PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 10, October 1996

ON A FIXED POINT PROBLEM OF REICH

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper, we give an affirmative answer to a fixed point problem of S. Reich.

1. INTRODUCTION

Let X be a metric space. $CB(X)$ stands for the set of all non-empty closed bounded subsets of X. $CB(X)$ is a metric space with the Hausdorff metric H. In [6], S. Reich presented the following

Problem. Let (X,d) be a complete metric space. Suppose that $F : X \to CB(X)$ satisfies $H(F(x),F(y)) \le K(d(x,y))d(x,y)$ for all x,y in X, $x \ne y$, where $K : (0,+\infty) \to [0,1)$ and $\lim_{r\to t^+} \sup K(r) < 1$ for all $0 < t < +\infty$. Does F have a fixed point?

In fact, this problem was raised by S. Reich in [4]. S. Reich [5] also gives an affirmative answer to this problem when Fx is non-empty compact for $x \in X$.

In this paper, we give an affirmative answer to this problem. We have the following results.

Theorem 1. *Let all the conditions of the above problem be satisfied. Then F has a fixed point if and only if there exists a closed subset $Y \subseteq X$, $Fx \cap Y \ne \varnothing$ for all $x \in Y$, such that for each closed subset $Z \subseteq Y$, if $Fx \cap Z \ne \varnothing$ for all $x \in Z$, then $d(x,F(x)\cap Z) = d(x,F(x)), \forall x \in Z$.*

Remark. When F is single valued, let $Y = X$; then for each subset $Z \subseteq X$, such that $Fx \in Z$ for all $x \in Z$, we must have $d(x,F(x)) = d(x,F(x)\cap Z)$.

Theorem 2. *Let (X,d) be a complete metric space. $F : X \to CB(X)$ satisfies the following conditions.*

(1) If $Y \subseteq X$ is a non-empty closed bounded subset, and $Fx \cap Y \ne \varnothing$ for all $x \in Y$, then $d(x,Fx) = d(x,Fx\cap Y)$ for all $x \in Y$;

(2) $H(F(x),F(y)) \le K(d(x,y))d(x,y)$, $\forall x,y \in X$, $x \ne y$, where $K : (0,+\infty) \to [0,1)$ and $\lim_{r\to t^+} \sup K(r) < 1$ for all $0 < t < +\infty$.

Then F has a fixed point.

Received by the editors September 14, 1994 and, in revised form, January 10, 1995 and March 27, 1995.

1991 *Mathematics Subject Classification.* Primary 47H06, 47H10; Secondary 54H25.

Key words and phrases. Fixed point, multivalued contraction, Hausdorff metric.

2. Proofs

We first prove Theorem 2, then we can prove Theorem 1 similarly to the proof of Theorem 2.

Proof of Theorem 2. Take $t_n \in (0, +\infty)$, $n = 1, 2, \dots$, and $t_1 > t_2 > \cdots > t_n \to 0$. Since $\lim_{r \to t_n^+} \sup K(r) < 1$, there exist $0 \le k_n < 1$ and $\delta_n > 0$, such that $K(r) \le k_n$, $\forall r \in (t_n, t_n + \delta_n)$, $n = 1, 2, \dots$.

Let $\eta_n = \min\{\frac{\delta_n}{4}, \frac{1}{n}\}$, $\varepsilon_n = t_n + \eta_n$; then $K(r) \le k_n$, $\forall r \in [\varepsilon_n - \frac{\eta_n}{2}, \varepsilon_n + \eta_n]$, $n = 1, 2, \dots$. It is easy to see $\varepsilon_n \to 0^+$ as $n \to \infty$.

