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ON A FIRST-ORDER WAVE THEORY FOR A RELAXING GAS FLOW

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SUMMARY

The motion created by withdrawal of a piston from an infinitely long tube containing a relaxing gas is examined by the method of perturbations in the plane of the characteristic parameters. It is shown that the technique fails to produce a uniformly valid first-order solution, except for the limiting cases of zero or infinite relaxation times and in certain portions of the general flow field. The analysis exemplifies the reasons for this failure.

CONTENTS

1. Introduction

The analytical study of reacting or relaxing gas flows has received considerable attention in the last few years, but has with few exceptions been confined within the framework of a purely linearised theory. That is to say, in general terms, the equations for a singly-relaxing gas have been set up in an orthogonal coordinate system and all non-linear terms which appear have subsequently been discarded. At least this is true, with few exceptions, in all cases where more than one dimension (or in other words, more than one independent variable) is concerned. In dealing with one-dimensional problems, such as plane shock-wave behaviour, or the flow through a nozzle it has often proved possible to solve the resulting ordinary non-linear equations with a minimum of additional restrictive assumptions although, to be sure, one has all too frequently been driven to use numerical methods in order to obtain results.

Naturally there will always be situations for which the basic notion of small disturbances (which leads to linearisation) is not reasonable, but we shall not be concerned with these cases here. Instead we shall concern ourselves with the failure of the formal linear theories to provide an adequate description of the flow field in regions remote from the primary source of the disturbance. The basic reasons for this failure are two-fold. Firstly, an infinitesimal disturbance propagates at the (variable) local and not at the (constant) undisturbed-field speed of sound and secondly, it does this relative to the fluid, from which it follows that the disturbance is also convected with the local gas velocity. For brevity we shall hereafter refer to both of these phenomena as 'convective effects'. It is the accumulation of these second-order influences which eventually leads to a lack of uniformity in the first-order theories.

There are several techniques for the development of uniformly valid first approximations to the flow of a compressible ideal gas. Two of these are epitomised by the work of Whitham (1952) and Lighthill (1949) for example but, principally because they involve an investigation of the second-order terms in the solution, these methods appear to be difficult to apply directly to the relaxing gas case. A third method, due to Lin (1954), would seem to be more readily adaptable to the problem in hand and it is this technique which we propose to examine here. Lin's method makes use of the equations in their characteristics form, so that our first task will be to set up the appropriate equations for a relaxing gas flow.

The specific problem to be studied is that of the flow created by the withdrawal of a piston from an infinitely long tube filled with a relaxing gas. In order not to complicate the situation with non-linear effects other than those of convection, we shall assume that the gas has a constant value of relaxation time τ' and constant values of the specific heats. We shall write C_{p_1}, C_{V_1} for the specific heats at constant pressure and constant volume, respectively, of the active molecular energy modes. The specific heat of the relaxing energy mode will be denoted by C_2. T_1 and T_2 are the translational and relaxing-mode temperatures.

2. Equations in characteristics form.

For a one-dimensional unsteady situation the equations of conservation of mass, momentum, energy and excitation of the internal mode can be written down as follows:

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} = 0 \quad , \tag{1}$$

$$\rho\,\frac{Du}{Dt} + \frac{\partial p}{\partial x} = 0 \quad , \tag{2}$$

$$C_{V_1}\,\frac{DT_1}{Dt} + C_2\frac{DT_2}{Dt} + \frac{p}{\rho}\,\frac{\partial u}{\partial x} = 0 \, , \tag{3}$$

$$\tau'\,\frac{DT_2}{Dt} + T_2 - T_1 = 0 \quad , \tag{4}$$

where the convective operator $D/Dt \equiv \partial/\partial t + u\partial/\partial x$. The thermal equation of state is

$$p = \rho R T_1 \tag{5}$$

and, writing

$$C_{V_1} = \frac{R}{\gamma_1 - 1} \quad , \tag{6}$$

where γ_1 is the active mode (or frozen) specific heat's ratio, it is clear that equations (1), (3) and (4) can be manipulated to give

$$\frac{Dp}{Dt} + \rho a_1{}^2\frac{\partial u}{\partial x} = -\rho\,(\gamma_1 - 1)\,C_2\,(T_1 - T_2)\,/\tau' \equiv Q \quad . \tag{7}$$

a_1 is the frozen sound speed,

$$a_1{}^2 = \gamma_1\, p/\rho \quad , \tag{8}$$

and the symbol Q is defined in equation (7) for later convenience.

Introducing a length variable y, where

$$y = a_{1\infty} t \tag{9}$$

and $a_{1\infty}$ is a constant (reference) frozen sound speed, which we will define more carefully at a later stage, equations (2) and (7) can be re-written as

$$\frac{\partial p}{\partial x} + \rho u\,\frac{\partial u}{\partial x} + \rho a_{1\infty}\frac{\partial u}{\partial y} = 0 \, , \tag{10}$$

$$u\,\frac{\partial p}{\partial x} + a_{1\infty}\frac{\partial p}{\partial y} + \rho\, a_1{}^2\,\frac{\partial u}{\partial x} = Q. \tag{11}$$

Equations (10) and (11) define a set of characteristic curves: denoting the characteristic parameters by α and β, these curves are given by

$$a_{1\infty}\frac{\partial x}{\partial \alpha} = (a_1 + u)\frac{\partial y}{\partial \alpha}\ ;\ \beta = \text{constant}, \tag{12}$$

$$a_{1\infty}\frac{\partial x}{\partial \beta} = -(a_1 - u)\frac{\partial y}{\partial \beta};\ \ \alpha = \text{constant}, \tag{13}$$

and equations (10) and (11) with α, β as independent variables become*

$$\frac{\partial p}{\partial \alpha} + \rho a_1 \frac{\partial u}{\partial \alpha} = \frac{Q}{a_{1\infty}}\frac{\partial y}{\partial \alpha} \quad , \tag{14}$$

$$\frac{\partial p}{\partial \beta} - \rho a_1 \frac{\partial u}{\partial \beta} = \frac{Q}{a_{1\infty}}\frac{\partial y}{\partial \beta} \quad . \tag{15}$$

We intend to use the characteristic parameters α and β as the new independent variables (this is the essence of Lin's method) and, accordingly, both x and y are hereafter treated as a pair of dependent variables which are to be found from equations (12) and (13) and some suitable boundary conditions which have yet to be discussed. In order to proceed, it is now necessary to eliminate Q from equations (14) and (15). To do this we note first that equations (3) and (4) give

$$\tau' C_{V_1}\frac{D}{Dt}\left(\frac{DT_2}{Dt}\right) + C_V\frac{DT_2}{Dt} + p/_\rho\ \frac{\partial u}{\partial x} = 0\ , \tag{16}$$

where

$$C_V = C_{V_1} + C_2\ , \tag{17}$$

whilst equations (9) and (7) give

$$\frac{Q}{\rho} = -(\gamma_1 - 1)\ C_2\frac{DT_2}{Dt} \quad . \tag{18}$$

Combining (16) and (18) we have

$$\tau' C_{V_1}\frac{D}{Dt}\left(\frac{Q}{\rho}\right) - \frac{p}{\rho}\ (\gamma_1 - 1)C_2\ \frac{\partial u}{\partial x} + C_V\frac{Q}{\rho} = 0 \tag{19}$$

It is easy to show that

$$\frac{D}{Dt} = \tfrac{1}{2}a_{1\infty}\left(\frac{1}{y_\alpha}\frac{\partial}{\partial\alpha} + \frac{1}{y_\beta}\frac{\partial}{\partial\beta}\right)\ :\ \frac{\partial}{\partial x} = \tfrac{1}{2}\frac{a_{1\infty}}{a_1}\left(\frac{1}{y_\alpha}\frac{\partial}{\partial\alpha} - \frac{1}{y_\beta}\frac{\partial}{\partial\beta}\right) \tag{20}$$

whence, substituting equations (14) and (15) into equation (19), we find that

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*The Jacobian of the transformation from x,y to α, β coordinates is tacitly assumed to be neither zero nor infinite. Since perturbations are to be assumed small u will never be permitted to approach a_1 in magnitude and it will appear later that the chosen piston paths are such as to avoid this difficulty in any finite part of the flow.

$$\frac{1}{2} a_{1\infty}\tau''\left(\frac{1}{y_\alpha} \frac{\partial}{\partial\alpha} + \frac{1}{y_\beta} \frac{\partial}{\partial\beta}\right) \left(\frac{1}{\rho y_\alpha} \left[p_\alpha + \rho a_1 u_\alpha \right] \right) + \frac{1}{\rho y_\alpha} \left[p_\alpha + \rho a_1 u_\alpha \right]$$

$$- \frac{(\gamma_1 - 1) C_2 p}{2\rho a_1 C_V} \left(\frac{1}{y_\alpha} u_\alpha - \frac{1}{y_\beta} u_\beta \right) = 0 \qquad (21)$$

$$\frac{1}{2} a_{1\infty}\tau''\left(\frac{1}{y_\alpha} \frac{\partial}{\partial\alpha} + \frac{1}{y_\beta} \frac{\partial}{\partial\beta}\right) \left(\frac{1}{\rho y_\beta} \left[p_\beta - \rho a_1 u_\beta \right] \right) + \frac{1}{\rho y_\beta} \left[p_\beta - \rho a_1 u_\beta \right]$$

$$- \frac{(\gamma_1 - 1) C_2 p}{2\rho a_1 C_V} \left(\frac{1}{y_\alpha} u_\alpha - \frac{1}{y_\beta} u_\beta \right) = 0, \qquad (22)$$

where

$$\tau'' = \tau' C_{V_1}/C_V. \qquad (23)$$

(The suffix notation for partial derivatives has been used above and will be employed in the work which follows when it proves to be convenient.) Equations (21), (22), (12), (13) and (8) constitute five equations for the six unknown quantities p, ρ, u, a_1, x and y as functions of α and β. The set is completed by using equation (1) which, with the aid of equation (20), can be written as follows:

$$\frac{a_1}{y_\alpha} \rho_\alpha + \frac{a_1}{y_\beta} \rho_\beta + \frac{\rho}{y_\alpha} u_\alpha - \frac{\rho}{y_\beta} u_\beta = 0 \quad . \qquad (24)$$

