y \in l_1(\Gamma)$, $\langle v^*, S^*y\rangle = 0$, i.e., $\langle Sv^*, y\rangle = 0$. This means $Sv^* = 0$ and the injectivity and linearity of S concludes the proof of the second part of our claim.

Let $\|\cdot\|$ be an original (an original dual) norm on V and let $|\cdot|$ be an equivalent dual LUR norm on $l_1(\Gamma)$. Such a norm exists according to Troyanski [10], or, simply, we can put $|\cdot|^2 = \|\cdot\|^2_{l_1} + \|\cdot\|^2_{l_2}$ (see [6]). Define

$$|\cdot|^2_n = \inf\{\|\cdot - S^*y\|^2 + |y|^2/n : y \in l_1(\Gamma)\}, \qquad n = 1, 2, \ldots,$$

and further

$$|||\cdot|||^2 = \sum_{n=1}^{\infty} 2^{-n}|\cdot|^2_n .$$

It is elementary to verify that the $|\cdot|_n$ are convex, positively homogeneous, and even equivalent norms on V with $|\cdot|_n \le \|\cdot\|$. Therefore $|||\cdot|||$ is also an equivalent norm on V. Later we shall show that it is LUR.

We claim that *for every $v \in V$ and every $n \in \mathbf{N}$ there is $y \in l_1(\Gamma)$ so that*

$$|v|^2_n = \|v - S^*y\|^2 + |y|^2/n .$$

To prove this take $v \in V$ and $n \in \mathbf{N}$ and find a sequence $\{y_m\} \subset l_1(\Gamma)$ such that

$$\|v - S^*y_m\|^2 + |y_m|^2/n \to |v|^2_n .$$

Then, surely, the sequence $\{y_m\}$ is bounded in $l_1(\Gamma)$, hence relatively weak* compact. Let $y \in l_1(\Gamma)$ be its weak* cluster point, so that there is a subsequence $\{y_{m_i}\}$ weak* converging to y. Then $S^*y_{m_i} \to S^*y$ weakly (as $S^*(l_1(\Gamma)) \subset V$) and the weak and weak* l.s.c. of $\|\cdot\|$ and $|\cdot|$ respectively yield

$$\begin{aligned}|v|_n^2 &\le \|v - S^*y\|^2 + |y|^2/n \\ &\le \liminf_i \|v - S^*y_{m_i}\|^2 + \liminf_i |y_{m_i}|^2/n \\ &\le \liminf_i (\|v - S^*y_{m_i}\|^2 + |y_{m_i}|^2/n) = |v|_n^2 .\end{aligned}$$

Further, we claim that $|\cdot|_n$ *and hence also* $|||\cdot|||$ *is weak* l.s.c. (that is, they are both dual norms) if* V *is a dual Banach space and* $\|\cdot\|$ *is a dual norm on* V. So let $\{v_\tau\}$ be a net in V weak* converging to some $v \in V$. By the last claim we find $y_\tau \in l_1(\Gamma)$ such that

$$|v_\tau|_n^2 = \|v_\tau - S^*y_\tau\|^2 + |y_\tau|^2/n .$$

Without loss of generality we may and do assume that the limit $\lim_\tau |v_\tau|_n$ exists and is finite. Then for some τ_0, $\{y_\tau : \tau > \tau_0\}$ is a bounded net; let $\{y_{\tau_\sigma}\}$ be a subnet weak* converging to some $y \in l_1(\Gamma)$. Then $S^*y_{\tau_\sigma} \to S^*y$ in the weak, and, a fortiori, in the weak* topology. So the weak* l.s.c. of $\|\cdot\|$ and $|\cdot|$ yields

$$\begin{aligned}|v|_n^2 \le \|v - S^*y\|^2 + |y|^2/n &\le \liminf_\sigma \|v_{\tau_\sigma} - S^*y_{\tau_\sigma}\|^2 + \liminf_\sigma |y_{\tau_\sigma}|^2/n \\ &\le \liminf_\sigma (\|v_{\tau_\sigma} - S^*y_{\tau_\sigma}\|^2 + |y_{\tau_\sigma}|^2/n) = \liminf_\sigma |v_{\tau_\sigma}|_n^2 = \lim_\tau |v_\tau|_n^2 ,\end{aligned}$$

which proves that $|\cdot|_n$ is weak* l.s.c. if so is $\|\cdot\|$.