Step 1. For $\varepsilon_1 > 0$, we prove there exists $x_1 \in X$, such that

$$Fx \cap B_1 \neq \varnothing, \quad \forall x \in B_1 = \{x | d(x, x_1) \le \varepsilon_1\}. \tag{2.1}$$

Suppose (2.1) is not true. Then for each $x \in X$, there exists $x_0 \in X$, $d(x, x_0) \le \varepsilon_1$, but

$$d(x, y) > \varepsilon_1, \quad \forall y \in Fx_0.$$

Case (a). If $d(x, x_0) < \varepsilon_1 - \frac{\eta_1}{2}$, then

$$\begin{aligned} d(x, Fx) &\ge d(x, Fx_0) - H(Fx_0, Fx) \ge \varepsilon_1 - K(d(x_0, x))d(x_0, x) \\ &> \varepsilon_1 - d(x, x_0) > \frac{\eta_1}{2}. \end{aligned}$$

Case (b). If $d(x, x_0) \ge \varepsilon_1 - \frac{\eta_1}{2}$, then

$$\begin{aligned} d(x, Fx) &\ge d(x, Fx_0) - H(Fx_0, Fx) \ge \varepsilon_1 - K(d(x_0, x))d(x_0, x) \\ &\ge \varepsilon_1 - k_1\varepsilon_1 = (1 - k_1)\varepsilon_1. \end{aligned}$$

From Cases (a) and (b), we have

$$d(x, Fx) \ge \min\left\{\frac{\eta_1}{2}, (1 - k_1)\varepsilon_1\right\} > 0, \quad \forall x \in X. \tag{2.2}$$

Now, fix $x_0 \in X$ and $x_1 \in Fx_0$. Since

$$d(x_1, Fx_1) \le H(Fx_0, Fx_1) \le K(d(x_0, x_1))d(x_0, x_1) < d(x_0, x_1),$$

there exists $x_2 \in Fx_1$, such that

$$d(x_1, Fx_1) - \frac{1}{2^2} \le d(x_1, x_2) \le d(x_0, x_1)$$

and

$$d(x_1, x_2) \le K(d(x_0, x_1))d(x_0, x_1) + \frac{1}{2^2}.$$

By induction, we get $x_n \in Fx_{n-1}$, $n \ge 3$, such that

$$d(x_{n-1}, Fx_{n-1}) - \frac{1}{2^n} \le d(x_{n-1}, x_n) \le d(x_{n-2}, x_{n-1})$$

and

$$d(x_{n-1}, x_n) \le K(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n}.$$

By the construction of $\{x_n\}$, we know $\lim_{n\to\infty} d(x_{n-1}, x_n) = S_0$ exists.

Suppose $S_0 > 0$. Then

$$\lim_{n\to\infty} d(x_{n-1}, x_n) \le \overline{\lim_{n\to\infty}} \left[K(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n}\right]$$
$$\le \overline{\lim_{r\to S_0^+}} K(r) \lim_{n\to\infty} d(x_{n-2}, x_{n-1}).$$

So $S_0 \le \overline{\lim}_{r\to S_0^+} K(r)S_0 < S_0$, a contradiction.

Hence we have $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$. This implies that

$$\lim_{n\to\infty} d(x_{n-1}, Fx_{n-1}) \le \lim_{n\to\infty} \left[d(x_{n-1}, x_n) + \frac{1}{2^n}\right] = 0,$$

a contradiction to (2.2). So (2.1) is true.

Step 2. For $\varepsilon_2 > 0$, we prove there exists $x_2 \in B_1$, such that

(2.3) $$Fx \cap B_2 \ne \varnothing, \quad \forall x \in B_2 = \{x \in B_1 | d(x, x_2) \le \varepsilon_2\}.$$

Suppose (2.3) is not true. For each $x \in B_1$, there exists $y_0 \in B_1$, such that $d(x, y_0) \le \varepsilon_2$, but

$$d(x, y) > \varepsilon_2, \quad \forall y \in Fy_0.$$

With the same argument of Cases (a) and (b) in Step 1, we get

(2.4) $$d(x, Fx) \ge \min\left\{\frac{\eta_2}{2}, (1 - k_2)\varepsilon_2\right\}, \quad \forall x \in B_1.$$

Now, fix $x_0 \in B_1$, $x_1 \in Fx_0 \cap B_1$. By assumption (1),

$$d(x_1, Fx_1 \cap B_1) = d(x_1, Fx_1) \le H(Fx_0, Fx_1) \le K(d(x_0, x_1))d(x_0, x_1).$$