In order to examine the piston problem mentioned in the Introduction we shall assume that the piston's displacement is given by

$$x = \epsilon D(y) \ ; \ y > 0 \ , \qquad (25)$$

$$= 0 \ ; \ y < 0,$$

and we shall assume that the gas occupies the infinite half-space to the right of this boundary. For withdrawal of the piston it follows that D(y) must be a negative function, but it is not too important to stress this restriction at this stage. From equations (9) and (25), the velocity condition becomes

$$u = \epsilon a_{1\infty} D'(y) \ ; \ y > 0, \ x = \epsilon D(y) \ , \qquad (26)$$

$$= 0 \ ; \ y < 0, \ x = 0.$$

Arbitrarily restricting $dD/dy \equiv D'$ to be at most of order unity, the positive constant ϵ is a measure of the maximum piston speed as a fraction of the reference sound speed $a_{1\infty}$. Small perturbations will therefore follow if $\epsilon \ll 1$. We shall assume throughout that

$$D(0) = 0 = D'(0). \qquad (27)$$

The line $\alpha = \beta$ is selected to represent the piston-path in the α, β - plane whence, remembering that x and y are functions of α and β we deduce from equation (25) that

$$x(\beta, \beta) = \epsilon D(y(\beta, \beta)). \qquad (28)$$

If we choose

$$y\ (\beta,\beta)\ =\ \beta \tag{29}$$

the conditions on u, x and y in the α,β-plane become

$$u = \epsilon a_{1\infty} D'(\beta), \tag{30}$$
$$x = \epsilon D(\beta), \tag{31}$$
$$y = \beta, \tag{32}$$

all when $\alpha = \beta$. It follows from equation 27 that the curves labelled $\alpha = 0$, $\beta = 0$ intersect at the origin ($x = 0$, $y = 0$) in the x, y - plane. If the gas in $x > \epsilon\ D(\beta)$ is in a uniform equilibrium state for $y < 0$, sufficient conditions for a solution of the equations derived above are found by requiring that all disturbances vanish for all $\beta < 0$. The configurations in the x, y and α, β planes are sketched in Figs. 1 (a) and (b).

3. Perturbation procedures.

In order to solve the equations set out in the previous Section the dependent variables, including x and y, will be expressed as power series expansions in the (small) parameter ϵ: i.e. we shall write

$$\Psi(\alpha,\beta) = \Psi^{(0)}(\alpha,\beta) + \epsilon\Psi^{(1)}(\alpha,\beta) + \ \ldots\ldots\ldots \tag{33}$$

where Ψ stands for either u, p, ρ, a_1, x or y.

Since the disturbances created by the piston motion propagate into an initially quiescent uniform gas we can write

$$u^{(0)} = 0\ ;\ p^{(0)} = p_\infty\ ;\ \rho^{(0)} = \rho_\infty;\ a_1^{(0)} = a_{1\infty}, \tag{34}$$

where $a_{1\infty}^2 = \gamma_1\, p_\infty/\rho_\infty$, and all of these quantities are constants. (It will be noticed that $a_{1\infty}$ has been chosen to be the frozen sound speed in the undisturbed gas.) Substituting the series 33 into equations (21), (22), (12), (13), (24) and (8), and equating coefficients of like powers of ϵ leads to the following set of equations:

$$\tfrac{1}{2}a_{1\infty}\tau''\left(\frac{1}{y_\alpha}(0)\frac{\partial}{\partial\alpha} + \frac{1}{y_\beta}(0)\frac{\partial}{\partial\beta}\right)\left(\frac{1}{y_\alpha}(0)\left[p_\alpha^{(1)} + \rho_\infty a_{1\infty}u_\alpha^{(1)}\right]\right) +$$
$$\frac{1}{y_\alpha}(0)\left[p_\alpha^{(1)} + \rho_\infty a_{1\infty}u_\alpha^{(1)}\right] - \left(\frac{\gamma_1 - 1}{2\gamma_1}\right)\frac{C_2}{C_V}\ \rho_\infty a_{1\infty}\left(\frac{1}{y_\alpha}(0)\ u_\alpha^{(1)} - \frac{1}{y_\beta}(0)\ u_\beta^{(1)}\right) = 0\ , \tag{35}$$

$$\tfrac{1}{2}a_{1\infty}\tau''\left(\frac{1}{y_\alpha}(0)\frac{\partial}{\partial\alpha} + \frac{1}{y_\beta}(0)\frac{\partial}{\partial\beta}\right)\left(\frac{1}{y_\beta}(0)\left[p_\beta^{(1)} - \rho_\infty a_{1\infty}u_\beta^{(1)}\right]\right) +$$
$$\frac{1}{y_\beta}(0)\left[p_\beta^{(1)} - \rho_\infty a_{1\infty}u_\beta^{(1)}\right] - \left(\frac{\gamma_1 - 1}{2\gamma_1}\right)\frac{C_2}{C_V}\ \rho_\infty a_{1\infty}\left(\frac{1}{y_\alpha}(0)\ u_\alpha^{(1)} - \frac{1}{y_\beta}(0)\ u_\beta^{(1)}\right) = 0\ , \tag{36}$$

$$x_\alpha^{(0)} = y_\alpha^{(0)} \quad , \tag{37}$$

$$x_\beta^{(0)} = -y_\beta^{(0)} \quad , \tag{38}$$

$$a_{1\infty} x_\alpha^{(1)} = a_{1\infty} y_\alpha^{(1)} + \left(a_1^{(1)} + u^{(1)}\right) y_\alpha^{(0)} \quad , \tag{39}$$

$$a_{1\infty} x_\beta^{(1)} = -a_{1\infty} y_\alpha^{(1)} - \left(a_1^{(1)} - u^{(1)}\right) y_\beta^{(0)} \quad , \tag{40}$$

$$a_{1\infty}\left(\frac{1}{y_\alpha(0)} \rho_\alpha^{(1)} + \frac{1}{y_\beta(0)} \rho_\beta^{(1)}\right) = -\left(\frac{1}{y_\alpha(0)} u_\alpha^{(1)} - \frac{1}{y_\beta(0)} u_\beta^{(1)}\right), \tag{41}$$

$$a_1^{(1)} = \tfrac{1}{2} a_{1\infty}\left(\frac{p^{(1)}}{p_\infty} - \frac{\rho^{(1)}}{\rho_\infty}\right) \quad . \tag{42}$$

Similarly the boundary conditions (30), (31) and (32) give

$$u^{(1)} = a_{1\infty} D'(\beta), \tag{43}$$

$$x^{(0)} = 0 \; ; \; y^{(0)} = \beta \; , \tag{44}$$

$$x^{(1)} = D(\beta) \; ; \; y^{(1)} = 0 \; , \tag{45}$$

all to be applied when $\alpha = \beta$. The causality condition (described in the last sentence of Section 2) can be expressed in the form

$$\psi^{(1)}(\alpha, \beta < 0) = 0 \tag{46}$$

where ψ has the same meanings as before.

Equations (35) to (46) inclusive are applicable to the first approximation (as indicated by the superscript $(_1)$) except, that is, for equations (37) and (38) which we shall discuss shortly. Clearly one could, in principle, proceed with approximations of higher order, but we shall be concerned with only the first approximation here.

Equations (37) and (38) can be integrated at once to give

$$x^{(0)} - y^{(0)} = f(\beta) \; ; \; x^{(0)} + y^{(0)} = g(\alpha) \quad , \tag{47}$$

where f and g are, as yet, arbitrary functions. Application of conditions (44) shows, however, that $g(\alpha) = \alpha$ and $f(\beta) = -\beta$, whence

$$\left.\begin{aligned} 2x^{(0)} &= \alpha - \beta \; , \\ 2y^{(0)} &= \alpha + \beta \; . \end{aligned}\right\} \tag{48}$$

Thus both $y_\alpha^{(0)}$ and $y_\beta^{(0)}$ are equal to $\frac{1}{2}$ and the remaining equations above can be simplified somewhat.

