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On a family of p-valently analytic functions missing initial Taylor coefficients

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Abstract. For $k \geq 0$, $0 \leq \gamma \leq 1$, and some convolution operator g, the object of this paper is to introduce a generalized family $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$ of p-valently analytic functions of complex order $b \in \mathbb{C} \setminus \{0\}$ and type $\alpha \in [0, p)$. Apart from studying certain coefficient, radii and subordination problems, we prove that $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$ is convex and derive its extreme points. Moreover, the closedness of this family under the modified Hadamard product is discussed. Several previously established results are obtained as particular cases of our theorems.

1. Introduction

Let $\mathbb{D} := \{\xi : |\xi| < 1\}$ be the open unit disk. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be p-valent ($p \in \mathbb{N}$) in $\mathbb{D}$ if it takes each of its values at most p times in $\mathbb{D}$. If $p = 1$, then it is said to be univalent in $\mathbb{D}$. The function $f(\xi) = \xi$ is univalent while $f(\xi) = \xi^2$ is 2-valent. For some important problems and recent works in the theory of p-valent functions, we refer to [4, 5, 9, 13, 14, 22–24, 26, 27, 29, 40] and the references therein. For $n, p \in \mathbb{N}$, let $\mathcal{A}_p(n)$ be the famiy of analytic p-valent functions $f : \mathbb{D} \to \mathbb{C}$ of the form

$$f(\xi) = \xi^p + \sum_{j=n+p}^{\infty} a_j \xi^j. \tag{1}$$

Set $\mathcal{A} := \mathcal{A}_1(1)$. A function $f \in \mathcal{A}_p(n)$ is in the family $\mathcal{S}_n^*(p, \alpha)$ of p-valently *starlike* functions of order $\alpha \in [0, p)$ if and only if $\Re\left(\xi f'(\xi)/f(\xi)\right) > \alpha$, $\xi \in \mathbb{D}$. The class $S_1^*(p, \alpha) = S^*(p, \alpha)$ was introduced in [28]. A function $f \in \mathcal{A}_p(n)$ is in the family $C_n(p, \alpha)$ of p-valently *convex* functions of order α if and only if $\Re\left(1 + \xi f''(\xi)/f'(\xi)\right) > \alpha$, $\xi \in \mathbb{D}$. Furthermore, $f \in \mathcal{A}_p(n)$ is in the family $\mathcal{K}_n(p, \alpha)$ of p-valently *close-to-convex* functions of order α if and only if there exists $h \in \mathcal{S}_n^*(p, \alpha)$ such that $\Re\left(\xi f'(\xi)/g(\xi)\right) > \alpha$, $\xi \in \mathbb{D}$. Since $h(\xi) = \xi^p$ is a member of $\mathcal{S}_n^*(p, \alpha)$, it follows that a function $f \in \mathcal{A}_p(n)$ which satisfies $\Re\left(f'(\xi)/\xi^{p-1}\right) > \alpha$ in $\mathbb{D}$ is a member of the class $\mathcal{K}_n(p, \alpha)$. We note that, $\mathcal{S}^* := \mathcal{S}_1^*(1, 0)$, $C := C_1(1, 0)$ and $\mathcal{K} := \mathcal{K}_1(1, 0)$ are, respectively, the families of starlike, convex and close-to-convex functions in $\mathbb{D}$.

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Functions of complex order.

A function $f \in \mathcal{A}_p(n)$ is in the family $\mathcal{S}_n^*(p, b, \alpha)$ of p-valently starlike functions of *complex order* b ($b \in \mathbb{C}\setminus\{0\}$) and type $\alpha \in [0, p)$ in $\mathbb{D}$ if

$$\Re\left(p + \frac{1}{b}\left(\frac{\xi f'(\xi)}{f(\xi)} - p\right)\right) > \alpha.$$

Further, let $f \in C_n(p, b, \alpha) \iff zf' \in \mathcal{S}_n^*(p, b, \alpha)$.

k-uniformly starlike functions.