It remains to prove that $|||\cdot|||$ is a LUR norm. Consider $v, v_1, v_2, \ldots$ in V such that

$$(*) \qquad 2|||v|||^2 + 2|||v_j|||^2 + |||v + v_j|||^2 \to 0 \quad \text{as } j \to \infty .$$

Fix any $n \in \mathbb{N}$. By the second claim we find $y, y_1, y_2, \ldots \in l_1(\Gamma)$ such that

$$|v|_n^2 = \|v - S^*y\|^2 + |y|^2/n, \quad |v_j|_n^2 = \|v_j - S^*y_j\|^2 + |y_j|^2/n, \quad j = 1, 2, \ldots$$

Using convexity, let us estimate

$$\begin{aligned}&2|v|_n^2 + 2|v_j|_n^2 - |v + v_j|_n^2 \\ &\quad\ge 2\|v - S^*y\|^2 + 2|y|^2/n + 2\|v_j - S^*y_j\|^2 + 2|y_j|^2/n \\ &\qquad - \|v + v_j - S^*(y + y_j)\|^2 - |y + y_j|^2/n \\ &\quad\ge (\|v - S^*y\| - \|v_j - S^*y_j\|)^2 + (2|y|^2 + 2|y_j|^2 - |y + y_j|^2)/n .\end{aligned}$$

Then from $(*)$ and from the definition of $|||\cdot|||$ we get, as $j \to \infty$,

$$\|v_j - S^*y_j\| \to \|v - S^*y\| \quad \text{and} \quad 2|y|^2 + 2|y_j|^2 - |y + y_j|^2 \to 0 .$$

But $|\cdot|$ is LUR. Hence $|y - y_j| \to 0$. Thus

$$\begin{aligned}\limsup_j \|v - v_j\| &\le \limsup_j (\|v - S^*y\| + \|S^*y - S^*y_j\| + \|v_j - S^*y_j\|) \\ &= 2\|v - S^*y\| \le 2|v|_n .\end{aligned}$$

On the other hand, as $n \to \infty$ we have

$$\begin{aligned}|v|_n^2 &\le \inf\{\|v - S^*z\|^2 + |z|^2/n : z \in l_1(\Gamma),\, |z| < n^{1/4}\} \\ &\le \inf\{\|v - S^*z\|^2 : z \in l_1(\Gamma), |z| < n^{1/4}\} + n^{1/2}/n \to 0\end{aligned}$$

since $S^*(l_1(\Gamma))$ is dense in V. Therefore $\|v - v_j\| \to 0$, which means that $|||\cdot|||$ is a LUR norm. ∎

3. Still the known case: Vašák space

THEOREM 2. *Let* V *be a Vašák space. Then it admits an equivalent LUR norm.*

Proof. The main difficulty now is that we have at hand no injective weak*-to-weak continuous mapping from V^* into $c_0(\Gamma)$. Instead we can construct many, in fact, uncountably many, mappings from V^* into $c_0(\Gamma)$, with varying Γ and having good properties. Of course, we can associate to each such mapping a norm on V. But then the problem arises how to add an uncountable family of norms. Fortunately, the countable determination of the Vašák spaces enables us to work only with a countable family of norms, which, in turn, approximate somehow each norm in the uncountable family. This is caused by the separability of Σ', that is, by the countability of the base for the topology on Σ'.

Let us proceed to the proof. We start as in the proof of [4, Theorem 2]. For our V we find $\Sigma' \subset \mathbb{N}^{\mathbb{N}}$ and a mapping φ according to the definition of Vašák space. Also, by Vašák [11] and after some transfinite induction argument [4] we can find an ordinal ν and a "long sequence" $\{P_\alpha : \alpha \in [\omega, \nu]\}$ of linear bounded projections on V such that $P_\omega \equiv 0$, $P_\alpha P_\beta = P_{\min(\alpha,\beta)}$ if $\alpha, \beta \in [\omega, \nu]$, $(P_{\alpha+1} - P_\alpha)V$ is separable for each $\alpha \in [\omega, \nu)$, each $v \in V$ lies in $\overline{\mathrm{sp}}\{(P_{\alpha+1} - P_\alpha)v : \alpha \in [\omega, \nu)\}$, and $V = \overline{\mathrm{sp}}\{\bigcup (P_{\alpha+1} - P_\alpha)V : \alpha \in [\omega, \nu)\}$.

Put $L = [\omega, \nu) \times \mathbb{N}$ and, almost according to Mercourakis [7], define a (Banach) space

$$C_1(\Sigma' \times L) = \{x \in l_\infty(\Sigma' \times L) : x\chi_{K \times L} \in c_0(\Sigma' \times L) \text{ for every compact } K \subset \Sigma'\}$$

endowed with the supremum norm. We can at once observe that $c_0(\Sigma' \times L)$ is a subspace of $C_1(\Sigma' \times L)$ and that each element of $l_1(\Sigma' \times L)$ belongs to the dual $C_1(\Sigma' \times L)^*$.

LEMMA 1. *Let* $K \subset \Sigma'$ *be a compact set and consider a sequence* $\Omega_1 \supset \Omega_2 \supset \ldots \supset K$ *of open sets in* Σ' *such that* $\bigcap_{m=1}^\infty \Omega_m = K$. *Let* $x \in C_1(\Sigma' \times L)$ *and* $\varepsilon > 0$ *be given. Then for large* $m \in \mathbb{N}$

$$x((\Omega_m \backslash K) \times L) \subset (-\varepsilon, \varepsilon) .$$

The proof is easy and can be found in [4].