If we write

$$\left(\frac{\gamma_1 - 1}{2\gamma_1}\right) \frac{C_2}{C_V} = \tfrac{1}{2}\left(1 - \frac{1}{a^2}\right) \; ; \; a^2 = \frac{\gamma_1}{\gamma_2} > 1 \quad , \tag{49}$$

where

$$\gamma_2 = C_p/C_v\ ;\ C_p = C_v + R = C_{p1} + C_2\ , \qquad (50)$$

equations (35) and (36) become

$$\left\{ a_{1\infty}\tau'' \left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} \right) + 1 \right\}\left\{ p_{\alpha}^{(1)} + \rho_\infty a_{1\infty} u_{\alpha}^{(1)} \right\} - \tfrac{1}{2}\left(1 - \frac{1}{a^2} \right)\rho_\infty a_{1\infty}\left(u_{\alpha}^{(1)} - u_{\beta}^{(1)} \right) = 0 \qquad (51)$$

$$\left\{ a_{1\infty}\tau'' \left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} \right) + 1 \right\}\left\{ p_{\beta}^{(1)} - \rho_\infty a_{1\infty} u_{\beta}^{(1)} \right\} - \tfrac{1}{2}\left(1 - \frac{1}{a^2} \right)\rho_\infty a_{1\infty}\left(u_{\alpha}^{(1)} - u_{\beta}^{(1)} \right) = 0 \qquad (52)$$

Note that a^2 in equation (49) is also equal to the square of the ratio of the frozen to equilibrium sound speeds. $p^{(1)}$ can readily be eliminated from equations (51) and (52), resulting in the following equation for $u^{(1)}$:

$$\Gamma\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} \right)\left(2\, u_{\alpha\beta}^{(1)} \right) + (a^2 + 1)\, u_{\alpha\beta}^{(1)} + \tfrac{1}{2}(a^2 - 1)\left(u_{\alpha\alpha}^{(1)} + u_{\beta\beta}^{(1)} \right) = 0, \qquad (53)$$

where

$$\Gamma = a_{1\infty} a^2 \tau'' = a_{1\infty} C_{p1}\, \tau'/C_p\ . \qquad (54)$$

(The second result in equation (59) follows from equations (23), (49) and (50).)

The solution of equation (53) is facilitated by a simple change of variables; we write

$$\beta = \xi\ ;\ \alpha - \beta = \eta\ , \qquad (55)$$

so that equation (53) becomes

$$\Gamma\left(u_{\xi\xi\eta}^{(1)} - u_{\xi\eta\eta}^{(1)} \right) + u_{\xi\eta}^{(1)} - u_{\eta\eta}^{(1)} + \tfrac{1}{4}(a^2 - 1)\, u_{\xi\xi}^{(1)} = 0 \qquad (56)$$

Apart from a factor of 2 associated with the variable η (see below) equation (56) is in precisely the form used by Der (1961) to study certain problems in the two-dimensional steady flow of a reacting or relaxing gas. The distinction between two-dimensional steady and one-dimensional unsteady cases is not important in the present context and is mainly concerned with differences in the definition of a^2. Der's equation was derived from a formal linearisation of a set of equations like equations (1) to (4) (see e.g. Vincenti, 1959; Clarke, 1960) by writing, in our notation,

$$\xi = y - x\ ;\ x = \tfrac{1}{2}\eta .$$

We remark that in the present case ξ and η are not so simply related to x and y but are, instead, rather similar functions of the characteristic parameters α and β.

4. First-order solutions

Equation (56) can be solved with the aid of the Laplace transform $\overline{u}(z;\eta)$ of the velocity perturbation $u^{(1)}(\xi,\eta)$, where

$$\overline{u}(z:\eta) = \int_0^{\infty} u^{(1)}(\xi,\eta)\, e^{-z\xi}\, d\xi \quad . \tag{57}$$

It readily follows that the required solution is

$$\overline{u} = A(z)\, e^{\frac{1}{2}z(1-x)\eta} \quad ; \; x^2 = (a^2 + \Gamma z)(1+\Gamma z)^{-1} \quad , \tag{58}$$

where A(z) is to be found from condition 43. In transform language this condition reads

$$\overline{u}(z:0) = a_{1\infty}\overline{D}'(z) \quad , \tag{59}$$

where we have written $\overline{D}'(z)$ for the transform of $D'(\xi) = D'(\beta)$. The inversion theorem for Laplace transforms therefore shows that

$$\frac{u^{(1)}}{a_{1\infty}} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\, e^{z\xi + \frac{1}{2}z(1-x)\eta}\, dz \quad , \tag{60a}$$

or alternatively, from equation (55),

$$\frac{u^{(1)}}{a_{1\infty}} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\, e^{\frac{1}{2}z(1-x)\alpha + \frac{1}{2}z(1+x)\beta}\, dz \quad . \tag{60b}$$

c is a constant, large enough to make the integration contour lie to the right of all the singularities in the integrand. The function x is responsible for the appearance of branch points, located at $z = {}^{-1}/_{\Gamma}$ and ${}^{-a^2}/_{\Gamma}$. With regard to $\overline{D}'(z)$, we shall assume that it behaves like $z^{-\nu}$, $\nu > 1$, as $|z| \to \infty$, which is sufficient to ensure that both $D(\xi)$ and $D'(\xi) \to 0$ as $\xi \to 0$ from above. One can then show that

$$\frac{u^{(1)}}{a_{1\infty}} = \frac{1}{2\pi i}\int_0^{\xi} \frac{d}{d\xi} D'(\xi_0) \int_{c-i\infty}^{c+i\infty} e^{z(\xi-\xi_0) + \frac{1}{2}z(1-x)\eta}\, \frac{dz}{z} \, . \, d\xi_0 \quad ,$$

or, integrating by parts,

$$\frac{u^{(1)}}{a_{1\infty}} = D'(\xi)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}z(1-x)\eta}\, \frac{dz}{z} - \int_0^{\xi} D'(\xi_0)\frac{1}{2\pi i}\frac{d}{d\xi_0}\int_{c-i\infty}^{c+i\infty} e^{z(\xi-\xi_0)+\frac{1}{2}z(1-x)\eta}\, \frac{dz}{z}\, d\xi_0 \tag{61}$$

Deforming the contour $c \pm i\infty$ into a loop contour $\mathcal{L}$ surrounding the branch points at $z = {}^{-1}/_{\Gamma}$ and ${}^{-a^2}/_{\Gamma}$ and the simple pole at $z = 0$, the ${}^{d}/_{d\xi_0}$ operation can be taken through the integral sign in the last term of equation (61) whilst the first integral there can be evaluated by expanding the integrand for large $|z|$. It follows that

$$\frac{u^{(1)}}{a_{1\infty}} = D'(\xi)e^{-\frac{1}{4\Gamma}(a^2-1)\eta} + \int_0^{\xi} D'(\xi_0)\frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\xi-\xi_0)+\frac{1}{2}z(1-x)\eta}\,dz.\,d\xi_0 . \qquad (62)$$

Equation (62) is a form of the solution for $u^{(1)}$ which is especially suitable for later developments.

It is now necessary to find $p^{(1)}$ and $\rho^{(1)}$. The latter is found from equation (41), simplified with the aid of equation (48) to read

$$a_{1\infty}\left(\rho_\alpha^{(1)} + \rho_\beta^{(1)}\right) = -\rho_\infty\left(u_\alpha^{(1)} - u_\beta^{(1)}\right) , \qquad (63)$$

and $p^{(1)}$ can be found from equations (14) and (15), the perturbation series 33, and equation (48), which together show that

$$p_\alpha^{(1)} - p_\beta^{(1)} = -\rho_\infty a_{1\infty}\left(u_\alpha^{(1)} + u_\beta^{(1)}\right) . \qquad (64)$$

It is easiest to use equation (60b) to eliminate the derivatives of $u^{(1)}$ from equations (63) and (64). Making sure that both $p^{(1)}$ and $\rho^{(1)}$ vanish for $\beta = \xi < 0$ it can be shown that

$$\frac{p^{(1)}}{p_\infty} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\,\frac{\gamma_1}{x}\,e^{\frac{1}{2}z(1-x)\alpha + \frac{1}{2}z(1+x)\beta}\,dz , \qquad (65a)$$

$$= \gamma_1 D'(\xi)e^{-\frac{1}{4\Gamma}(a^2-1)\eta} + \int_0^{\xi} D'(\xi_0)\frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\gamma_1}{x}\,e^{z(\xi-\xi_0)+\frac{1}{2}z(1-x)\eta}\,dz\,d\xi_0 , \qquad (65b)$$

$$\frac{\rho^{(1)}}{\rho_\infty} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\,x\,e^{\frac{1}{2}z(1-x)\alpha + \frac{1}{2}z(1+x)\beta}\,dz , \qquad (66a)$$

$$= D'(\xi)e^{-\frac{1}{4\Gamma}(a^2-1)\eta} + \int_0^{\xi} D'(\xi_0)\frac{1}{2\pi i}\int_{\mathcal{L}} x\,e^{z(\xi-\xi_0)+\frac{1}{2}z(1-x)\eta}\,dz\,d\xi_0 . \qquad (66b)$$

It is now possible to use equation (42) to find $a^{(1)}$ and then to solve equations (39) and (40) for $x^{(1)}$ and $y^{(1)}$. Equations (65a) and (66a) are best for this task and, initially, we find that

$$2\left(x_\alpha^{(1)} - y_\alpha^{(1)}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\left\{\frac{\gamma_1}{2x} - \frac{x}{2} + 1\right\}e^f\,dz , \qquad (67)$$

$$-2\left(x_\beta^{(1)} + y_\beta^{(1)}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\left\{\frac{\gamma_1}{2x} - \frac{x}{2} - 1\right\}e^f\,dz , \qquad (68)$$

where, for brevity, we define

$$f = \tfrac{1}{2}z(1-x)\alpha + \tfrac{1}{2}z(1+x)\beta = z\xi + \tfrac{1}{2}z(1-x)\eta . \qquad (69)$$