Extending the classes introduced in [8, 16, 21, 30, 31], Kanas and Wisniowska [20] introduced the family of *k-uniformly starlike* functions as

$$k - \mathcal{ST} := \left\{ f \in \mathcal{A} : \Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > k\left|\frac{\xi f'(\xi)}{f(\xi)} - 1\right|\right\}, \quad k \geq 0.$$

In geometrical terms, $f \in k - \mathcal{ST}$ if and only if $Q_f(\mathbb{D}) \subset \Omega_k$, where $Q_f(\xi) = \xi f'(\xi)/f(\xi)$ and Ω_k is the *conic region* $\Omega_k := \left\{(x, y) : x^2 > k^2\left((x-1)^2 + y^2\right),\ x > 0\right\}$. Further, $f \in k - \mathcal{UC}$, the corresponding *k-uniformly convex* family, if and only if $zf' \in k - \mathcal{ST}$. Several generalizations and unifications of these families have been introduced into the literature for which we refer to [3, 12, 18, 19, 25, 35, 36, 44] and the references therein.

Definition 1.1 (Hadamard Product). *For $f(\xi) = \xi + \sum_{j=2}^{\infty} a_j \xi^j$ and $g(\xi) = \xi + \sum_{j=2}^{\infty} b_j \xi^j$, the Hadamard product (or convolution) of f and g, denoted by $f * g$, is defined as*

$$(f * g)(\xi) = \xi + \sum_{j=2}^{\infty} a_j b_j \xi^j, \qquad \xi \in \mathbb{D}.$$

The convex function $p(\xi) := \xi/(1-\xi) = \xi + \sum_{n=2}^{\infty} \xi^n$ plays the role of identity element under the operation of Hadamard product. Using Hadamard product, Aouf et al. [6, 7] introduced the family $\mathcal{S}_\gamma(g, \alpha, k)$ as

$$\mathcal{S}_\gamma(g, \alpha, k) := \left\{ f \in \mathcal{A} : \Re\left(\Phi_\gamma(f, g, \xi) - \alpha\right) > k\left|\Phi_\gamma(f, g, \xi) - 1\right|\right\},$$

where

$$\Phi_\gamma(f, g, \xi) := \frac{\xi(f * g)'(\xi) + \gamma\xi^2(f * g)''(\xi)}{(1-\gamma)(f * g)(\xi) + \gamma\xi(f * g)'(\xi)}, \quad \xi \in \mathbb{D}, \tag{2}$$

with $k \geq 0$, $0 \leq \gamma \leq 1$, $-1 \leq \alpha < 1$, and the function g given by

$$g(\xi) = \xi + \sum_{j=2}^{\infty} b_j \xi^j \qquad (b_j \geq 0;\ \xi \in \mathbb{D}). \tag{3}$$

The authors in [7] discussed several characteristic properties of a subfamily of $S_\gamma(g, \alpha, k)$. Recently, Bukhari et al. [10] extended the idea of Aouf et al. [7] to introduce a new analytic function-family $\mathcal{U}(g, \gamma, b, k)$ involving *complex order*. For $b \in \mathbb{C} \setminus \{0\}$, Bukhari et al. [10] defined $\mathcal{U}(g, \gamma, b, k)$ as

$$\mathcal{U}(g, \gamma, b, k) := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{1}{b}\left(\Phi_\gamma(f, g, \xi) - 1\right)\right) > k\left|\frac{1}{b}\left(\Phi_\gamma(f, g, \xi) - 1\right)\right|\right\},$$

where $\Phi_\gamma(f, g, \xi)$ and $g(\xi)$ are given by (2) and (3), respectively.

Motivated by the above works, in this paper, we extend $\mathcal{U}(g, \gamma, b, k)$ to introduce a novel family $\mathcal{U}_p^n(g, \gamma, k, b, \alpha)$ consisting of p-valently analytic functions of complex order b and type α with initial Taylor coefficients missing. We define this function family as follows:

Definition 1.2. *Let $f \in \mathcal{A}_p(n)$ be defined as in* (1). *Let $k \geq 0$, $\gamma \in [0,1]$, $b \in \mathbb{C} \setminus \{0\}$, and $\alpha \in [0,p)$. Then $f \in \mathcal{U}_p^n(g,\gamma,k,b,\alpha)$, if for some $g \in \mathcal{A}_p(n)$ given by*

$$g(\xi) = \xi^p + \sum_{j=n+p}^{\infty} b_j \xi^j \qquad (b_j \geq 0). \tag{4}$$

satisfying $(f * g)(\xi) \neq 0$ we have

$$\Re\left(p + \frac{1}{b}\left(\Phi_\gamma(f,g,\xi) - p\right)\right) > k\left|\frac{1}{b}\left(\Phi_\gamma(f,g,\xi) - p\right)\right| + \alpha. \tag{5}$$

Functions with negative coefficients.