For each $\alpha \in [\omega,\nu)$ we choose a sequence $\{h_n^\alpha : n \in \mathbb{N}\} \subset (P_{\alpha+1} - P_\alpha)V$, $\|h_n^\alpha\| \le 1/n$, such that $\overline{\mathrm{sp}}\{h_{n_\alpha}^\alpha : n \in \mathbb{N}\} = (P_{\alpha+1} - P_\alpha)V$. Further, for each $\alpha \in [\omega,\nu)$ and each $n \in \mathbb{N}$ we find $\sigma_n^\alpha \in \Sigma'$ so that $h_n^\alpha \in \varphi(\sigma_n^\alpha)$. Then for $v^* \in V^*$ and $(\sigma,\alpha,n) \in \Sigma' \times L$ we put

$$Sv^*(\sigma,\alpha,n) = \begin{cases} \langle v^*, h_n^\alpha\rangle & \text{if } \sigma = \sigma_n^\alpha,\ \alpha \in [\omega,\nu),\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2. *S is a linear bounded mapping from V^* into $C_1(\Sigma' \times L)$.*

The proof is easy (see the proof of [4, Theorem 2]). Of course, the properties of φ serve here as a crucial information.

LEMMA 3. *The set*

$$M = \{S^*y : y \in l_1(\Sigma' \times L),\ \mathrm{supp}\, y \subset K \times L \text{ for some compact } K \subset \Sigma'\}$$

is linear, lies in V and is dense in V.

Proof. The linearity of M is quite obvious. Take further any $y \in l_1(\Sigma' \times L)$ with $\mathrm{supp}\, y \subset K \times L$ for some compact $K \subset \Sigma'$. We are to prove that $S^*y \in V$. Let $\{v_\tau^*\}$ be any net in V^* weak* converging to some $v^* \in V^*$. We shall be done when we show that $\langle S^*y, v_\tau^*\rangle \to \langle S^*y, v^*\rangle$. And this is true because, by the definition of S^*, we have

$$\begin{aligned}\langle S^*y, v_\tau^*\rangle &= \langle y, Sv_\tau^*\rangle = \sum\{y(\sigma,\alpha,n)(Sv_\tau^*)(\sigma,\alpha,n) : (\sigma,\alpha,n) \in \Sigma' \times L\} \\ &= \sum\{y(\sigma_n^\alpha,\alpha,n)\langle v_\tau^*, h_n^\alpha\rangle : \alpha \in [\omega,\nu), n \in \mathbb{N}\} \\ &= \Big\langle v_\tau^*, \sum\{y(\sigma_n^\alpha,\alpha,n)h_n^\alpha : \alpha \in [\omega,\nu), n \in \mathbb{N}\}\Big\rangle\end{aligned}$$

as $y \in l_1(\Sigma' \times L)$ and $\|h_n^\alpha\| \le 1$

$$\begin{aligned}&\to \Big\langle v^*, \sum\{y(\sigma_n^\alpha,\alpha,n)h_n^\alpha : \alpha \in [\omega,\nu), n \in \mathbb{N}\}\Big\rangle \\ &= \langle y, Sv^*\rangle = \langle S^*y, v^*\rangle .\end{aligned}$$

This means that S^*y lies in V and $M \subset V$.

It remains to prove that M is dense in V. Assume by contradiction this is not so. Then, since M is linear, there is $0 \ne v^* \in V^*$ such that $\langle v^*, S^*y\rangle = 0$ whenever $y \in l_1(\Sigma' \times L)$, and $\mathrm{supp}\, y$ is in $K \times L$ for some compact $K \subset \Sigma'$. Thus, in particular, for each $\alpha \in [\omega,\nu)$ and each $n \in \mathbb{N}$

$$\begin{aligned}0 &= \langle v^*, S^*\chi_{\{(\sigma_n^\alpha,\alpha,n)\}}\rangle = \langle \chi_{\{(\sigma_n^\alpha,\alpha,n)\}}, Sv^*\rangle \\ &= Sv^*(\sigma_n^\alpha,\alpha,n) = \langle v^*, h_n^\alpha\rangle .\end{aligned}$$

And since $\overline{\mathrm{sp}}\{h_n^\alpha : \alpha \in [\omega,\nu),\ n \in \mathbb{N}\} = V$ by a property of $\{P_\alpha\}$, we conclude $v^* = 0$, a contradiction. ∎