Using conditions (45) equations (67) and (68) can be integrated and rearranged a little to give

$$x^{(1)} - y^{(1)} = \frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} \bar{D}'(z)\, e^{f} \frac{dz}{z} + \frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} \bar{D}'(z) \left(\frac{\gamma_1}{2x} + \frac{x}{2}\right) \frac{\left[e^{f} - e^{z\beta}\right]}{z(1 - x)}\, dz \quad , \tag{70}$$

$$x^{(1)} + y^{(1)} = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \bar{D}'(z) \left(\frac{\gamma_1}{2x} + \frac{x}{2}\right) \frac{\left[e^{f} - e^{z\alpha}\right]}{z(1 + x)}\, dz \quad . \tag{71}$$

The formal solution of the first-order problem, according to the present method, is now complete. However, it is clearly not in a form which admits of either a ready assessment of its physical significance or of its validity in any special circumstances. Therefore we find it necessary to consider a number of special cases which permit simplification of the complex integrals found in the foregoing solutions.

5. Frozen flow.

When Γ, and hence τ', is infinite the flow is said to be frozen because the internal energy mode plays no part in the gas-dynamical processes. Referring to equation (58) it can be seen that as $\Gamma \to \infty$, $x \to 1$ and (from equation (69)) $f \to z\beta = z\xi$.

Using equations (60), (65a) and (66a), it can readily be seen that in the limit

$$u^{(1)} = a_{1\infty} D'(\xi) \quad , \tag{72}$$

$$p^{(1)} = \gamma_1 p_\infty D'(\xi) = \rho_\infty a_{1\infty} u^{(1)} \quad , \tag{73}$$

$$\rho^{(1)} = \rho_\infty D'(\xi) = \left(\rho_\infty / a_{1\infty}\right) u^{(1)} \quad , \tag{74}$$

These are precisely the results which arise from a formal linear theory of frozen flow, except that $x - y$ would appear in place of ξ for the argument of the function D'. The present solution shows that, to first order, the flow is of the simple-wave type, with all quantities constant on the (frozen) Mach lines $\xi = \beta$ = constant. However β is not equal to $x - y$ and these Mach lines are not the undisturbed field characteristics. Using equations (33) and (48) and taking proper limiting values in equations (70) and (71), it is not difficult to show that

$$x - y = -\beta + \epsilon D(\beta) + \tfrac{1}{4}\epsilon(\gamma_1 + 1) D'(\beta)\,(\alpha - \beta) \quad , \tag{75}$$

$$x + y = \alpha + \epsilon D(\beta) + \tfrac{1}{4}\epsilon(\gamma_1 + 1)\left[D(\alpha) - D(\beta)\right] \quad . \tag{76}$$

Adding these two equations gives

$$2\left[x - \epsilon D(\beta)\right] = \alpha - \beta + \tfrac{1}{4}\epsilon(\gamma_1 + 1)\left\{ D'(\beta)(\alpha - \beta) + D(\alpha) - D(\beta) \right\} \quad .$$

Using the mean value theorem we can write

$$D(\alpha) - D(\beta) = \int_{\beta}^{\alpha} D'(\hat{\beta})\, d\hat{\beta} = D'(\bar{\beta})\,(\alpha - \beta) \quad ,$$

where $\alpha \geqslant \bar{\beta} \geqslant \beta$. Since $D'(\beta)$ must be bounded, if only from physical considerations, it follows that

$$2 \left[x - \epsilon D(\beta) \right] = (\alpha - \beta)(1 + 0(\epsilon)) ,$$

whence equation (75) shows that

$$\left[x - \epsilon D(\beta) \right] \left[1 - \tfrac{1}{2}\epsilon(\gamma_1 + 1) D'(\beta) \right] - y = \beta \quad , \qquad (77)$$

correct to $0(\epsilon)$.

The Mach line β = constant is therefore a straight line, passing through the point $\epsilon D(\beta), \beta$ on the piston path, with slope

$$\left(\frac{dy}{dx}\right)_{\beta} = 1 - \tfrac{1}{2}\epsilon(\gamma_1 + 1) D'(\beta) \qquad (78)$$

When the piston moves to the left $D(\beta)$ is a negative function: if we also ensure that $D'(\beta)$ is always less than zero and decreases monotonically to some final value $D'(\delta)$ in the interval $0 \leqslant \beta \leqslant \delta$, thereafter remaining constant for all $\beta > \delta$, it is clear that each successive Mach line (β = constant) has a greater slope than its predecessors and no intersections of characteristics occur. The mapping from the α, β to the x, y plane is therefore single valued. The situation arising when the image of the characteristics plane is no longer a single-sheeted surface in x, y coordinates, and the associated question of shock wave formation, is discussed by Miss Fox (1955) and we shall not pursue this matter any further here. The paper by Miss Fox just referred to also establishes the convergence of the series for x and y and we may take it that the results of this section constitute a uniformly valid first-order estimate of the frozen flow behaviour for the case treated.

The case of a piston suddenly accelerated to a constant velocity $\epsilon a_{1\infty} D'(\delta) < 0$ can be solved by letting $\delta \to 0+$. Then $D(\beta) \to 0$ in $0 \leqslant \beta \leqslant \delta$ and we find from equation (77) that

$$y = x \left[1 - \tfrac{1}{2}\epsilon(\gamma_1 + 1) D'(\beta) \right] = x \left[1 - \tfrac{1}{2}(\gamma_1 + 1) \frac{u}{a_{1\infty}} \right] \qquad (79)$$

within this range of β. Equation (79) describes the configuration of the centred simple-wave, through which u decreases from zero to its final value of $\epsilon a_{1\infty} D'(\delta)$ in between the diverging characteristic lines

$$y = x \text{ and } y = x \left[1 - \tfrac{1}{2}\epsilon(\gamma_1 + 1) D'(\delta) \right] \quad . \qquad (80)$$

Within this expansion fan the velocity gradient is given by

$$\frac{\partial u}{\partial y} = \frac{-2a_{1\infty}}{(\gamma_1 + 1)x} \qquad (81)$$

and hence is constant, to first order, for any given x.

The results for $\Gamma = \infty$ given above will provide useful comparisons later on. It should also be pointed out that all of the results obtained specifically for the case $\Gamma = \infty$ will be equally true for the case $a^2 = 1$, no matter what the value of Γ may be. Putting a^2 equal to unity implies (see equation 49) that the internal mode contains no communicable energy and it is therefore not surprising that the value of Γ is irrelevant.

6. Conditions at the wave head.

It is clear from equations (62), (65) and (66) that the flow disturbances vanish as $\xi \to 0$, from above as well as from below, on account of the assumed continuity of the function $D'(\xi)$ at this point. However, we may profitably examine the gradients of the perturbations along this same line.

We may note first that, if all perturbations vanish on $\beta = 0$, equation (12) shows directly that

$$\frac{\partial}{\partial \alpha}(x - y) = 0 \quad ; \beta = 0 .$$

Then equations (28) and (29) show that the line $\beta = 0$ is simply the straight line

$$y = x, \tag{82}$$

a result which is true to any order of accuracy given the proviso made in the previous sentence.

Since the flow disturbances (i.e. $u^{(1)}$, $p^{(1)}$, $\rho^{(1)}$) all vanish on $\beta = 0$ it is clear that their derivatives with respect to α (β being held constant) are also zero on this line. We can find $\partial u^{(1)}/\partial \xi$ at fixed η (or $u_\xi^{(1)}$ for short) from equation (62) for example. The result is

$$u_\xi^{(1)} = a_{1\infty} D''(\xi) e^{-\frac{1}{4\Gamma}(a^2 - 1)\eta} + \int_0^\xi D''(\xi_0) \frac{1}{2\pi i} \int_{\mathcal{L}} e^{z(\xi - \xi_0) + \frac{1}{2}z(1 - x)\eta} \, dz \, d\xi_0 \tag{83}$$

from which it is clear that

$$u_\xi^{(1)} (\xi = \beta = 0) = a_{1\infty} D''(0) e^{-\frac{1}{4\Gamma}(a^2 - 1)\eta} = a_{1\infty} D''(0) e^{-\frac{1}{4\Gamma}(a^2 - 1)\alpha} . \tag{84}$$

$D''(0)$ is not zero (in general) and the velocity gradient immediately downstream of the leading characteristic depends on the initial curvature of the piston path. Comparing this general result with that for frozen flow, namely

$$u_\xi^{(1)} (\xi = \beta = 0 \; ; \; \Gamma = \infty) = a_{1\infty} D''(0) , \tag{85}$$

it can be seen that the relaxation effects lead to a decay in the velocity gradient with increasing distance from the corner, (i.e. with increasing α.) Equations (65b) and (66b) show that

$$p_\xi^{(1)} (\xi = 0) = \gamma_1 p_\infty D''(0) e^{-\frac{1}{4\Gamma}(a^2 - 1)\alpha} = \rho_\infty a_{1\infty} u_\xi^{(1)} (\xi = 0) \tag{86}$$

$$\rho_\xi^{(1)} (\xi = 0) = \rho_\infty D''(0) e^{-\frac{1}{4\Gamma}(a^2 - 1)\alpha} = \rho_\infty / a_{1\infty} \, u_\xi^{(1)} (\xi = 0) \tag{87}$$

From these results we see (compare with equations (73) and (74)) that the relationships between $u^{(1)}$, $p^{(1)}$ and $\rho^{(1)}$ are those that would arise in a completely frozen flow. These results apply, to the current order of accuracy, under all circumstances.

