TOKYO J. MATH.
VOL. 25, NO. 2, 2002

On a Fluctuation Identity for Multidimensional Lévy Processes

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1. Introduction.

Fluctuation identities for one-dimensional Lévy processes, often called Wiener Hopf factorizations, were investigated by many authors (e.g. Baxter and Donsker [1], Rogozin [9], Gusak and Korolyuk [5], Pecherskii and Rogozin [8], Borovkov [3], Greenwood and Pitman [4], Skorohod [11, §**4.3** and **4.4**], Sato [10, Chapter 9], Bertoin [2, Chapter VI]). There are several types of identities, for instance, those concerning supremum processes mainly and including ladder processes too. We are particularly interested in those given by Pecherskii and Rogozin [8], involving supremum processes, and developed in Sato [10, Chapter 9]. Our aim is to give some extension of their results to multidimensional Lévy processes. Our problem might be discussed from the view-point of Millar's general results ([6] [7]) on the decomposition of Markov processes at splitting times but detailed computations would be needed to arrive at our result. In this paper, we employ an elementary method starting from random walks; it may be a straightforward extension of the method developed in Sato[10, pp. 333–345] but we emphasize that there is a crucial point concerning a careful definition of $X''(H^-(t)*)$ and $X''(H^+(t)*)$ (see §**2** and Lemma 5.4). Another emphasis is that a relevant choice of an approximating compound Poisson process much simplifies the argument of deriving the result for general Lévy processes from that for compound Poisson processes (see (5.2) and Lemma 5.4; compare it with the arguments of [11, pp. 207–213] and [10, pp. 342–345]). The method of approximation is, in the sense of analysis, the same as the well-known method, often called Yosida's approximation, in the theory of semigroups of linear operators which makes use of a bounded operator $\varepsilon^{-1}(\varepsilon^{-1}I - A)^{-1}A$ to approximate an unbounded infinitesimal generator A ([12]).

2. The main result.

Given a Lévy process $X(t)$ taking values on $\mathbf{R}^d$, $d \geq 2$, with $X(0) = 0$, we denote by $X'(t)$ and $X''(t)$ the first and the other components of $X(t)$, respectively, and so $X(t)$ can be

Received December 10, 2001
2000 Mathematics Subject Classification. 60G51.
Keywords and phrases. Multidimensional Lévy process, compound Poisson prosess, random walk, fluctuation identity.

expressed as

$$X(t) = (X'(t), X''(t))$$

where $X'(t)$ is a one-dimensional Lévy process and $X''(t)$ is a $(d-1)$-dimensional Lévy process. The sample paths of these Lévy processes are assumed to be right continuous and have left limits. We write $a \vee b$ (resp. $a \wedge b$) for the maximum (resp. minimum) of a and b and set

$$\begin{aligned}
&M'(t) = \sup\{X'(s) : 0 \le s \le t\}, \\
&H^-(t) = \inf\{s \in [0,t] : X'(s-) \vee X'(s) = M'(t)\}, \\
&H^+(t) = \sup\{s \in [0,t] : X'(s-) \vee X'(s) = M'(t)\}, \\
&X''(H^-(t)*) = \begin{cases} X''(H^-(t)) & \text{if } X'(H^-(t)) = M'(t), \\ X''(H^-(t)-) & \text{if } X'(H^-(t)-) = M'(t), \end{cases} \\
&X''(H^+(t)*) : \text{defined similarly}, \\
&M^-(t) = (M'(t), X''(H^-(t)*)), \\
&M^+(t) = (M'(t), X''(H^+(t)*)).
\end{aligned}$$

In order that $X''(H^-(t)*)$ is well defined, $X''(\cdot)$ should be continuous at $H^-(t)$ if $X'(H^-(t)-) = X'(H^-(t)) = M'(t)$, a.s. This will be proved in Lemma 5.3.

In this paper u, v and w are always expressed as

$$u = (\xi + i\xi',\ i\xi''), \quad v = (\eta + i\eta',\ i\eta''), \quad w = (\zeta + i\zeta',\ i\zeta'')$$

with $\xi, \xi', \eta, \eta', \zeta, \zeta' \in \mathbf{R}$ and $\xi'', \eta'', \zeta'' \in \mathbf{R}^{d-1}$, $(i = \sqrt{-1})$. ξ, η and ζ are often denoted by $\Re_1 u$, $\Re_1 v$ and $\Re_1 w$, respectively. The notations $u \cdot M$ and $\langle \xi'', X'' \rangle$ stand for the inner products in $\mathbf{C}^d$ and in $\mathbf{R}^{d-1}$, respectively.