Section 2 suggests that it would be natural to put $|\cdot|_n = \inf\{\|\cdot - S^*y\|^2 + |y|^2/n : y \in l_1(\Sigma' \times L)\}$. However, now it may happen that S^* restricted to $l_1(\Sigma' \times L)$ would not be weak*-to-weak continuous. Indeed, otherwise every Vašák space would be weakly compactly generated. This means we must proceed more carefully. From now on let $|\cdot|$ denote an equivalent dual LUR norm on $l_1(\Sigma' \times L)$ such that $|\cdot|_{l_1} \le |\cdot|$. For $n \in \mathbb{N}$ and $X \subset \Sigma'$ we put

$$|\cdot|_{n,X}^2 = \inf\{\|\cdot - S^*y\|^2 + |y|^2/n : y \in l_1(\Sigma' \times L),\ \mathrm{supp}\, y \subset X \times L\} .$$

Note that $c\|\cdot\| \le |\cdot|_{n,X} \le \|\cdot\|$ with an appropriate $c > 0$, so that $|\cdot|_{n,X}$ is an equivalent norm on V.

LEMMA 4. *For each $v \in V$, each $n \in \mathbb{N}$, and each compact $K \subset \Sigma'$ there is $y \in l_1(\Sigma' \times L)$, with $\mathrm{supp}\, y \subset K \times L$, such that*

$$|v|_{n,K}^2 = \|v - S^*y\|^2 + |y|^2/n .$$

Proof. Fix v, n and K as in the lemma and find y_m in $l_1(\Sigma' \times L)$, $\mathrm{supp}\, y_m \subset K \times L$, $m = 1,2,\ldots$, such that

$$|v|_{n,K}^2 = \lim_m (\|v - S^*y_m\|^2 + |y_m|^2/n) .$$

As $\{y_m\}$ is a bounded sequence in $l_1(\Sigma' \times L)$, it has a subsequence $\{y_{m_i}\}$ which converges in $w(l_1(\Sigma' \times L), c_0(\Sigma' \times L))$ to some $y \in l_1(\Sigma' \times L)$. Clearly $\mathrm{supp}\, y \subset K \times L$. By the definition of $C_1(\Sigma' \times L)$, we conclude that $\langle y_{m_i}, x\rangle \to \langle y, x\rangle$ for each x from $C_1(\Sigma' \times L)$; that is, $y_{m_i} \to y$ in $w(C_1(\Sigma' \times L)^*, C_1(\Sigma' \times L))$, too. Then $S^*y_{m_i} \to S^*y$ weak*, that is, weakly (as $S^*y_{m_i}$, S^*y belong to V). Now the weak l.s.c. of $\|\cdot\|$ and the weak* l.s.c. of $|\cdot|$ yield

$$\begin{aligned}|v|_{n,K}^2 &\le \|v - S^*y\|^2 + |y|^2/n \le \liminf_i \|v - S^*y_{m_i}\|^2 + \liminf_i |y_{m_i}|^2/n \\ &\le \liminf_i (\|v - S^*y_{m_i}\|^2 + |y_{m_i}|^2/n) = |v|_{n,K}^2\end{aligned}$$

and the lemma is proved. ∎

LEMMA 5. *Let K be a compact set in Σ' and consider a sequence $\Omega_1 \supset \Omega_2 \supset \ldots \supset K$ of open sets such that $\bigcap_{m=1}^\infty \Omega_m = K$. Then for each $v \in V$ and each $n \in \mathbb{N}$*

$$|v|_{n,K} = \lim_m |v|_{n,\Omega_m} .$$

Proof. Take $v \in V$ and $n \in \mathbb{N}$ and for $m = 1,2,\ldots$ find y_m in $l_1(\Sigma' \times L)$, $\mathrm{supp}\, y_m \subset \Omega_m \times L$, so that

$$|v|_{n,\Omega_m}^2 + 1/m > \|v - S^*y_m\|^2 + |y_m|^2/m .$$

Here the sequence $\{y_m\}$ is bounded, say $|y_m| < c$ for all m. So it has a $w(l_1(\Sigma' \times L), c_0(\Sigma' \times L))$ cluster point $y \in l_1(\Sigma' \times L)$. Assume for simplicity that $y_m \to y$ in this topology. Then clearly $\mathrm{supp}\, y \subset K \times L$. We shall show that $y_m \to y$ in $w(C_1(\Sigma' \times L)^*, C_1(\Sigma' \times L))$, too. So take any $x \in C_1(\Sigma' \times L)$ and any $\varepsilon > 0$. By Lemma 1, $x((\Omega_m \setminus K) \times L) \subset (-\varepsilon,\varepsilon)$ for large $m \in \mathbb{N}$.