7. The Quantities $y_\alpha^{(1)}$ and $y_\beta^{(1)}$

The two equations (51) and (52) from which the single equation (56) for $u^{(1)}$ and thence all the solutions presented in the last three sections have been derived are themselves derived from equations (21) and (22) with the aid of the perturbation series and the zero-th approximation to y (in particular $y_\alpha^{(0)}$ and $y_\beta^{(0)}$). We note the implication that $y_\alpha^{(1)}$ and $y_\beta^{(1)}$ shall both be no more than 0(1), so that we should examine these two latter quantities in order to check whether this is indeed so.

Equations (70) and (71) show that

$$2\,y^{(1)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\,\tfrac{1}{2}\left(\frac{\gamma_1}{x} + x\right)\left\{\frac{e^{z\alpha} - e^{f}}{1+x} + \frac{e^{z\beta} - e^{f}}{1-x}\right\}\frac{dz}{z} \tag{88}$$

from which it readily follows that

$$2y_\alpha^{(1)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\,\tfrac{1}{2}\left(\frac{\gamma_1}{x} + x\right)\left(e^{z\alpha} - e^{f}\right)\frac{dz}{1+x}\ , \tag{89}$$

$$2y_\beta^{(1)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{D}'(z)\,\tfrac{1}{2}\left(\frac{\gamma_1}{x} + x\right)\left(e^{z\beta} - e^{f}\right)\frac{dz}{1-x}\ . \tag{90}$$

The integral for $y_\alpha^{(1)}$ can be re-arranged in the form

$$\begin{aligned} 2y_\alpha^{(1)} = {} & \tfrac{1}{4}(\gamma_1 + 1)\,D'(\alpha) + \int_0^\alpha D'(\xi_0)\,\frac{1}{2\pi i}\int_{\mathcal{L}} \tfrac{1}{2}\left(\frac{\gamma_1}{x} + x\right)\frac{e^{z(\alpha-\xi_0)}}{1+x}\,dz\,d\xi_0 \\ & -\tfrac{1}{4}(\gamma_1 + 1)\,D'(\beta)\,e^{-\frac{1}{4l}(a^2-1)\eta} - \int_0^\beta D'(\xi_0)\,\frac{1}{2\pi i}\int_{\mathcal{L}} \tfrac{1}{2}\left(\frac{\gamma_1}{x} + x\right)\frac{e^{f - z\xi_0}}{1+x}\,dz\,d\xi_0 \end{aligned} \tag{91}$$

and it is apparent that $y_\alpha^{(1)}$ is 0(1) everywhere (since D' is limited to this same order of magnitude.)

With regard to $y_\beta^{(1)}$ we note first that

$$z^2\overline{D}'(z) = \overline{D}'''(z) + D''(0).$$

where $\overline{D}'''$ is the transform of the third derivative of D. Putting this result into equation (90), using the convolution theorem and integrating by parts gives

$$\begin{aligned} 2y_\beta^{(1)} = {} & D''(\beta)\,\frac{1}{2\pi i}\int_{\mathcal{L}} \tfrac{1}{2}\left(\frac{\gamma_1}{x} + x\right)\left(1 - e^{\frac{1}{2}z(1-x)\eta}\right)\frac{dz}{z^2(1-x)} \\ & + \int_0^\beta D''(\xi_0)\,\frac{1}{2\pi i}\int_{\mathcal{L}} \tfrac{1}{2}\left(\frac{\gamma_1}{x} + x\right)\left(e^{z(\beta-\xi_0)} - e^{f - z\xi_0}\right)\frac{dz}{z(1-x)}\,d\xi_0\ . \end{aligned}$$

The first complex integral can be evaluated and we find that

$$y_\beta^{(1)} = -\tfrac{1}{8}(\gamma_1 + 1) D''(\beta)\left(\frac{4\Gamma}{a^2 - 1}\right)\left(1 - e^{-(a^2-1)\eta/4\Gamma}\right)$$

$$+\int_0^\beta D''(\xi_0)\frac{1}{2\pi i}\int_{\mathcal{L}} \tfrac{1}{4}\left(\frac{\gamma_1}{x} + x\right)\left(e^{z(\beta - \xi_0)} - e^{f - z\xi_0}\right)\frac{dz}{z(1-x)}\, d\xi_0 \qquad (92)$$

In the limit as $\Gamma \to \infty$ equations (91) and (92) give the same results as would be obtained from equations (75) and (76).

The behaviour of $y_\beta^{(1)}$ is clearly very different from that of $y_\alpha^{(1)}$, depending as it does on D'' rather than on D'. At this stage two observations are in order. First, the parameters α and β which define the network of <u>frozen</u> Mach lines are only appropriate so long as $\Gamma > 0$. In the special case for which $\Gamma = 0$ the characteristics change discontinuously to the equilibrium Mach lines and the compatability conditions 12 and 13, on which the whole of the subsequent analysis is based, cease to apply. The case $\Gamma = 0$ is developed briefly in the Appendix. Second, we note that when $\Gamma = \infty$, so that

$$y_\beta^{(1)} = -\tfrac{1}{8}(\gamma_1 + 1) D''(\beta)\eta\ ,$$

$y_\beta^{(1)}$ grows without limit as η increases for a fixed β. If we write the last term in equation (92) in the form

$$\int_0^\beta D''(\xi_0)\frac{1}{2\pi i}\int \tfrac{1}{8}\left(\frac{\gamma_1}{x} + x\right)\left\{\int_\alpha^\beta e^{\frac{1}{2}z(1-x)\hat{\alpha} + \frac{1}{2}z(1+x)\beta}\, d\hat{\alpha}\right\}e^{-z\xi_0}\, dz.\, d\xi_0\ ,$$

it is apparent that $y_\beta^{(1)}$ will in general be proportional to $\eta D''$ for any value of Γ, although the proportionality factor and the argument of the function D'' will be much more complicated expressions than the simple ones which arise when $\Gamma = \infty$. When $\beta = 0$, so that

$$y_\beta^{(1)} = -\tfrac{1}{8}(\gamma_1 + 1)D''(0)\left(\frac{4\Gamma}{a^2 - 1}\right)\left(1 - e^{-(a^2-1)\alpha/4\Gamma}\right) \qquad ,$$

we note that we cannot permit $D''(0)\Gamma$ to approach large values for any value of α which makes $(a^2 - 1)\alpha/\Gamma \geq O(1)$, as it would also not then be possible to write

$$\frac{1}{y_\beta} \approx \frac{1}{y_\beta^{(0)}}\left(1 - \epsilon y_\beta^{(1)}/y_\beta^{(0)} \ \ldots\ldots\ \right)$$

as we have had to do in the derivation of equations (35) and (36).

When $\Gamma = \infty$, the success of the present method in producing a uniformly valid first-order solution is not in question, even though $y_\beta^{(1)}$ does behave like $D''(\beta)\eta$, since neither $u^{(1)}$ nor $p^{(1)}$ depend explicitly on y_α and y_β in this case. Since we have no need to look further at the case $\Gamma = \infty$ and cannot examine the case $\Gamma = 0$ by present methods, we must concentrate our efforts on the situation for which $0 < \Gamma < \infty$. In doing so we must take note of the restrictions imposed by the form of $y_\beta^{(1)}$, namely that there will be a limitation on the size of η above which the solutions cease to be valid, (this value will depend on D'') and that $\Gamma D''$ also cannot be allowed to become large. If these restrictions are not adhered to the perturbation method will break down. Unfortunately this would seem to preclude a discussion of both the suddenly withdrawn piston and of the flow far from the piston face. However, despite these rather disappointing limitations we can still produce some results of interest and this we now proceed to do.

8. Conditions on the piston face

The piston face is located on the line $\alpha = \beta$, or $\eta = 0$, in the α, β plane. Consequently the pressure and density perturbations are simply

$$\frac{p^{(1)}}{p_\infty} = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \bar{D}'(z) \frac{\gamma_1}{x} e^{z\beta} dz \quad , \tag{93}$$

$$\frac{\rho^{(1)}}{\rho_\infty} = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \bar{D}'(z) \, x \, e^{z\beta} dz \quad , \tag{94}$$

(see equations (65a) and (66a)). When $\eta = 0$ we know from the boundary conditions that $\beta = y$, so that equations (93) and (94) give the pressure and density perturbations directly as functions of y on $x = \epsilon D(\beta) = \epsilon D(y)$.