Let $\mathcal{T}_p(n)$ denote the subfamily of $\mathcal{A}_p(n)$ whose members are of the form

$$f(\xi) = \xi^p - \sum_{j=n+p}^{\infty} a_j \xi^j, \qquad a_j \geq 0. \tag{6}$$

The class $\mathcal{T}_1(1) = \mathcal{T}$ was introduced by Silverman [34] and later on studied extensively by a number of authors including the ones in [3, 35, 36, 45–47]. The importance of the class $\mathcal{T} \subset \mathcal{A}$ in the theory of univalent functions is due to the fact that some conditions which are only sufficient for the members of $\mathcal{A}$ prove to be both necessary and sufficient for the members of $\mathcal{T}$. The coefficient characterization makes several computations in $\mathcal{T}$ manageable which can be very messy and difficult for the whole of $\mathcal{A}$.

The Family $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$.

We now define the family $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$ by

$$\mathcal{TU}_p^n(g,\gamma,k,b,\alpha) := \mathcal{U}_p^n(g,\gamma,k,b,\alpha) \cap \mathcal{T}_p(n). \tag{7}$$

This paper studies several geometric and analytic properties of the family $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$. In Section 2, apart from solving the coefficient problem, we determine the radius of close-to-convexity, starlikeness, and convexity for the members of $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$. Section 3 proves that the family $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$ is convex and investigates its extreme points. A subordination problem involving the concept of *subordinating factor sequences* is discussed in Section 4. In Section 5, we prove that the family $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$ is closed under the *modified* Hadamard product. Finally, Section 6 summarizes the paper and provides certain future prospects.

2. Coefficient and Radii Problems

Theorem 2.1. *Let $f(\xi)$ be of the form* (6) *and $g(\xi)$ be as in* (4). *Let $k \geq 0$, $0 \leq \gamma \leq 1$, $0 \leq \alpha < p$, and $b \in \mathbb{C} \setminus \{0\}$. Then $f \in \mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$ if and only if*

$$\sum_{j=n+p}^{\infty} \Big((k+1)(j-p) + (p-\alpha)|b|\Big)\Big[1 + \gamma(j-1)\Big] a_j b_j \leq (p-\alpha)\Big[1 + \gamma(p-1)\Big]|b|. \tag{8}$$

Proof. Let $f \in \mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$, then (5) holds. Upon using the series forms (6) and (4) in the expression (2), and then letting $\xi \to 1^-$ along the real axis, we obtain

$$p - \frac{1}{|b|}\left(\frac{\sum_{j=n+p}^{\infty}(j-p)[1+\gamma(j-1)]a_j b_j}{[1+\gamma(p-1)] - \sum_{j=n+p}^{\infty}[1+\gamma(j-1)]a_j b_j}\right) > \frac{k}{|b|}\left(\frac{\sum_{j=n+p}^{\infty}(j-p)[1+\gamma(j-1)]a_j b_j}{[1+\gamma(p-1)] - \sum_{j=n+p}^{\infty}[1+\gamma(j-1)]a_j b_j}\right) + \alpha.$$

A simplification of the above expression yields the desired inequality (8). Conversely, assume that (8) holds. Then, in view of Definition 1.2, it is sufficient to prove that the inequality

$$k\left|\frac{1}{b}\left(\Phi_\gamma(f,g,\xi)-p\right)\right|-\Re\left(\frac{1}{b}\left(\Phi_\gamma(f,g,\xi)-p\right)\right)\le p-\alpha \tag{9}$$

holds for each $\xi\in\mathbb{D}$. For $\xi\in\partial\mathbb{D}$, the boundary of $\mathbb{D}$, we have

$$k\left|\frac{1}{b}\left(\Phi_\gamma(f,g,\xi)-p\right)\right|-\Re\left(\frac{1}{b}\left(\Phi_\gamma(f,g,\xi)-p\right)\right)\le\frac{(k+1)\left|\Phi_\gamma(f,g,\xi)-p\right|}{|b|}.$$