Our theorem is then stated as follows.

THEOREM 1. *Let $X(t) = (X'(t), X''(t))$ be a d-dimensional Lévy process with $X(0) = 0$ $(d \ge 2)$, and set*

$$\begin{aligned}
A(t) &= u \cdot M^-(t) + w \cdot (M^+(t) - M^-(t)) \\
&\quad + v \cdot (X(t) - M^+(t)) - \alpha H^-(t) - \beta H^+(t),
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
B(t) &= \mathrm{e}^{-(\lambda+\alpha+\beta)t} E\{\mathrm{e}^{u \cdot X(t)} - 1;\ X'(t) > 0\} \\
&\quad + \mathrm{e}^{-(\lambda+\beta)t} E\{\mathrm{e}^{i\langle \zeta'', X''(t)\rangle} - 1;\ X'(t) = 0\} \\
&\quad + \mathrm{e}^{-\lambda t} E\{\mathrm{e}^{v \cdot X(t)} - 1;\ X'(t) < 0\} \\
&\quad + \{\mathrm{e}^{-(\lambda+\alpha+\beta)t} - \mathrm{e}^{-\lambda t}\} P\{X'(t) > 0\} \\
&\quad + \{\mathrm{e}^{-(\lambda+\beta)t} - \mathrm{e}^{-\lambda t}\} P\{X'(t) = 0\}.
\end{aligned} \tag{2.2}$$

Then we have

$$\int_0^\infty e^{-\lambda t} E\{e^{A(t)}\}dt = \frac{1}{\lambda}\exp\left\{\int_0^\infty B(t)\frac{dt}{t}\right\}. \tag{2.3}$$

for any $\lambda,\ \alpha,\ \beta,\ u,\ v$ *and* w *satisfying the following condition*:

$$\begin{cases} \lambda > 0,\ \alpha \geq 0,\ \beta \geq 0 \quad \text{and} \quad u = (\xi + i\xi', i\xi''), \\ v = (\eta + i\eta', i\eta''),\ w = (\zeta + i\zeta', i\zeta'') \quad \text{with} \\ \xi \leq 0,\ \eta \geq 0,\ \xi', \eta', \zeta, \zeta' \in \mathbf{R} \quad \text{and} \quad \xi'', \eta'', \zeta'' \in \mathbf{R}^{d-1}. \end{cases} \tag{2.4}$$

The identity (2.3) can be rewritten in a slightly better form. We set

$$\begin{aligned} \tilde{A}(t) &= u \cdot M^-(t) - \alpha H^-(t) \\ &\quad + w \cdot (M^+(t) - M^-(t)) - \beta(H^+(t) - H^-(t)) \\ &\quad + v \cdot (X(t) - M^+(t)) - \gamma(t - H^+(t)), \\ \tilde{B}(t) &= e^{-(\lambda+\alpha)t} E\{e^{u \cdot X(t)} - 1;\ X'(t) > 0\} \\ &\quad + \{e^{-(\lambda+\alpha)t} - e^{-\lambda t}\} P\{X'(t) > 0\} \\ &\quad + e^{-(\lambda+\beta)t} E\{e^{i\langle \zeta'', X''(t)\rangle} - 1;\ X'(t) = 0\} \\ &\quad + \{e^{-(\lambda+\beta)t} - e^{-\lambda t}\} P\{X'(t) = 0\} \\ &\quad + e^{-(\lambda+\gamma)t} E\{e^{v \cdot X(t)} - 1;\ X'(t) < 0\} \\ &\quad + \{e^{-(\lambda+\gamma)t} - e^{-\lambda t}\} P\{X'(t) < 0\}, \end{aligned}$$

and introduce an exponential random time T with mean $1/\lambda$ and independent of $\{X(t)\}$.