Hence for those m

$$\begin{aligned}|\langle y_m - y, x\rangle| &\le |\langle y_m - y, x\chi_{K\times L}\rangle| + |\langle y_m - y, x\chi_{(\Omega_m\setminus K)\times L}\rangle| \\ &\le |\langle y_m - y, x\chi_{K\times L}\rangle| + (|y_m| + |y|)\varepsilon\end{aligned}$$

(as $|\cdot|_{l_1} \le |\cdot|$) and thus, recalling that $x\chi_{K\times L} \in c_0(\Sigma' \times L)$, we get

$$\limsup_m |\langle y_m - y, x\rangle| \le 2c\varepsilon .$$

Therefore $\langle y_m, x\rangle \to \langle y, x\rangle$ since $\varepsilon > 0$ was arbitrary. This means that $y_m \to y$ in $w(C_1(\Sigma' \times L)^*, C_1(\Sigma' \times L))$. Hence $S^* y_m \to S^* y$ weakly and thus, using the l.s.c. of $\|\cdot\|$ and $|\cdot|$ with respect to appropriate topologies, we conclude that

$$\begin{aligned}|v|^2_{n,K} &\le \|v - S^*y\|^2 + |y|^2/n \le \liminf_m \|v - S^*y_m\|^2 + \liminf_m |y_m|^2/n \\ &\le \liminf_m (|v|^2_{n,\Omega_m} + 1/m) \le \limsup_m |v|^2_{n,\Omega_m} \le |v|^2_{n,K} . \ \blacksquare\end{aligned}$$

Let now $\{U_m\}$ be a countable base for the topology in Σ' which is moreover closed under finite unions and finite intersections. We define

$$|||\cdot|||^2 = \sum_{n,m=1}^{\infty} 2^{-n-m} |\cdot|^2_{n,U_m} .$$

Since the $|\cdot|_{n,U_m}$ are equivalent norms on V and $|\cdot|_{n,U_m} \le \|\cdot\|$, the norm $|||\cdot|||$ is also equivalent. We shall prove that $|||\cdot|||$ is LUR and thus we shall complete the proof of our theorem. So consider v and a sequence $\{v_j\}$ in V such that

$$(*) \qquad 2|||v|||^2 + 2|||v_j|||^2 + |||v + v_j|||^2 \to 0 \quad \text{as } j \to \infty .$$

We are to show that $\|v - v_j\| \to 0$. Fix $\varepsilon > 0$. From Lemma 3 we know that there are a compact set $K \subset \Sigma'$ and y_0 in $l_1(\Sigma' \times L)$, with $\operatorname{supp} y_0 \subset K \times L$, such that $\|v - S^*y_0\| < \varepsilon$. Fix $n \in \mathbf{N}$ so large that $|y_0|^2/n < \varepsilon^2$. Then

$$|v|^2_{n,K} \le \|v - S^*y_0\|^2 + |y_0|^2/n < \varepsilon^2 + \varepsilon^2 = 2\varepsilon^2 .$$

By Lemma 4 we find $y \in l_1(\Sigma' \times L)$, with $\operatorname{supp} y \subset K \times L$, so that

$$|v|^2_{n,K} = \|v - S^*y\|^2 + |y|^2/n .$$

Let $\{m_i\} \subset \mathbf{N}$ be a sequence such that $U_{m_1} \supset U_{m_2} \supset \ldots \supset K$ and $\bigcap_{i=1}^{\infty} U_{m_i} = K$; this is possible since K is compact and $\{U_m\}$ is a base for the topology on Σ', which is closed under finite unions and intersections. By Lemma 5, we find i so that

$$|v|^2_{n,K} < |v|^2_{n,U_{m_i}} + \varepsilon/n ;$$

in what follows we shall denote this m_i by m. Further, for $j = 1, 2, \ldots$ find $y_j \in l_1(\Sigma' \times L)$, with $\operatorname{supp} y_j \subset U_m \times L$, such that

$$|v_j|^2_{n,U_m} + \varepsilon/n > \|v_j - S^*y_j\|^2 + |y_j|^2/n .$$

(Here we cannot use Lemma 4 but this does not matter.) From convexity and $(*)$ we have

$$(**) \qquad 2|v|^2_{n,U_m} + |v_j|^2_{n,U_m} - |v + v_j|^2_{n,U_m} \to 0 \quad \text{as } j \to \infty .$$

Take j_0 so large that

$$2|v|^2_{n,U_m} + |v_j|^2_{n,U_m} - |v + v_j|^2_{n,U_m} < \varepsilon/n \quad \text{whenever } j > j_0 .$$