The last expression of the above inequality is bounded above by $p-\alpha$ provided (8) holds. Applying maximum modulus principle, we establish that the inequality holds true for each $\xi\in\mathbb{D}$ whenever (8) holds. □

Corollary 2.2. *Let $f(\xi)$ be given by* (6). *If $f\in\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$, then*

$$a_j\le\frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j},\qquad(j\ge n+p). \tag{10}$$

The equality in (10) *is attained for $f(\xi)$ given by*

$$f(\xi)=\xi^p-\frac{(p-\alpha)[1+\gamma(p-1)]|b|}{\left((k+1)(j-p)+(p-\alpha)|b|\right)[1+\gamma(j-1)]b_j}\xi^j,\qquad(j\ge n+p). \tag{11}$$

Remark 2.3. *The condition* (8) *is only sufficient for the family $\mathcal{U}_p^n(g,\gamma,k,b,\alpha)$,*

Definition 2.4 (Radius Problems). *Let $\mathcal{F}$ and $\mathcal{G}$ be two subfamilies of $\mathcal{A}$. Then the $\mathcal{F}$-radius of $\mathcal{G}$, denoted by $\mathscr{R}_{\mathcal{F}}(\mathcal{G})$, is the largest number ρ $(0<\rho<1)$ such that $r^{-1}f(r\xi)\in\mathcal{F}$ for all $f\in\mathcal{G}$, where $0<r\le\rho$. The problem of finding the number ρ is called a radius problem. Further, if we can find an $f_0\in\mathcal{G}$ such that $r^{-1}f_0(r\xi)\notin\mathcal{F}$ whenever $r>\rho$, then the number ρ is said to be sharp.*

Goodman [15, Chapter 13] listed, systematically, several radii results concerning some classical subfamilies of $\mathcal{S}$. For some recent works on radius problems, we refer to [1, 2, 11, 49] and the references therein. In the following theorems, we find the radii of p-valent close-to-convexity, starlikeness, and convexity for the members of the family $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$.

Theorem 2.5. *Let $f(\xi)$ defined in* (6) *be a member of $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$ and let $g(\xi)$ be of the form* (4). *Then $f(\xi)$ is p-valently close-to-convex of order δ $(0\le\delta<p)$ in $|\xi|<r_1$, where*

$$r_1:=\inf_{j\ge n+p}\left\{\left(\frac{p-\delta}{j}\right)\frac{\left((k+1)(j-p)+(p-\alpha)|b|\right)[1+\gamma(j-1)]b_j}{(p-\alpha)[1+\gamma(p-1)]|b|}\right\}^{1/(j-p)}.$$

The result is sharp for $f(\xi)$ defined in (11).

Proof. Let $f\in\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$. Then it is easy to etablish that the inequality $\left|f'(\xi)/\xi^{p-1}-p\right|\le p-\delta$ holds whenever $|\xi|<r_1$. □

Theorem 2.6. *Let $f(\xi)$ defined by* (6) *be in the class $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$ and let $g(\xi)$ be of the form* (4). *Then $f(\xi)$ is:*

(i) *p-valently starlike of order δ $(0\le\delta<p)$ in $|\xi|<r_2$, where*

$$r_2:=\inf_{j\ge n+p}\left\{\left(\frac{p-\delta}{j-\delta}\right)\frac{\left((k+1)(j-p)+(p-\alpha)|b|\right)[1+\gamma(j-1)]b_j}{(p-\alpha)[1+\gamma(p-1)]|b|}\right\}^{1/(j-p)},$$

(ii) *p-valently convex of order* δ $(0 \le \delta < p)$ *in* $|\xi| < r_3$, *where*

$$r_3 := \inf_{j \ge n+p} \left\{ \left(\frac{p-\delta}{j(j-\delta)} \right) \frac{\Big((k+1)(j-p) + (p-\alpha)|b|\Big)[1+\gamma(j-1)]b_j}{(p-\alpha)[1+\gamma(p-1)]|b|} \right\}^{1/(j-p)}.$$

Both the results are sharp for $f(\xi)$ *defined in* (11).