Some reduction of the complex integrals in equations (93) and (94) is possible, using the convolution theorem and the known inverse transformations of the functions $(xz)^{-1}$ and xz^{-1}. The results are quite well known and will not therefore be repeated here (see Clarke, 1960, and Der, 1961, for example). However we note that for $\xi = \beta \to 0$

$$p^{(1)} \to \gamma_1 \, p_\infty D'(0) \quad ; \quad \rho^{(1)} \to \rho_\infty D'(0) \tag{95}$$

whilst for $\xi = \beta \to \infty$

$$p^{(1)} \to \gamma_1 \, a^{-1} p_\infty D'(\infty) \quad ; \quad \rho^{(1)} \to a\rho_\infty D'(\infty) \, . \tag{96}$$

Since $u^{(1)}$ is equal to $a_{1\infty} D'(\beta)$, equations (95) and (96) show that the perturbations are related as in a frozen or in an equilibrium state, respectively, according as to whether one is near to or far from the point at which the piston begins to accelerate.

9. Conditions in regions where η/Γ is large

The previous section demonstrates that the perturbations can be readily evaluated when $\eta = 0$. Unfortunately no such simple solutions are available for $\eta > 0$ but it is possible to make certain useful approximations in regions where $\eta \gg \Gamma$.

Let us first consider $u^{(1)}$, using the form of the solution written down in equation (62). Since $|D'|$ is 0(1) the first term there becomes very small when $(a^2 - 1)\eta \, / 4\Gamma \gg 1$. We remark that it is therefore necessary to deal only with the case for which $a^2 > 1$ but as we already know the solution for $a = 1$ this causes no hardship (see Section 5.) We may also observe that a^2 is never very much greater than unity, whence it follows that η/Γ will be large if $(a^2 - 1)\eta/4\Gamma$ is large. In future we shall take $\eta/\Gamma \gg 1$ to be a sufficient condition for the neglect of terms like the first one in equation (62). Under this condition, the major contribution to $u^{(1)}$ will arise from the second (integral) term of 62, as we shall shortly show, always provided that ξ is not too small. The situation arising at $\xi = 0$ is dealt with in Section 6.

In order to justify our assertion that the integral term in (62) is dominant under the stated conditions we shall start by examining the inner (complex) integral there. We note that it can be written in the form

$$J = \frac{1}{2\pi i \Gamma} \int_{\mathcal{L}} e^{w(\xi' - \xi_0') + \frac{1}{2}w(1 - x)\eta'} dw \tag{97}$$

where

$$w = z\Gamma \; ; \; \xi' = \xi/\Gamma \; ; \; \xi_0' = \xi_0/\Gamma \; ; \; \eta' = \eta/\Gamma \; ,$$

$$x^2 = \frac{a^2 + w}{1 + w} \quad ,$$

and $\mathcal{L}$ is a loop contour surrounding the branch points at $w = -a^2$ and -1. When $\eta' >> 1$ the major contributions to J will arise from those parts of $\mathcal{L}$ which can be made to pass through saddle points of the function

$$w\mu + w(1 - x) \; ; \; \mu \equiv 2(\xi' - \xi_0')/\eta' \; , \tag{98}$$

the value of μ being fixed. The saddle points occur at values of w, written as w_0, for which

$$\mu + F'(w_0) = 0 \quad ; \quad F(w) \equiv w(1 - x) \quad . \tag{99}$$

Thus w_0 is a function of μ. The appropriate dominant part of J will be proportional to

$$\exp\left\{ \left[w_0\mu + F(w_0) \right] \tfrac{1}{2}\eta' \right\},$$

and with η' fixed this exponential term will itself be a maximum for some particular value of μ. The latter value can be found by noting that a maximum for the function

$$w_0\mu + F(w_0)$$

occurs when $w_0 = 0$ (note equation (99)). Hence the required value of μ is given by

$$\mu = -F'(0) = a - 1 .$$

We are therefore led to write J in the form

$$J = \frac{1}{2\pi i \Gamma} \int_S e^{w\Delta' + w(a - x)\frac{1}{2}\eta'} dw \tag{100}$$

where

$$\Delta' = \xi' - \xi_0' - \tfrac{1}{2}(a - 1)\eta'$$

and S is the steepest path through the col at $w = 0$. It is now a straightforward matter to evaluate equation (100) asymptotically, leading to the expression

$$J \sim \frac{e^{-\Delta'^2/\eta'(a - 1/a)}}{\Gamma\sqrt{\pi\eta'(a - 1/a)}} \quad , \tag{101}$$

correct to within a factor $1 + 0(\eta'^{-\frac{1}{2}})$.

Putting (101) into (62) we find that

$$\frac{u^{(1)}}{a_{1\infty}} \sim \int_0^{\xi} D'(\xi_0) \exp\left\{ - \frac{[\xi - \xi_0 - \frac{1}{2}(a-1)\eta]^2}{\Gamma\eta(a - 1/a)} \right\} \frac{d\xi_0}{\sqrt{\pi\Gamma\eta(a - 1/a)}} \tag{102}$$

under the stated conditions. A similar result has been given by Whitham (1959) in connection with his studies of a general equation like our equation (56).

A useful alternative form of equation (102) is derived by writing

$$\xi - \tfrac{1}{2}(a-1)\eta = B \tag{103}$$

and putting

$$B - \xi_0 = \lambda s \ ; \ \lambda \equiv \sqrt{\Gamma\eta(a - 1/a)} \ . \tag{104}$$

Then (102) becomes

$$\frac{u^{(1)}}{a_{1\infty}} \sim \frac{1}{\sqrt{\pi}} \int_{-B/\lambda}^{\frac{1}{2}(a-1)\eta/\lambda} D'(B + \lambda s) e^{-s^2} \, ds \ . \tag{105}$$

The upper limit in equation (105) is proportional to $\sqrt{\eta/\Gamma}$, so that replacing it by infinity will lead only to errors of order $\exp(-\eta/\Gamma)$ and such terms have already been neglected.

In order to make further progress it is convenient at this stage to be a little more specific about the nature of the function $D'(\xi)$. We have remarked previously that it is a bounded function and we shall now assume that it is continuous and tends monotonically to the value $D'(\xi)$ as ξ increases from 0 to δ. For $\xi > \delta$ we assume that $D'(\xi) = D'(\delta) =$ constant, its magnitude being $0(1)$. The implications of these assumptions are that the second derivative D'' is at worst piece-wise continuous and bounded. The mean value can now be used to write

$$\left.\begin{aligned} &D'(B + \lambda s) = D'(B) + \lambda s D''(B + \theta\lambda s) \ ; \ 0 < \theta < 1 \ ; \ -B < \lambda s < \delta - B \\ &D'(B + \lambda s) = D'(\delta); \ \lambda s > \delta - B. \end{aligned}\right\} \tag{106}$$

Using equations (106) in (105) we distinguish between the two possible cases

$$B \lessgtr \delta - \tfrac{1}{2}(a - 1)\eta \tag{107}$$

for which the quantity $B + \lambda s$ either does not or does pass through the value δ within the integration interval, respectively.

Firstly, when $B < \delta - \frac{1}{2}(a-1)\eta$, we find that

$$\frac{u^{(1)}}{a_{1\infty}} \sim D'(B)\, \tfrac{1}{2}\left\{ 1 + \text{erf}\,(B/\lambda) \right\} + \frac{\lambda}{2\sqrt{\pi}} D''(\bar{\xi}) e^{-B^2/\lambda^2} \ . \tag{108}$$

Terms of order $\exp(-\eta/\Gamma)$ have been neglected and $\bar{\xi}$ in the function $D''(\bar{\xi})$ is a suitable mean value, lying between 0 and ξ, which depends in general on both ξ and η. Since equation (108) applies essentially to the case $\eta/\Gamma >> 1$ we observe that if B is to lie in the range around and above zero it is essential to have $\delta/\Gamma >> 1$ too if the result is to be at all meaningful in the circumstances. This important supplementary condition for the validity of equation (108) can be re-interpreted if we note the monotonicity of D' in the interval $0 \leq \xi \leq \delta$, for we can then reasonably assume that D'' is $0(1/\delta)$ within this same interval. Consequently the requirement that $\delta/\Gamma >> 1$ is equivalent to saying that $\Gamma D'' << 1$. As we have remarked at the end of Section 7, the present theory is only valid if $\Gamma D''$ is not allowed to become large and so the circumstances leading up to equation (108) are quite in line with the inherent limitations of the present theory (resulting from the behaviour of the functions y_α and y_β.)

Secondly, when $B > \delta - \frac{1}{2}(a - 1)\eta$, the integration interval in equation (105) can be split into two sub-intervals namely, $-B \leq \lambda s \leq \delta - B$ and $\delta - B \leq \lambda s \leq \frac{1}{2}(a - 1)\eta$. It then follows that

$$\frac{u^{(1)}}{a_{1\infty}} \sim D'(B)\,\tfrac{1}{2}\left\{\operatorname{erf}\left(\frac{\delta - B}{\lambda}\right) + \operatorname{erf}\left(\frac{B}{\lambda}\right)\right\} + \frac{\lambda}{2\sqrt{\pi}}\, D''(\bar{\delta})\left\{e^{-B^2/\lambda^2} - e^{-(\delta - B)^2/\lambda^2}\right\}$$

$$+ D'(\delta)\,\tfrac{1}{2}\left\{1 - \operatorname{erf}\left(\frac{\delta - B}{\lambda}\right)\right\} , \qquad (109)$$

where $\bar{\delta}$ in $D''(\bar{\delta})$ is the appropriate value of $\bar{\xi}$ (defined above) when $\xi = \delta$. That equations (108) and (109) are equivalent when $B = \delta - \frac{1}{2}(a - 1)\eta$ can be seen by writing $\delta - B = \frac{1}{2}(a - 1)\eta$ in equation (109) and ignoring terms in $\exp(-\eta/\Gamma)$. (To this order of accuracy $\operatorname{erf}[(\delta - B)/\lambda] = 1$.)