COROLLARY. (i) *For any* $\gamma \geq 0$ *and any* $\lambda, \alpha, \beta, u, v, w$ *satisfying the condition* (2.4) *we have*

$$E\{e^{\tilde{A}(T)}\} = \exp\left\{\int_0^\infty \tilde{B}(t)\frac{dt}{t}\right\}, \tag{2.5}$$

(ii) *The three random vectors*

$$\begin{aligned} U &= (M^-(T), H^-(T)) \\ V &= (M^+(T) - M^-(T), H^+(T) - H^-(T)) \\ W &= (X(T) - M^+(T), T - H^+(T)) \end{aligned}$$

are independent.

3. The case of random walks in $\mathbf{R}^d$.

In this section we prove a theorem for random walks in $\mathbf{R}^d$ which is analogous to Theorem 1.

Let $\{S_n,\ n \geq 0\}$ be a random walk in $\mathbf{R}^d,\ d \geq 2$. It is expressed as $S_n = \sum_{k=1}^n X_k,\ S_0 = 0$, where X_k's are *i.i.d.* random variables in $\mathbf{R}^d$. As in §**2** we use the expression $S_n = (S_n',\ S_n'')$ where $S_n' = \sum_{k=1}^n X_k'$ and $S_n'' = \sum_{k=1}^n X_k''$ are random walks in $\mathbf{R}$ and in $\mathbf{R}^{d-1}$, respectively. S_n is often written as $S(n)$, in particular, when n is replaced by certain random variables such as $H_n^-,\ H_n^+$, etc. We set

$$\begin{aligned} M_n' &= \max\{S_k' :\ 0 \leq k \leq n\}\,, \\ H_n^- &= \min\{\ell :\ 0 \leq \ell \leq n,\ S_\ell' = M_n'\}\,, \\ H_n^+ &= \max\{\ell :\ 0 \leq \ell \leq n,\ S_\ell' = M_n'\}\,, \\ M_n^- &= S(H_n^-) = (S'(H_n^-),\ S''(H_n^-)) = (M_n',\ S''(H_n^-))\,, \\ M_n^+ &= S(H_n^+) = (S'(H_n^+),\ S''(H_n^+)) = (M_n',\ S''(H_n^+))\,. \end{aligned}$$

The following is a random walk analogue of Theorem 1.

THEOREM 2. *Let* $u = (\xi + i\xi',\ i\xi''),\ v = (\eta + i\eta',\ i\eta''),\ w = (\zeta + i\zeta',\ i\zeta'')$ *with* $\Re_1 u = \xi \leq 0,\ \Re_1 v = \eta \geq 0$, *and set*

$$A_n = u \cdot M_n^- + w \cdot (M_n^+ - M_n^-) + v \cdot (S_n - M_n^+), \tag{3.1}$$

$$\begin{aligned} B_n = {} & (sr\rho)^n E\{e^{u \cdot S_n} - 1;\ S_n' > 0\} \\ & + (s\rho)^n E\{e^{i\langle \zeta'', S_n''\rangle} - 1;\ S_n' = 0\} \\ & + s^n E\{e^{v \cdot S_n} - 1;\ S_n' < 0\} \\ & + \{(sr\rho)^n - s^n\} P\{S_n' > 0\} \\ & + \{(s\rho)^n - s^n\} P\{S_n' = 0\}\,. \end{aligned} \tag{3.2}$$

Then for any $s,\ r$ *and* ρ *satisfying* $|s| < 1,\ |r| \leq 1$ *and* $|\rho| \leq 1$ *we have*

$$\sum_{n=0}^{\infty} s^n E\{e^{A_n} r^{H_n^-} \rho^{H_n^+}\} = (1-s)^{-1} \exp\left(\sum_{n=1}^{\infty} \frac{B_n}{n}\right). \tag{3.3}$$

Evidently both sides do not depend on ζ *and* ζ'.

For proving the theorem we employ the same method as in Lemma 45.6 of Sato [10, p. 336]. We set

$$\begin{aligned} T_+ &= \min\{n \geq 1 :\ S_n' > 0\}\,, \quad T_{+0} = \min\{n \geq 1 :\ S_n' \geq 0\}\,, \\ T_- &= \min\{n \geq 1 :\ S_n' < 0\}\,, \quad T_{-0} = \min\{n \geq 1 :\ S_n' \leq 0\}\,, \end{aligned}$$