Proof. (i). The result is proved by verifying that $\left|\xi f'(\xi)/f(\xi) - p\right| \le p - \delta$ for $|\xi| < r_2$. On using the fact that $f(\xi)$ is p-valently convex $\Longleftrightarrow$ $\xi f'(\xi)/p$ is p-valently starlike, the proof of (*ii*) follows immediately. □

3. Convexity and Extreme points of $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$

Theorem 3.1. *Let the functions* f_i $(i = 1, 2, \ldots, m)$ *be defined as*

$$f_i(\xi) = \xi^p - \sum_{j=n+p}^{\infty} a_{j,i}\xi^j \qquad (a_{j,i} \ge 0;\ n, p \in \mathbb{N}).$$

Suppose that $f_i \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$ *for all* $1 \le i \le m$. *Then the function* h *given by*

$$h(\xi) = \sum_{i=1}^{m} \lambda_i f_i(\xi), \qquad \left(\lambda_i \ge 0,\ \sum_{i=1}^{m} \lambda_i = 1\right),$$

also belongs to the family $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$.

Proof. Since $f_i \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$, it follows from Theorem 2.1 that

$$\sum_{j=n+p}^{\infty} \frac{[(k+1)(j-p) + (p-\alpha)|b|][1+\gamma(j-1)]}{(p-\alpha)[1+\gamma(p-1)]|b|} a_{j,i} b_j \le 1, \quad (1 \le i \le m). \tag{12}$$

From the definition of $h(\xi)$, we have

$$h(\xi) = \sum_{i=1}^{m} \lambda_i f_i(\xi) = \xi^p - \sum_{j=n+p}^{\infty} \left(\sum_{i=1}^{m} \lambda_i a_{j,i} \right) \xi^j = \xi^p - \sum_{j=n+p}^{\infty} d_j \xi^j,$$

where $d_j = \sum_{i=1}^{m} \lambda_i a_{j,i}$, $j \ge n+p$. Now, in view of (12),

$$\sum_{j=n+p}^{\infty} \frac{[(k+1)(j-p) + (p-\alpha)|b|][1+\gamma(j-1)]}{(p-\alpha)[1+\gamma(p-1)]|b|} d_j b_j \le \sum_{i=1}^{m} \lambda_i = 1.$$

Therefore, by Theorem 2.1, we conclude that $h \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$. □

Theorem 3.2. *If* $f \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$, *then* $f(\epsilon\xi)/\epsilon^p \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$, *where* $0 \le \epsilon \le 1$.

Extreme Points.

Let $\mathcal{X}$ be a topological vector space over $\mathbb{C}$ and suppose that $U \subset \mathcal{X}$. Then U is *convex* if $sx_1 + (1-s)x_2 \in U$, whenever $x_1, x_2 \in U$ and $s \in (0, 1)$. The *closed convex hull* $H(U)$ of U is the intersection of all closed convex sets containing U. A point $u \in U \subset \mathcal{X}$ is said to be an *extreme point* of U if it can not be written as $u = sx + (1-s)y$ for distinct $x_1, x_2 \in U$ and $0 < s < 1$. Let $\mathbb{E}(U)$ be the set of all extreme points of U. Since the family $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$ is convex, we determine its extereme points. The following lemma will be useful.

Lemma 3.3 ([32]). *Let $\mathcal{X}$ be a topological vector space. If $\emptyset \neq \Omega \subset \mathcal{X}$ is compact, then $\Omega \subset H(\mathbb{E}(\mathcal{X}))$. Further, if Ω is convex in $\mathcal{X}$, then $\Omega = H(\mathbb{E}(\mathcal{X}))$.*

Theorem 3.4. *Let $f_1(\xi) = \xi^p$ and*

$$f_j(\xi) = \xi^p - \frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}\xi^j, \quad j \geq n+p,$$

where $k \geq 0$, $0 \leq \gamma \leq 1$, $0 \leq \alpha < p$, $b \in \mathbb{C} \setminus \{0\}$ and $\xi \in \mathbb{D}$. Then $f \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha$ if and only if

$$f(\xi) = \vartheta_1 f_1(\xi) + \sum_{j=n+p}^{\infty} \vartheta_j f_j(\xi), \tag{13}$$

where $\vartheta_1 \geq 0$, $\vartheta_j \geq 0$ for all $j \geq n+p$ and $\vartheta_1 + \sum_{j=n+p}^{\infty} \vartheta_j = 1$.