Then putting together the above relations, we get for $j > j_0$

$$\begin{aligned}3\varepsilon/n &> 2(|v|^2_{n,U_m} + \varepsilon/n) + 2|v_j|^2_{n,U_m} - |v + v_j|^2_{n,U_m} \\ &> 2|v|^2_{n,K} + 2|v_j|^2_{n,U_m} - |v + v_j|^2_{n,U_m} \\ &> 2\|v - S^*y\|^2 + 2|y|^2/n + 2\|v_j - S^*y_j\|^2 + 2|y_j|^2/n - 2\varepsilon/n \\ &\quad - \|v + v_j - S^*(y + y_j)\|^2 - |y + y_j|^2/n \\ &\ge (2|y|^2 + 2|y_j|^2 - |y + y_j|^2)/n - 2\varepsilon/n ;\end{aligned}$$

hence

$$2|y|^2 + 2|y_j|^2 - |y + y_j|^2 < 5\varepsilon \quad \text{for } j > j_0 .$$

Set

$$\delta(\Delta) = \sup\{|y - z| : 2|y|^2 + 2|z|^2 - |y + z|^2 < \Delta,\ z \in l_1(\Sigma' \times L)\}, \quad \Delta > 0 .$$

Then for $j > j_0$ we get

$$\begin{aligned}\|v - v_j\| &\le \|v - S^*y\| + \|S^*(y - y_j)\| + \|v_j - S^*y_j\| \\ &\le |v|_{n,K} + \|S^*\|\delta(5\varepsilon) + (|v_j|^2_{n,U_m} + \varepsilon/n)^{1/2} .\end{aligned}$$

And remembering that $(**)$ implies

$$|v_j|_{n,U_m} \to |v|_{n,U_m} \le |v|_{n,K} \quad \text{as } j \to \infty ,$$

we obtain

$$\begin{aligned}\limsup_j \|v - v_j\| &\le |v|_{n,K} + \|S^*\|\delta(5\varepsilon) + (|v|^2_{n,K} + \varepsilon)^{1/2} \\ &< 2\varepsilon + \|S^*\|\delta(5\varepsilon) + (2\varepsilon^2 + \varepsilon)^{1/2} .\end{aligned}$$

Now, since $|\cdot|$ is LUR, $\delta(\Delta) \to 0$ as $\Delta \downarrow 0$. Therefore, letting $\varepsilon > 0$ go to 0 we get $\|v - v_j\| \to 0$. $\blacksquare$

4. Finally the new case: dual Vašák space

THEOREM 3. *Let V be a Vašák space which is moreover dual. Then it admits an equivalent dual LUR norm.*

Proof. Let $\|\cdot\|$ be a dual equivalent norm on V. We shall use the notation of the last section. We can see no reason why the norm $|||\cdot|||$ constructed there should be dual. Therefore we refine that construction.

LEMMA 6. *Let $n \in \mathbf{N}$ and a compact set $K \subset \Sigma'$ be given. Then the norm $|\cdot|_{n,K}$ is weak* l.s.c., that is, dual.*

Proof. Let $\{v_\tau\} \subset V$ be a net weak* converging to some $v \in V$. By Lemma 4 we find $y_\tau \in l_1(\Sigma' \times L)$, $\operatorname{supp} y_\tau \subset K \times L$, such that

$$|v_\tau|^2_{n,K} = \|v_\tau - S^*y_\tau\|^2 + |y_\tau|^2/n .$$

Without loss of generality we may and do assume that $\lim_\tau |v_\tau|_{n,K}$ exists and is finite. Then the $|y_\tau|$ are also bounded for large τ and so $\{y_\tau\}$ has a $w(l_1(\Sigma'\times L), c_0(\Sigma'\times L))$ cluster point $y \in l_1(\Sigma'\times L)$, with $\operatorname{supp} y \subset K \times L$. For simplicity, assume $y_\tau \to y$ in this topology. Then clearly $y_\tau \to y$ in $w(C_1(\Sigma'\times L)^*, C_1(\Sigma'\times L))$, too. Hence $S^*y_\tau \to S^*y$ weakly, and, a fortiori, weakly*. Thus the weak* l.s.c. of $\|\cdot\|$ and $|\cdot|$ yields

$$|v|^2_{n,K} \le \|v - S^*y\|^2 + |y|^2/n \le \liminf_\tau \|v_\tau - S^*y_\tau\|^2 + \liminf_\tau |y_\tau|^2/n$$
$$\le \liminf_\tau (\|v_\tau - S^*y_\tau\|^2 + |y_\tau|^2/n) = \lim_\tau |v_\tau|^2_{n,K} . \ \blacksquare$$

For $n \in \mathbf{N}$ and $X \subset \Sigma'$ let $\|\cdot\|_{n,X}$ be the weak* lower semicontinuous regularization of $|\cdot|_{n,X}$, that is,

$$\|v\|_{n,X} = \lim_{W\in\mathcal{W}} \inf\{|u|_{n,X} : u \in v + W\}, \quad v \in V,$$

where $\mathcal{W}$ is the filter of convex weak* neighbourhoods of 0 in V.