$$f_s(u) = \sum_{n=0}^{\infty} s^n E\{e^{u\cdot S_n};\ T_{-0} > n\}\,,$$

$$g_s(v) = \sum_{n=0}^{\infty} s^n E\{e^{v\cdot S_n};\ T_{+0} > n\}\,,$$

$$h_s(w) = \sum_{n=0}^{\infty} s^n E\{e^{w\cdot S_n};\ T_{+} > n,\ S'_n = 0\}\,,$$

$$\gamma_s^+(u) = \exp\sum_{n=1}^{\infty} \frac{s^n}{n} E\{e^{u\cdot S_n};\ S'_n > 0\}\,,$$

$$\gamma_s^-(v) = \exp\sum_{n=1}^{\infty} \frac{s^n}{n} E\{e^{v\cdot S_n};\ S'_n < 0\}\,,$$

$$c_s(\zeta'') = \exp\sum_{n=1}^{\infty} \frac{s^n}{n} E\{e^{i\langle\zeta'',\ S''_n\rangle};\ S'_n = 0\}\,.$$

LEMMA 3.1. *For* $|s| < 1,\ |r| \le 1,\ |\rho| \le 1,\ \Re_1 u = \xi \le 0$ *and* $\Re_1 v = \eta \ge 0$, *we have*

$$f_{sr\rho}(u)h_{s\rho}(w)g_s(v) = \sum_{n=0}^{\infty} s^n E\{e^{A_n} r^{H_n^-} \rho^{H_n^+}\}\,. \tag{3.4}$$

PROOF. Let $0 \le m \le \ell$. Then

$$\begin{aligned} &E\{e^{w\cdot S_{\ell-m}};\ T_+ > \ell - m,\ S'_{\ell-m} = 0\} \\ &\quad = E\{e^{w\cdot S_{\ell-m}};\ S'_k \le 0\ \ (0 \le \forall k \le \ell - m),\ S'_{\ell-m} = 0\}\,. \end{aligned} \tag{3.5}$$

Since $(X_1,\ X_2,\ \cdots,X_{\ell-m})$ is identical in law to $(X_{m+1}, X_{m+2},\cdots,X_\ell)$, the joint distribution of $S_{\ell-m}, S'_k(1 \le k \le \ell-m),\ S'_{\ell-m}$ is the same as that of $S_\ell - S_m,\ S'_{m+k} - S'_m(1 \le k \le \ell - m),\ S'_\ell - S'_m$. Therefore (3.5) is equal to

$$\begin{aligned} &E\{e^{w\cdot(S_\ell - S_m)};\ S'_{m+k} - S'_m \le 0\ \ (0 \le \forall k \le \ell - m),\ S'_\ell - S'_m = 0\} \\ &\quad = E\{e^{w\cdot(S_\ell - S_m)};\ S'_m \ge S'_k\ \ (m \le \forall k \le \ell),\ S'_m = S'_\ell\}\,. \end{aligned}$$

Similarly, by using the law identity of $(X_1, X_2,\cdots,X_{n-\ell})$ and $(X_{\ell+1}, X_{\ell+2},\cdots,X_n)$ we have

$$\begin{aligned} &E\{e^{v\cdot S_{n-\ell}};\ T_{+0} > n-\ell\} = E\{e^{v\cdot S_{n-\ell}};\ S'_k < 0\ \ (1 \le \forall k \le n-\ell)\} \\ &\quad = E\{e^{v\cdot(S_n - S_\ell)};\ S'_\ell > S'_k\ \ (\ell + 1 \le \forall k \le n)\}\,. \end{aligned}$$

Moreover again by using the law identity of $(X_1, X_2, \cdots, X_m)$ and $(X_m, X_{m-1}, \cdots, X_1)$ we have

$$E\{e^{u\cdot S_m};\ T_{-0} > m\} = E\{e^{u\cdot S_m};\ S'_m > S'_j\ \ (0 \le j \le m-1)\}\,.$$