Proof. Let $f(\xi)$ be of the form (13), then

$$f(\xi) = \vartheta_1 f_1(\xi) + \sum_{j=n+p}^{\infty} \vartheta_j f_j(\xi) = \xi^p - \sum_{j=n+p}^{\infty} \Phi_j \xi^j,$$

where

$$\Phi_j = \vartheta_j \frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}.$$

Since

$$\sum_{j=n+p}^{\infty} \frac{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]}{(p-\alpha)[1+\gamma(p-1)]|b|}\Phi_j b_j \leq 1,$$

it follows from Theorem 2.1 that $f \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$. Conversely, suppose that $f(\xi)$ defined by (6) is a member of the family $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$. Then

$$a_j \leq \frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}, \qquad (j \geq n+p).$$

Setting

$$\vartheta_j = \frac{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]}{(p-\alpha)[1+\gamma(p-1)]|b|}a_j b_j$$

and $\vartheta_1 = 1 - \sum_{j=n+p}^{\infty} \vartheta_j$, it can be easily seen that $f(\xi)$ is expressible as (13). □

Let us define the set U as

$$U := \left\{ f_j(\xi) : f_1(\xi) = \xi^p \text{ and } f_j(\xi) = \xi^p - \kappa_j \xi^j, \ j \geq n+p \right\}. \tag{14}$$

where $\kappa_j := \frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}$. Clearly U is a subset of $\mathbb{E}\left(\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)\right)$, the set of extreme points of $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$. Also, from Theorem 3.4, we conclude that $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha) = H(U)$. Using the fact that U is compact and then applying Lemma 3.3, the following result follows:

Theorem 3.5. *The set U given by (14) is the set of extreme points of the function family $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$, that is, $\mathbb{E}\left(\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)\right) = U$.*

4. Subordination Problem

Definition 4.1 (Subordination). *Let f_1 be analytic and f_2 be univalent in $\mathbb{D}$. Then we say that f_1 is subordinate to f_2 in $\mathbb{D}$, written as $f_1 \prec f_2$, if and only if $f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$.*

Definition 4.2 (Subordinating factor sequence). *A sequence $\{s_j\}_{j=1}^{\infty}$ in $\mathbb{C}$ is called a subordinating factor sequence if for every convex univalent function*

$$h(\xi) = \xi + \sum_{j=2}^{\infty} c_j \xi^j, \qquad (\xi \in \mathbb{D}) \tag{15}$$

we have the subordination $\sum_{j=1}^{\infty} s_j c_j \xi^j \prec h(\xi)$, $\xi \in \mathbb{D}$, $c_1 = 1$.

Lemma 4.3 ([50, Theorem 2]). *The sequence $\{s_j\}_{j=1}^{\infty}$ is a subordinating factor sequence if and only if $\Re(1 + 2\sum_{j=1}^{\infty} s_j \xi^j) > 0$.*

Theorem 4.4. *Let $f \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$ be as in (6) and let*

$$\tau = \frac{[n(k+1) + (p-\alpha)|b|][1 + \gamma(n+p-1)]b_{n+p}}{2\Big\{(p-\alpha)[1+\gamma(p-1)]|b| + [n(k+1) + (p-\alpha)|b|][1 + \gamma(n+p-1)]b_{n+p}\Big\}}.$$

Then for every convex $h(\xi)$ given by (15) we have

$$\Big(\tau f(\xi)/\xi^{p-1}\Big) * h(\xi) \prec h(\xi), \qquad \xi \in \mathbb{D}. \tag{16}$$

Furthermore,

$$\Re\left(\xi^{1-p} f(\xi)\right) > -1/2\tau, \qquad \xi \in \mathbb{D}. \tag{17}$$

If p and n are odd, then the constant factor τ in (16) and (17) is best possible.