So there exists $x_2 \in Fx_1 \cap B_1$, such that

$$d(x_1, Fx_1) - \frac{1}{2^2} \le d(x_1, x_2) \le d(x_0, x_1)$$

and

$$d(x_1, x_2) \le K(d(x_0, x_1))d(x_0, x_1) + 1/2^2.$$

Generally, we get $x_n \in Fx_{n-1} \cap B_1$, $n \ge 3$, such that

$$d(x_{n-1}, Fx_{n-1}) - \frac{1}{2^n} \le d(x_{n-1}, x_n) \le d(x_{n-2}, x_{n-1})$$

and

$$d(x_{n-1}, x_n) \le K(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) + \frac{1}{2^n}.$$

So $\lim_{n\to\infty} d(x_{n-1}, x_n)$ exists and equals zero.

We get $\lim_{n\to\infty} d(x_{n-1}, Fx_{n-1}) = 0$, a contradiction to (2.4). So (2.3) is true.

Step 3. By induction, we get $x_{n+1} \in B_n$, such that

(2.5) $$Fx \cap B_{n+1} \ne \varnothing, \quad \forall x \in B_{n+1} = \{x \in B_n | d(x, x_{n+1}) \le \varepsilon_{n+1}\},\ n \ge 2.$$

It is obvious that $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \supseteq B_n \supseteq \dots$, and

$$\lim_{n\to\infty} \operatorname{diam}(B_n) = 0.$$

So there exists only one point $x \in \bigcap_{n\ge1} B_n$, and $x \in Fx$. This completes the proof.

Proof of Theorem 1. Necessity: If F has a fixed point, let $Y = \{x \in X | x \in Fx\}$. Then $Y \neq \varnothing$ is closed and it is the desired subset.

Sufficiency: Suppose $Y \subseteq X$ is non-empty closed, $Fx \cap Y \neq \varnothing$, $\forall x \in Y$, and for each closed subset $Z \subseteq Y$, if $Fx \cap Z \neq \varnothing$, $\forall x \in Z$, then $d(x, F(x)) = d(x, Fx \cap Z)$, $\forall x \in Z$.

Let $\{\varepsilon_n\}$ be as in the proof of Theorem 2.

Step 1. Take $B_1 = Y$; then $d(x, F(x)) = d(x, F(x) \cap B_1)$, $\forall x \in B_1$.

Step 2. With the same argument of Step 2 in the proof of Theorem 2, we get $x_2 \in B_1$, such that $Fx \cap B_2 \neq \varnothing$, $\forall x \in B_2 = \{x \in B_1 | d(x, x_2) \leq \varepsilon_2\}$.

By induction, we get $x_{n+1} \in B_n$, such that

$$Fx \cap B_{n+1} \neq \varnothing, \quad \forall x \in B_{n+1} = \{x \in B_n | d(x, x_{n+1}) \leq \varepsilon_{n+1}\},\ n \geq 2.$$

So $\bigcap_{n \geq 1} B_n$ has only one point; it is the fixed point of F.

Acknowledgement

I am grateful to the referee for his (or her) valuable suggestions.

References

1. T. Hu, *Fixed point theorems for multi-valued mappings*, Canad. Math. Bull. **23** (1980), 193–197. MR **82d:**54052
2. H. M. Ko and Y. H. Tsai, *Fixed point theorems with a localized property*, Tamkang. Ja. Math. **8** (1977), 81–85. MR **57:**7566
3. S. B. Nadler, *Some results on multi-valued contraction mappings*, Lect. Notes. Math., vol. 171, Springer-Verlag, 1970, pp. 64–69. MR **43:**1148
4. S. Reich, *Some fixed point problems*, Atti Acad. Naz. Lincei. **57** (1974), 194–198. MR **53:**1346
5. S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital. **5** (1972), 26–42. MR **46:**8206
6. S. Reich, *Some problems and results in fixed point theory*, Contemporary Math. **21** (1983), 179–187. MR **85e:**47082
7. S. Reich, *A fixed point theorem for locally contractive multi-valued functions*, Rev. Roumaine. Math. Pures. Appl. **17** (1972), 569–572. MR **47:**7721

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