Therefore, for $|s| < 1,\ |\gamma| \le 1,\ |\rho| \le 1,\ \Re_1 u \le 0,\ \Re_1 v \ge 0$ and $w = (\zeta + i\zeta',\ i\zeta'')$ we have

$$\begin{aligned}
f_{sr\rho}(u)h_{s\rho}(w)g_s(v) &= \sum_{m=0}^{\infty}(sr\rho)^m E\{e^{u\cdot S_m};\ T_{-0} > m\} \\
&\quad \cdot \sum_{\ell=m}^{\infty}(s\rho)^{\ell-m}E\{e^{w\cdot S_{\ell-m}};\ T_+ > \ell - m,\ S'_{\ell-m} = 0\} \\
&\quad \cdot \sum_{n=\ell}^{\infty} s^{n-\ell}E\{e^{v\cdot S_{n-\ell}};\ T_{+0} > n-\ell\} \\
&= \sum_{0\le m\le \ell\le n<\infty} s^n r^m \rho^\ell E\{e^{u\cdot S_m};\ S'_m > S'_j\ \ (0 \le \forall j \le m-1)\} \\
&\quad \cdot E\{e^{w\cdot(S_\ell - S_m)};\ S'_m \ge S'_k\ \ (m \le \forall k \le \ell),\ S'_m = S'_\ell\} \\
&\quad \cdot E\{e^{v\cdot(S_n - S_\ell)};\ S'_\ell > S'_{k'}\ \ (\ell + 1 \le \forall k' \le n)\} \\
&= \sum_{0\le m\le \ell\le n<\infty} s^n r^m \rho^\ell E\{e^{u\cdot S_m + w\cdot(S_\ell - S_m) + v\cdot(S_n - S_\ell)}; \\
&\qquad S'_m > S'_j\ \ (0 \le \forall j \le m-1),\ S'_m \ge S'_k\ \ (m \le \forall k \le \ell), \\
&\qquad\qquad S'_m = S'_\ell,\ S'_\ell > S'_{k'}\ \ (\ell + 1 \le \forall k' \le n)\} \\
&= \sum_{0\le m\le \ell\le n<\infty} s^n r^m \rho^\ell E\{e^{u\cdot S_m + w\cdot(S_\ell - S_m) + v\cdot(S_n - S_\ell)};\ H_n^- = m,\ H_n^+ = \ell\} \\
&= \sum_{n=0}^{\infty} s^n E\{e^{A_n} r^{H_n^-}\rho^{H_n^+}\}\,.
\end{aligned}$$

This proves the lemma.

LEMMA 3.2. *For* $|s| < 1$

$$f_s(u) = \gamma_s^+(u) \quad on\ \{\Re_1 u \le 0\}\,, \tag{3.6}$$

$$g_s(v) = \gamma_s^-(v) \quad on\ \{\Re_1 v \ge 0\}\,, \tag{3.7}$$

$$h_s(w) = c_s(\zeta'') \quad for\ w = (\zeta + i\zeta',\ i\zeta'')\,. \tag{3.8}$$

PROOF. Fixing s and ξ'' ($|s| < 1,\ \xi'' \in \mathbf{R}^{d-1}$) we regard $f_s(u), g_s(u), h_s(u), \gamma_s^+(u)$ and $\gamma_s^-(u)$ as functions of $z = \xi + i\xi'$ alone and denote them by $f_{s,\xi''}(z)$, $g_{s,\xi''}(z)$, $h_{s,\xi''}(z)$, $\gamma^+_{s,\xi''}(z)$ and $\gamma^-_{s,\xi''}(z)$, respectively. Now suppose $\Re_1 u = \xi = 0$ for the time being. Then by putting $v = w = u$ and $r = \rho = 1$ in (3.4) we have

$$f_s(u)h_s(u)g_s(u) = \sum_{n=0}^{\infty} s^n E\{e^{u\cdot S_n}\} = \{1 - sE(e^{u\cdot S_1})\}^{-1}$$

$$= \exp[-\log\{1 - sE(e^{u\cdot S_1})\}] = \exp\sum_{n=1}^{\infty}\frac{s^n}{n}E(e^{u\cdot S_n})$$

$$= c_s(\xi'')\gamma_s^+(u)\gamma_s^-(u)\,,$$

or equivalently

$$f_{s,\xi''}(z)h_{s,\xi''}(z)g_{s,\xi''}(z) = c_s(\xi'')\gamma^+_{s,\xi''}(z)\gamma^-_{s,\xi''}(z) \quad \text{on } \{\Re z = 0\}\,, \tag{3.9}$$

and hence

$$\frac{f_{s,\xi''}(z)}{\gamma^+_{s,\xi''}(z)} = \frac{c_s(\xi'')\gamma^-_{s.\xi}(z)}{h_{s,\xi''}(z)g_{s,\xi''}(z)} \quad \text{on } \{\Re z = 0\}\,. \tag{3.10}$$