Proof. From the representations (6) and (15) of the functions $f(\xi)$ and $h(\xi)$, respectively, we have $\Big(\tau f(\xi)/\xi^{p-1}\Big) * h(\xi) = \tau\left(\xi - \sum_{j=n+p+1}^{\infty} a_{j-1} c_j \xi^j\right) = \sum_{j=1}^{\infty} d_j c_j \xi^j$, where

$$d_j := \begin{cases} \tau, & j = 1 \\ 0, & 2 \le j \le n+p \\ -\tau a_{j-1}, & j \ge n+p+1. \end{cases}$$

Thus, in view of Definition 4.2, it follows that the expression (16) will hold true if $\{d_j\}_{j=1}^{\infty}$ is a subordinating factor sequence. In view of Lemma 4.3, the sequence $\{d_j\}$ is a subordinating factor sequence if we show that $\Re(1 + 2\sum_{j=1}^{\infty} d_j \xi^j) > 0$. Consider the function $\Psi(j)$ given by $\Psi(j) = \Big((k+1)(j-p) + (p-\alpha)|b|\Big)[1 + \gamma(j-1)]b_j$, $j \ge n+p$. It is easy to verify that $\Psi(j)$ is an increasing function of j and hence $\Psi(j) \ge \Psi(n+p)$, $j \ge n+p$. Therefore for $|\xi| = r < 1$, we obtain after some simplifications that

$$\Re\left(1 + 2\sum_{j=1}^{\infty} d_j \xi^j\right) = \Re\left(1 + 2\tau\xi - 2\sum_{j=n+p}^{\infty} \tau a_j \xi^{j+1}\right) \ge 1 - r > 0$$

Hence the subordination result (16) is established. Now taking $h(\xi) = \xi/(1-\xi)$ in the subordination (16) and noting that this function maps $\mathbb{D}$ onto $\Re(w) > -1/2$, the result (17) follows easily. For the sharpness τ, consider the function

$$f_0(\xi) = \xi^p - \frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p) + (p-\alpha)|b|][1+\gamma(j-1)]b_{n+p}} \xi^{n+p},$$

which is a member of $\mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$. Thus from (16), we have $\tau f_0(\xi)/\xi^{p-1} \prec \xi/(1-\xi)$. Moreover, it can be easily verified that if p and n are odd then for $\xi = -1$, $(f_0(\xi)/\xi^{p-1}) = -1/2\tau$. This proves that the constant factor τ cannot be improved further. □

5. Closedness under modified Hadamard product

For $x = 1, 2$ and $n, p \in \mathbb{N}$, define

$$f_x(\xi) = \xi^p - \sum_{j=n+p}^{\infty} a_{j,x}\xi^j \qquad (a_{j,x} \geq 0;\ x = 1, 2). \tag{18}$$

The modified Hadamard product of the functions f_1 and f_2, denoted by $f_1 \circledast f_2$, is defined as

$$(f_1 \circledast f_2)(\xi) = \xi^p - \sum_{j=n+p}^{\infty} a_{j,1}a_{j,2}\xi^j.$$

Theorem 5.1. *Let the functions $f_x(\xi)$, $x = 1, 2$, be defined as in* (18). *If $f_1, f_2 \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$, then $f_1 \circledast f_2 \in \mathcal{TU}_p^n(g, \gamma, k, b, \beta)$, where*

$$\beta \leq p - \frac{n(k+1)(p-\alpha)^2[1+\gamma(p-1)]|b|}{[n(k+1)+(p-\alpha)|b|]^2[1+\gamma(n+p-1)]b_{n+p} - (p-\alpha)^2[1+\gamma(p-1)]|b|^2}.$$

The result is sharp for the functions $f_x(\xi)$ given by

$$f_x(\xi) = \xi^p - \frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}\xi^j, \quad (x = 1, 2;\ j \geq n+p).$$

Proof. We will make use of the technique adopted by the authors in [33]. Accordingly, we need to determine the largest β so that

$$\sum_{j=n+p}^{\infty} \frac{[(k+1)(j-p)+(p-\beta)|b|][1+\gamma(j-1)]b_j}{(p-\beta)[1+\gamma(p-1)]|b|}a_{j,1}a_{j,2} \leq 1. \tag{19}$$