LEMMA 7. *For each $n \in \mathbf{N}$ and each $X \subset \Sigma'$, $\|\cdot\|_{n,X}$ is an equivalent dual norm on V.*

Proof. The verification that $\|\cdot\|_{n,X}$ is positively homogeneous and subadditive is elementary. Moreover, $\|\cdot\|_{n,X} \le |\cdot|_{n,X}$ and $\|\cdot\|_{n,X} \ge c\|\cdot\|$ with an appropriate constant $c > 0$ since $|\cdot|_{n,X}$ is an equivalent norm and $\|\cdot\|$ is weak* l.s.c. Therefore $\|\cdot\|_{n,X}$ is an equivalent norm. We now show that $\|\cdot\|_{n,X}$ is weak* l.s.c. So fix any $v \in V$ and $\varepsilon > 0$. From the definition of $\|v\|_{n,X}$ there is a convex $W \in \mathcal{W}$ such that

$$\inf\{|u|_{n,X} : u \in v + W\} > \|v\|_{n,X} - \varepsilon .$$

Choose any $z \in v + \frac{1}{2}W$. Then

$$\|z\|_{n,X} \ge \inf\{|u|_{n,X} : u \in z + \tfrac{1}{2}W\} \ge \inf\{|u|_{n,X} : u \in v + W\} > \|v\|_{n,X} - \varepsilon .$$

This means that $\|\cdot\|_{n,X}$ is weak* l.s.c., that is, it is a dual norm. $\blacksquare$

LEMMA 8. *Under the assumptions of Lemma 5 for every $v \in V$ and every $n \in \mathbf{N}$*

$$|v|_{n,K} = \lim_m \|v\|_{n,\Omega_m} .$$

Proof. Fix v and n. Let W be any fixed convex weak* neighbourhood of 0 in V. For each $m = 1, 2, \ldots$ we find $v_m \in v + W$ such that

$$\|v\|^2_{n,\Omega_m} + 1/m > |v_m|^2_{n,\Omega_m}$$

and further, using the definition of $|\cdot|_{n,\Omega_m}$, we find y_m in $l_1(\Sigma' \times L)$, $\operatorname{supp} y_m \subset \Omega_m \times L$, satisfying

$$\|v\|^2_{n,\Omega_m} + 1/m > \|v_m - S^*y_m\|^2 + |y_m|^2/n .$$

As $\|v\|_{n,\Omega_m} \le |v|_{n,\Omega_m} \le |v|_{n,K}$, we can show exactly as in the proof of Lemma 5 that a subsequence of $\{y_m\}$, say, for simplicity, the whole sequence, converges to some $y_W \in l_1(\Sigma' \times L)$, $\operatorname{supp} y_W \subset K \times L$, with respect to $w(l_1(\Sigma' \times L), c_0(\Sigma' \times L))$ as well as to $w(C_1(\Sigma' \times L)^*, C_1(\Sigma' \times L))$. Also the sequence $\{v_m\}$ is bounded in V, so it has a weak* cluster point $v_W \in V$. Let $\{v_{m_\tau}\}$ be a subnet weak* converging to v_W. Then

$$\begin{aligned}|v_W|^2_{n,K} &\le \|v_W - S^*y_W\|^2 + |y_W|^2/n \\ &\le \liminf_\tau \|v_{m_\tau} - S^*y_{m_\tau}\|^2 + \liminf_\tau |y_{m_\tau}|^2/n \\ &\le \liminf_\tau (\|v_{m_\tau} - S^*y_{m_\tau}\|^2 + |y_{m_\tau}|^2/n) \\ &\le \liminf_\tau (\|v\|^2_{n,\Omega_{m_\tau}} + 1/m_\tau) \\ &= \lim_m \|v\|^2_{n,\Omega_m} \le \lim_m |v|^2_{n,\Omega_m} = |v|^2_{n,K} .\end{aligned}$$

Recall now that $v_m \in v + W$, so $v_W \in v + W + W$. If W runs through the whole family of convex weak* neighbourhoods of 0, we find that $v_W \to v$ weak*. But $|\cdot|_{n,K}$ is weak* l.s.c. according to Lemma 6. Therefore

$$|v|^2_{n,K} \le \lim_W |v_W|^2_{n,K} \le \lim_m \|v\|^2_{n,\Omega_m} \le |v|^2_{n,K} . \ \blacksquare$$

Next let $\{U_m\}$ be the countable base in Σ' from Section 3. We define

$$|||\cdot|||^2 = \sum_{n,m=1}^{\infty} 2^{-n-m} \|\cdot\|^2_{n,U_m} .$$

This is clearly an equivalent dual norm on V. We now show it is LUR. Assume, as usual, that

$$(*) \qquad 2|||v|||^2 + 2|||v_j|||^2 - |||v + v_j|||^2 \to 0 \quad \text{as } j \to \infty .$$