Moreover, for fixed s and ξ'' ($|s| < 1/2,\ \xi'' \in \mathbf{R}^{d-1}$) the left hand side of (3.10) is bounded and continuous on $\{\Re z \le 0\}$ and holomorphic in $\{\Re z < 0\}$; the right hand side of (3.10) is bounded and continuous on $\{\Re z \ge 0\}$ and holomorphic in $\{\Re z > 0\}$. Therefore the function

$$\varphi(z) = \begin{cases} \text{the left hand side of (3.10) on } \{\Re z \le 0\}\,, \\ \text{the right hand side of (3.10) in } \{\Re z > 0\}\,, \end{cases} \tag{3.11}$$

is holomorphic and bounded in the whole $\mathbf{C}$ and hence equal identically some constant c. The constant c must be 1 because we can easily prove that $f_{s,\xi''}(z) \to 1$ and $\gamma^+_{s,\xi''}(z) \to 1$ as $z = \xi$ (so z is on the real axis) $\to -\infty$. Therefore $\varphi(z) \equiv 1$ and this implies $f_{s,\xi''}(z) = \gamma^+_{s,\xi''}(z)$ on $\{\Re z \le 0\}$ and $h_{s,\xi''}(z)g_{s,\xi''}(z) = c_s(\xi'')\gamma^-_{s,\xi''}(z)$ on $\{\Re z \ge 0\}$, or what is the same thing,

$$f_s(u) = \gamma_s^+(u) \quad \text{for } |s| < 1/2 \text{ and } \Re_1 u \le 0\,, \tag{3.12}$$

$$h_s(v)g_s(v) = c_s(\eta'')\gamma_s^-(v) \quad \text{for } |s| < 1/2 \text{ and } \Re_1 v \ge 0\,. \tag{3.13}$$

Since all the functions in (3.12) and (3.13) are holomorphic functions of s in $\{|s| < 1\}$, (3.12) and (3.13) hold for $|s| < 1$. Finally to prove (3.7) and (3.8) we notice that (3.13) yields

$$\frac{g_s(v)}{\gamma_s^-(v)} = \frac{c_s(\eta'')}{h_s(v)} \text{ depends only on } s \text{ and } \eta'' \tag{3.14}$$

$$(v = (\eta + i\eta', i\eta'') \text{ with } \eta \ge 0)\,,$$

and it is easy to see that the left hand side of (3.14) tends to 1 as $\eta \to \infty$. Therefore the common value of (3.14) must be equal to 1 identically, so (3.7) and (3.8) hold. The proof of lemma is finished.

The proof of Theorem 2 is completed as follows. By Lemma 3.1 and Lemma 3.2 we have

$$\sum_{n=0}^{\infty} s^n E\{e^{A_n} r^{H_n^-} \rho^{H_n^+}\} = f_{sr\rho}(u) h_{s\rho}(w) g_s(v) = \gamma_{sr\rho}^+(u) c_{s\rho}(\zeta'') \gamma_s^-(v)\,,$$

and it is not hard to verify that the last term equals the right hand side of (3.3).

4. The case of compound Poisson processes.

When $X(t)$ is a compound Poisson process, Theorem 1 can be derived from Theorem 2 in the same way as in Theorem 45.5 of Sato [10, p. 335].

A compound Poisson process $\{X(t)\}$ can be expressed as

$$X(t) = S_{\pi(t)}, \quad t \geq 0\,, \tag{4.1}$$

where $\{S_n\}$ is a random walk in $\mathbf{R}^d$ with $S_0 = 0$ and $\{\pi(t)\}$ is a Poisson process with intensity c; it is assumed that $\{S_n\}$ and $\{\pi(t)\}$ are independent. Let $0 < \tau_1 < \tau_2 < \cdots$ be the jumping times of $\{\pi(t)\}$ and let $\tau_0 = 0$. Then $\tau_n - \tau_{n-1},\ n \geq 1$, are exponential random variables with mean $1/c$. Let $A(t)$ and A_n be defined by (2.1) and (3.1), respectively. We have the following relations. If $\tau_n < t < \tau_{n+1}$, then

$$\pi(t) = n,\ X(t) = S_n,\ M'(t) = M_n'\,, \tag{4.2}$$

$$H^-(t) = \tau_{H_n^-},\ H^+(t) = \begin{cases} \tau_{H_n^+ + 1} & \text{if } H_n^+ < n\,, \\ t & \text{if } H_n^+ = n\,, \end{cases} \tag{4.3}$$

$$M^-(t) = M_n^-\,, \quad M^+(t) = M_n^+\,, \quad A(t) = A_n - \alpha H^-(t) - \beta H^+(t)\,. \tag{4.4}$$