The conditions in equation (107) have a simple physical interpretation, which can be seen by noting that they are (from equation (103) defining B) equivalent to

$$\xi = \beta \lessgtr \delta .$$

Since the β = constant Mach lines delineate the limits of any upstream influence of changes in the piston motion, it is clear that the condition decides whether the flow field is unaware, or aware, respectively, of the fact that the piston has stopped accelerating.

Before proceeding to further discussion of the results in equations (108) and (109) we note two other results of considerable importance. If we treat equations (65b) and (66b) for $p^{(1)}$ and $\rho^{(1)}$ in precisely the same way as we have just treated equation (62) for $u^{(1)}$ it can quickly be seen that, when $\eta/\Gamma >> 1$,

$$\left.\begin{aligned} p^{(1)} &\simeq p_\infty \frac{\gamma_1}{a}\,\frac{u^{(1)}}{a_{1\infty}} = \rho_\infty a_{2\infty} u^{(1)} , \\ \rho^{(1)} &\simeq \rho_\infty\, a\, \frac{u^{(1)}}{a_{1\infty}} = \rho_\infty \frac{u^{(1)}}{a_{2\infty}} . \end{aligned}\right\} \qquad (110)$$

$a_{2\infty}$ (equal to $a_{1\infty}/a$) is the undisturbed-flow equilibrium sound speed, and equations (110) are the first-order results for an equilibrium flow. The important point is that they arise solely as a consequence of the assumptions which lead to equation (105) for $u^{(1)}$ (these assumptions enable one to show that the inner integral terms in equations (65b) and (66b) are equal to $\gamma_1 J/a$ and aJ respectively) and they do not depend on any additional factors.

Returning now to equations (108) and (109), it is possible to derive a number of interesting conclusions from them, but it is with one of these in particular that we wish to concern ourselves here. Thus, if $B >> \lambda$ equation (108) shows that

$$\frac{u^{(1)}}{a_{1\infty}} \sim D'(B) \ ; \ B < \delta - \tfrac{1}{2}(a-1)\eta \qquad (111)$$

since $\mathrm{erf}(B/\lambda) \to 1$ and the last term is small by hypothesis. When $B > \delta - \frac{1}{2}(a-1)\eta$ we must use equation (109) and a first estimate under the condition $B >> \lambda$ can be written as follows:

$$\frac{u^{(1)}}{a_{1\infty}} \sim D'(B) + \tfrac{1}{2}\left[D'(\delta) \sim D'(B)\right] \left\{1 - \mathrm{erf}\left(\frac{\delta - B}{\lambda}\right)\right\} - \frac{\lambda D''(\bar{\delta})}{2\sqrt{\pi}}\, e^{-(\delta - B)^2/\lambda^2} \ . \qquad (112)$$

The condition $B > \delta - \frac{1}{2}(a-1)\eta$ is certainly satisfied if $B > \delta$, since η is positive in the disturbed flow region. Since $D'(B) = D'(\delta)$ for $B > \delta$ the second term in (112) vanishes. The order of magnitude of the last term in (112) is less than or equal to λ/δ and is therefore negligible if $\lambda/\delta << 1$. When $0 < \delta - B < \frac{1}{2}(a-1)\eta$ we can write

$$D'(\delta) - D'(B) = 0(\left[\delta - B\right]/\delta)$$

and so the second term in (112) has an order of magnitude equal to

$$\left(\frac{\delta - B}{\delta}\right)\left\{1 - \mathrm{erf}\left(\frac{\delta - B}{\lambda}\right)\right\} \ .$$

We can show that when $\lambda \gg \Gamma$ this quantity has a maximum value and this value occurs for $\delta - B < \lambda$. Consequently the whole term is less than λ/δ in magnitude and so, if the condition $\lambda/\delta << 1$ is satisfied, equation (112) gives

$$\frac{u^{(1)}}{a_{1\infty}} \sim D'(B) \qquad (113)$$

once again.

To reiterate, $u^{(1)}$ will be given by (113) if all conditions $\eta \gg \Gamma$, $B \gg \lambda$ and $\delta \gg \lambda$ are satisfied. Since $\lambda \gg \Gamma$ is necessarily true when $\eta \gg \Gamma$, the last of these conditions merely confirms that $\delta \gg \Gamma$.

Since $p^{(1)}$, $\rho^{(1)}$ and $u^{(1)}$ are related as in equations (110) we have now proved that there are parts of the flow field in which the flow is, to a good degree of accuracy, in an equilibrium state and, furthermore, that the flow variables are constant along lines of constant B. We could now translate this information about the constancy of $u^{(1)}$, etc., on lines of constant B into the physical (x, y) plane by calculating $x^{(1)}$, $y^{(1)}$ and so on. This is a fairly lengthy procedure and we can elicit sufficient information for present purposes by looking instead at the behaviour of (dy/dx) on a line of constant B. By making use of the compatibility conditions (12) and (13) it is easy to show that

$$\left(\frac{dy}{dx}\right)_B = \frac{a_{1\infty}(a+1)y_\alpha + a_{1\infty}(a-1)y_\beta}{(a_1 + u)(a+1)y_\alpha - (a_1 - u)(a-1)y_\beta} \ . \qquad (114)$$

Using the series expansions (33) for the variables involved here, noting that $y_\alpha^{(0)} = \frac{1}{2} = y_\beta^{(0)}$, and neglecting terms of order higher than ϵ, we find that

$$\left(\frac{dy}{dx}\right)_B \simeq a\left\{1 - \epsilon\frac{a_1^{(1)}}{a_{1\infty}} - \epsilon a\frac{u^{(1)}}{a_{1\infty}} + \epsilon\left(\frac{a^2-1}{a}\right)\left(y_\beta^{(1)} - y_\alpha^{(1)}\right)\right\} \tag{115}$$

The assumption that $y_\alpha^{(1)}$ and $y_\beta^{(1)}$ are less than 0(1) is again implicit in the derivation of (115).

In the regions where (110) and (113) are valid approximate solutions we can readily show that the terms

$$\frac{\epsilon}{a_{1\infty}}\left(a_1^{(1)} + au^{(1)}\right)$$

in (115) are equal to

$$\tfrac{1}{2}\epsilon a(\gamma_2 + 1)\frac{u^{(1)}}{a_{1\infty}} = \tfrac{1}{2}\epsilon a(\gamma_2 + 1)D'(B).$$

Thus (115) can be re-written in the form

$$\left(\frac{dy}{dx}\right)_B \simeq a\left\{1 - \tfrac{1}{2}\epsilon a(\gamma_2 + 1)\,D'(B) + \epsilon\left(\frac{a^2-1}{a}\right)\left(y_\beta^{(1)} - y_\alpha^{(1)}\right)\right\} \tag{116}$$

in these regions.

Now, although we should not let $\Gamma = 0$ within the framework of the present theory, we can let $\Gamma \to 0$. If we do this, the conditions which lead to (116) are then such as to make it apparent that the given approximate solutions within their region of validity should be the same as those arising in an a priori equilibrium flow. The first-order equilibrium flow solutions are given in the Appendix and, identifying B with B_e, we can see that the relationships between $p^{(1)}$, $\rho^{(1)}$, $u^{(1)}$ and B are of precisely the correct form (i.e. compare equations (A6) with equations (110) and (113)). However, it is quite clear that $y_\beta^{(1)} - y_\alpha^{(1)}$ is not in general equal to zero and equations (116) and (A10) fail to agree.

One is forced to conclude that the extension of Lin's technique to the relaxing gas problem does not lead to a uniformly valid first-order solution in the event that $0 < \Gamma < \infty$. Certainly the behaviour of $y_\beta^{(1)}$, which was discussed in Section 7, intimated that we could expect such a breakdown in regions where $\eta D''$ became too large, but the results just presented show that the actual deficiencies in the theory are rather more serious.

This is not to say that there is absolutely no advantage to be gained by using the characteristic-parameter plane in place of a formal linearisation in x, y coordinates. For example, if we let $\Gamma \to 0$ and $\eta \to 0$ in such a way that $\eta/\Gamma \to \infty$ then $y_\beta^{(1)} - y_\alpha^{(1)}$ is zero (see equations (89) and (90)) and $(dy/dx)_B$ is equal to $(dy/dx)_{B_e}$. In this case the "lines of constant B" are, locally, coincident with the true equilibrium characteristic lines in quite the proper way. In addition, if $B \geq \delta$, so that both α and β are necessarily also greater than or equal to δ we can show that both $y_\alpha^{(1)}$ and $y_\beta^{(1)} \to 0$ as $\Gamma \to 0$. Once again we find that $(dy/dx)_B$ is equal to $(dy/dx)_{B_e}$ and the "final" equilibrium flow zone is correctly predicted to first-order.