Fix any $\varepsilon > 0$. In the same way as in the last section we find a compact set $K \subset \Sigma'$ and $n \in \mathbf{N}$ such that

$$|v|^2_{n,K} < 2\varepsilon^2 .$$

We also find $y \in l_1(\Sigma' \times L)$, $\operatorname{supp} y \subset K \times L$, so that

$$|v|^2_{n,K} = \|v - S^*y\|^2 + |y|^2/n .$$

Further, using Lemma 8, we find $m \in \mathbf{N}$ so that

$$|v|^2_{n,K} < \|v\|^2_{n,U_m} + \varepsilon/n .$$

From convexity and $(*)$ we find j_0 so that

$$(**) \qquad 2\|v\|^2_{n,U_m} + 2\|v_j\|^2_{n,U_m} - \|v + v_j\|^2_{n,U_m} < \varepsilon/n \quad \text{whenever } j > j_0 .$$

Fix for a moment any convex weak* neighbourhood W of 0 in V and any $j \in \{j_0+1, j_0+2, \ldots\}$. From the definition of $\|\cdot\|_{n,U_m}$ we find a convex weak* neighbourhood $W_j \subset W$ of 0 such that

$$\inf\{|u|^2_{n,U_m} : u \in v + v_j + W_j\} > \|v + v_j\|^2_{n,U_m} - \varepsilon/n\,.$$

Further, there is $v_j^W \in v_j + W_j$ ($\subset v_j + W$) such that

$$\|v_j\|^2_{n,U_m} + \varepsilon/n > |v_j^W|^2_{n,U_m}\,.$$

Then $v + v_j^W \in v + v_j + W_j$ and so

$$|v + v_j^W|^2_{n,U_m} > \|v + v_j\|^2_{n,U_m} - \varepsilon/n\,.$$

Also, we find $y_j^W \in l_1(\Sigma' \times L)$, $\operatorname{supp} y_j^W \subset U_m \times L$, so that

$$|v_j^W|^2_{n,U_m} + \varepsilon/n > \|v_j^W - S^* y_j^W\|^2 + |y_j^W|^2/n\,.$$

Then, taking into account the above relations, we can estimate

$$\begin{aligned} 8\varepsilon/n &> 2(\|v\|^2_{n,U_m} + \varepsilon/n) + 2(\|v_j\|^2_{n,U_m} + 2\varepsilon/n) - (\|v + v_j\|^2_{n,U_m} - \varepsilon/n) \\ &> 2|v|^2_{n,K} + 2(|v_j^W|^2_{n,U_m} + \varepsilon/n) - |v + v_j^W|^2_{n,U_m} \\ &> 2\|v - S^*y\|^2 + 2|y|^2/n + 2\|v_j^W - S^* y_j^W\|^2 + 2|y_j^W|^2/n \\ &\quad - \|v + v_j^W - S^*(y + y_j^W)\|^2 - |y + y_j^W|^2/n \\ &\ge (2|y|^2 + 2|y_j^W|^2 - |y + y_j^W|^2)/n\,. \end{aligned}$$

Thus, using the symbol $\delta(\Delta)$ from Section 3 we get

$$|y - y_j^W| \le \delta(8\varepsilon)\,.$$

We can also estimate

$$\|v_j^W - S^* y_j^W\|^2 \le |v_j^W|^2_{n,U_m} + \varepsilon/n < \|v_j\|^2_{n,U_m} + 2\varepsilon/n\,.$$

And, as (**) yields that

$$(\|v_j\|_{n,U_m} - \|v\|_{n,U_m})^2 < \varepsilon/n < \varepsilon\,,$$

we get

$$\begin{aligned} \|v_j^W - S^* y_j^W\|^2 &\le (\|v\|_{n,U_m} + \sqrt{\varepsilon})^2 + 2\varepsilon/n \\ &\le (|v|_{n,U_m} + \sqrt{\varepsilon})^2 + 2\varepsilon \le (|v|_{n,K} + \sqrt{\varepsilon})^2 + 2\varepsilon\,. \end{aligned}$$

Now, since $|v|^2_{n,K} < 2\varepsilon^2$, we conclude that

$$\begin{aligned} \|v - v_j^W\| &\le \|v - S^*y\| + \|S^*(y - y_j^W)\| + \|v_j^W - S^* y_j^W\| \\ &< 2\varepsilon + \|S^*\|\delta(8\varepsilon) + [(2\varepsilon + \sqrt{\varepsilon})^2 + 2\varepsilon]^{1/2} \equiv \alpha(\varepsilon) \end{aligned}$$

for all $j > j_0$ and all convex weak* neighbourhoods W of 0.

Finally, we recall that $v_j^W \in v_j + W$. It follows that $v_j^W \to v_j$ weak* as W runs over all convex weak* neighbourhoods of 0. And, since $\|\cdot\|$ is weak* l.s.c., we deduce that for $j > j_0$

$$\|v - v_j\| \le \liminf_{W \in \mathcal{W}} \|v - v_j^W\| \le \alpha(\varepsilon)\,.$$

Here $\alpha(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. Therefore $\|v - v_j\| \to 0$. ∎

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