Since $f_x(\xi) \in \mathcal{TU}_p^n(g, \gamma, k, b, \alpha)$ for $x = 1, 2$, the inequalities

$$\sum_{j=n+p}^{\infty} \frac{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}{(p-\alpha)[1+\gamma(p-1)]|b|}a_{j,1} \leq 1 \tag{20}$$

and

$$\sum_{j=n+p}^{\infty} \frac{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}{(p-\alpha)[1+\gamma(p-1)]|b|}a_{j,2} \leq 1. \tag{21}$$

hold in light of Theorem 2.1. By means of the well known Cauchy-Schwarz inequality, we obtain from (20) and (21) that

$$\sum_{j=n+p}^{\infty} \frac{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}{(p-\alpha)[1+\gamma(p-1)]|b|}\sqrt{a_{j,1}a_{j,2}} \leq 1. \tag{22}$$

Therefore, the inequality (19) will hold true if

$$\frac{[(k+1)(j-p)+(p-\beta)|b|]}{(p-\beta)}a_{j,1}a_{j,2} \leq \frac{[(k+1)(j-p)+(p-\alpha)|b|]}{(p-\alpha)}\sqrt{a_{j,1}a_{j,2}}\ ,$$

that is, if

$$\sqrt{a_{j,1}a_{j,2}} \leq \frac{[(k+1)(j-p)+(p-\alpha)|b|]}{(p-\alpha)} \times \frac{(p-\beta)}{[(k+1)(j-p)+(p-\beta)|b|]}, \tag{23}$$

for $j \geq n+p$. Also from (22), we obtain

$$\sqrt{a_{j,1}a_{j,2}} \leq \frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j}, \quad (j \geq n+p). \tag{24}$$

Thus, in view of (24), the inequality (23) will hold true if

$$\frac{(p-\alpha)[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|][1+\gamma(j-1)]b_j} \leq \frac{[(k+1)(j-p)+(p-\alpha)|b|]}{(p-\alpha)} \times \frac{(p-\beta)}{[(k+1)(j-p)+(p-\beta)|b|]},$$

for $j \geq n+p$. On simplification, the above expression yields

$$\beta \leq p - \frac{(k+1)(j-p)(p-\alpha)^2[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|]^2[1+\gamma(j-1)]b_j - (p-\alpha)^2[1+\gamma(p-1)]|b|^2}.$$

For $j \geq n+p$, define the function $M(j)$ as

$$M(j) = p - \frac{(k+1)(j-p)(p-\alpha)^2[1+\gamma(p-1)]|b|}{[(k+1)(j-p)+(p-\alpha)|b|]^2[1+\gamma(j-1)]b_j - (p-\alpha)^2[1+\gamma(p-1)]|b|^2}.$$

It can be observed that $M(j+1) > M(j)$ for $j \geq n+p$ with $n, p \in \mathbb{N}$. Therefore, we have $\beta \leq M(n+p)$. This completes the proof. □

6. Conclusion

In this paper, we use Hadamard product to introduce a novel family $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$ of p-valently analytic functions with missing initial Taylor coefficients and involving complex order. Several interesting geometric and analytic properties of this family are discussed. Since $\mathcal{TU}_p^n(g,\gamma,k,b,\alpha)$ is a generalization to several other recently introduced function families, the earlierly proved results can be easily obtained as special cases. In particular, setting $p=1$ and $\alpha=0$, we obtain the results of Bukhari et al. [10].

In light of the recent works of Srivastava et al. [41–43, 48], we note that the results presented in this paper has several future prospects for q-extensions. Moreover, we reiterate that by applying some obvious parametric and argument variations, the q-extensions can easily (and possibly trivially) be translated into the corresponding results for the (p,q)-analogues $(0 < q < p \leq 1)$, the additional parameter p being redundant. For comprehensive details, we refer to the survey-cum-expository review article Srivastava [38, p. 340] (also see Srivastava [37, 39]) which encourage and motivate significant further developments on q-calculus and other related topics.

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