Inspection of the results shows that it is only in regions where y_α and y_β are exactly equal to $\frac{1}{2}$ that the solutions succeed. One must infer that the perturbation techniques, which effectively replace the coefficients $1/y_\alpha$ and $1/y_\beta$ in equations (21) and (22) by $1/y_\alpha^{(0)}$ and $1/y_\beta^{(0)}$, (as in equations (35) and (36)), lead to too drastic an oversimplification of the true dependence of $p^{(1)}$, $u^{(1)}$ etc. on the parameters α and β.

It is significant that when $0 < \Gamma < \infty$ the problem cannot be set up without explicitly including the third set of characteristics, namely the streamlines or particle paths. Equations (4), (14) and (15) make this quite clear and the non-trivial existence of this additional set of characteristic lines (over and above the Mach lines α and β) is directly responsible for the appearance of y_α and y_β in equations (21) and (22) for p and u. In the limiting cases $\Gamma = 0$ and $\Gamma = \infty$ it can be seen that the problem is completely defined without the need for specific reference to the rate of variation of quantities along particle paths. The de-coupling of the equations for $p(\alpha,\beta)$ and $u(\alpha,\beta)$ from those for $x(\alpha,\beta)$ and $y(\alpha,\beta)$ which results has a direct bearing on the success of the method in these cases.

The relaxing gas problem has a considerable similarity to that of the flow of an ideal gas which contains distributed heat sources: in fact equations (12), (13), (14) and (15) describe the latter situation precisely, with Q related directly to the heat source terms, which are assumed known. Even with the simplifications resulting from some prior knowledge of Q it is still not possible to produce a uniformly valid first-order solution for precisely the same reasons as those that we have met with above, namely that it is impossible to avoid reference to the rate of entropy rise along particle paths, with the resultant coupling between the equations for x and y and those for p and u.

In conclusion, whilst the characteristics plane perturbation scheme does lead to some minimal gains in accuracy when compared with straightforward linearisations in orthogonal coordinate systems, it does not succeed in producing a uniformly valid solution over the entire field of an entropy-producing flow.

REFERENCES

Clarke, J. F.	1960	J. Fluid Mech., 7, 557.
Der, J. J.	1961	NASA TR - R - 119.
Fox, P. A.	1955	J. Math. and Phys., XXXIV, 133.
Lighthill, M. J.	1949	Phil. Mag., 7, 1179.
Lin, C. C.	1954	J. Math. and Phys., XXXIII, 117.
Vincenti, W. G.	1959	J. Fluid Mech., 6, 481.
Whitham, G. B.	1952	Comm. on Pure and Appl. Math., 5, 301.
Whitham, G. B.	1959	Comm. on Pure and Appl. Math., 12, 113.

APPENDIX

With the initial assumption that the flow is to remain in equilibrium throughout (so that $\tau' = 0 = \Gamma$) it becomes necessary to make some changes in the basic equations of Section 2. Without going into the matter too exhaustively these changes are as follows. Firstly, the compatibility conditions (12) and (13) must be replaced by

$$a_{1\infty}\frac{\partial x}{\partial A_e} = (a_2 + u)\frac{\partial y}{\partial A_e} \quad , \; B_e = \text{constant}, \tag{A1}$$

$$a_{1\infty}\frac{\partial x}{\partial B_e} = -(a_2 - u)\frac{\partial y}{\partial B_e} \; , \; A_e = \text{constant}, \tag{A2}$$

thereby defining two new characteristic parameters A_e and B_e. (NB. a_2 is the local equilibrium sound speed, equal to a_1/a. The variable y is as defined in (9).) Secondly, in place of (14) and (15) we have

$$\frac{\partial p}{\partial A_e} + \rho a_2 \frac{\partial u}{\partial A_e} = 0, \tag{A3}$$

$$\frac{\partial p}{\partial B_e} - \rho a_2 \frac{\partial u}{\partial B_e} = 0. \tag{A4}$$

Using the perturbation series (33) together with the boundary conditions

$$u = \epsilon\, a_{1\infty} D'(B_e) \; ; \; x = \epsilon D(B_e) \; ; \; y = B_e \text{ when } A_e = B_e \tag{A5}$$

we can show that

$$\left.\begin{aligned} & u^{(1)} = a_{1\infty} D'(B_e) \, , \\ & p^{(1)} = \rho_\infty a_{2\infty} u^{(1)} \quad ; \quad \rho^{(1)} = \rho_\infty u^{(1)}/a_{2\infty} \, , \\ & ax^{(0)} - y^{(0)} = -B_e \quad ; \quad ax^{(0)} + y^{(0)} = A_e \, , \end{aligned}\right\} \tag{A6}$$

$$\left.\begin{aligned} a_{1\infty} x_{A_e}^{(1)} &= a_{2\infty} y_{A_e}^{(1)} + \left(a_2^{(1)} + u^{(1)}\right) y_{A_e}^{(0)} \, , \\ a_{1\infty} x_{B_e}^{(1)} &= -a_{2\infty} y_{B_e}^{(1)} - \left(a_2^{(1)} - u^{(1)}\right) y_{B_e}^{(0)} \, , \end{aligned}\right\} \tag{A7}$$

where

$$a_2^{(1)} = \tfrac{1}{2} a_{2\infty}\left(\frac{p^{(1)}}{p_\infty} - \frac{\rho^{(1)}}{\rho_\infty}\right) \; . \tag{A8}$$

We can now use the results in (A6) and (A8), together with conditions (A5) to show that

$$\left.\begin{aligned} a\left[x - \epsilon D(B_e)\right] - y &= -B_e + \epsilon \frac{a}{4}(\gamma_2 + 1)D'(B_e)(A_e - B_e) , \\ a\left[x - \epsilon D(B_e)\right] + y &= A_e + \epsilon \frac{a}{4}(\gamma_2 + 1)\left[D(A_e) - D(B_e)\right] , \end{aligned}\right\} \qquad \text{(A9)}$$

and thus to complete the first-order equilibrium flow solution.

Equations (A9) show that, to $0(\epsilon)$,

$$\left(\frac{dy}{dx}\right)_{B_e} = a\left[1 - \epsilon\frac{a}{2}(\gamma_2 + 1)D'(B_e)\right] \qquad \text{(A10)}$$

We remark that the foregoing analysis represents a uniformly valid solution of the equilibrium flow problem, to first order, for the reasons given in the paper by Miss Fox (1955).

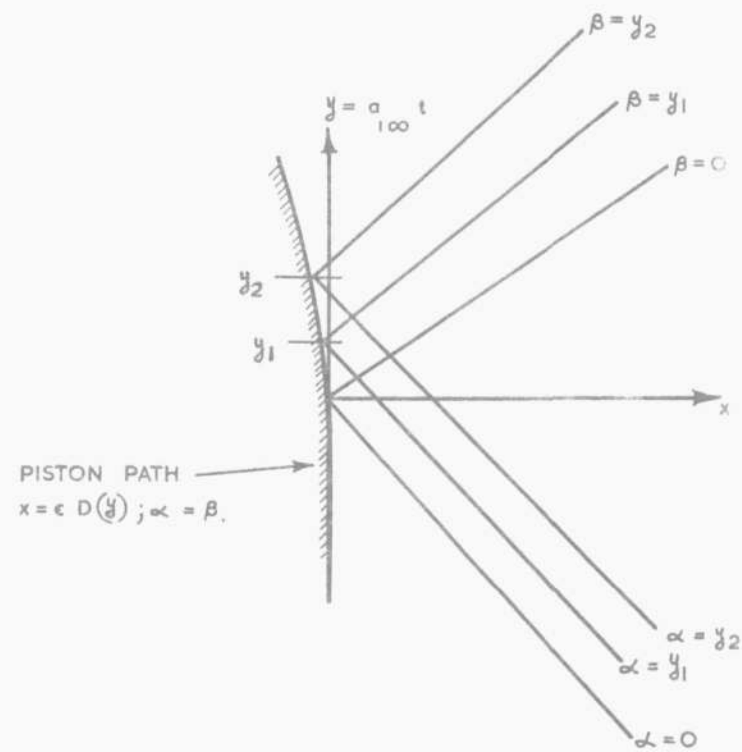

FIG. 1a. THE x – y – PLANE SHOWING PISTON PATH AND CHARACTERISTIC LINES

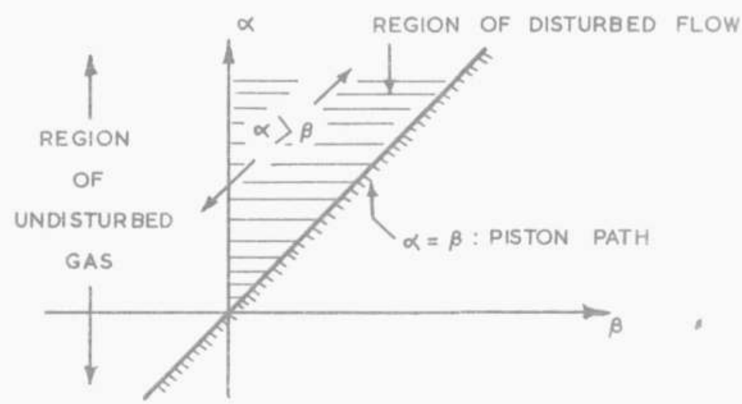

FIG. 1b. THE α – β OR CHARACTERISTIC, PLANE