Therefore, for any $\lambda > 0,\ \alpha \geq 0,\ \beta \geq 0$ and for any $u,\ v,\ w$ satisfying $\Re_1 u = \xi \leq 0,\ \Re_1 v = \eta \geq 0$ we have

$$\begin{aligned} &\int_0^\infty E\{e^{-\lambda t + A(t)}\} dt \\ &= \sum_{n=0}^{\infty} \int_0^\infty E\{e^{-\lambda t + A(t)};\ \tau_n < t < \tau_{n+1}\} dt \\ &= \sum_{n=0}^{\infty} \int_0^\infty E\{e^{-\lambda t + A_n - \alpha H^-(t) - \beta H^+(t)};\ \tau_n < t < \tau_{n+1}\} dt\,, \end{aligned} \tag{4.5}$$

$$
\begin{aligned}
(4.6)\qquad & E\{\mathrm{e}^{-\lambda t+A_n-\alpha H^-(t)-\beta H^+(t)};\ \tau_n<t<\tau_{n+1}\} \\
&= E\{\mathrm{e}^{-\lambda t+A_n-A'_n};\ \tau_n<t<\tau_{n+1},\ H_n^+<n\} \\
&\quad + E\{\mathrm{e}^{-\lambda t+A_n-A''_n(t)};\ \tau_n<t<\tau_{n+1},\ H_n^+=n\} \\
&\text{(with } A'_n=\alpha\tau_{H_n^-}+\beta\tau_{H_n^+ +1} \text{ and } A''_n(t)=\alpha\tau_{H_n^-}+\beta t) \\
&= \sum_{0\le m\le \ell\le n-1} E\left\{e^{-\lambda t+A(m,\ell,n)-\alpha\tau_m-\beta\tau_{\ell+1}};\ \begin{matrix}\tau_n<t<\tau_{n+1}\\ H_n^-=m,\ H_n^+=\ell\end{matrix}\right\} \\
&\quad + \sum_{0\le m\le n} E\left\{e^{-\lambda t+A(m,n,n)-\alpha\tau_m-\beta t};\ \begin{matrix}\tau_n<t<\tau_{n+1}\\ H_n^-=m,\ H_n^+=n\end{matrix}\right\} \\
&\text{(with } A(m,\ell,n)=u\cdot S_m+w\cdot(S_\ell-S_m)+v\cdot(S_n-S_\ell)) \\
&= \sum_{0\le m\le \ell\le n} E\{\mathrm{e}^{A(m,\ell,n)};\ H_n^-=m,\ H_n^+=\ell\}E(t;m,\ell,n)\,,
\end{aligned}
$$

where

$$
E(t;m,\ell,n)=\begin{cases} E\{\mathrm{e}^{-\lambda t-\alpha\tau_m-\beta\tau_{\ell+1}};\ \tau_n<t<\tau_{n+1}\} & \text{for } \ell\le n-1\,,\\ E\{\mathrm{e}^{-(\lambda+\beta)t-\alpha\tau_m};\ \tau_n<t<\tau_{n+1}\} & \text{for } \ell=n\,.\end{cases}
$$

On the other hand we have, for $0\le m\le \ell\le n-1$ with $n\ge 1$,

$$
\begin{aligned}
(4.7)\qquad & \int_0^\infty E(t;\ m,\ell,n)dt \\
&= E\left\{\int_{\tau_n}^{\tau_{n+1}} \mathrm{e}^{-\lambda t-\alpha\tau_m-\beta\tau_{\ell+1}}dt\right\} \\
&= \frac{1}{\lambda}E\{\mathrm{e}^{-\alpha\tau_m-\beta\tau_{\ell+1}-\lambda\tau_n}-\mathrm{e}^{-\alpha\tau_m-\beta\tau_{\ell+1}-\lambda\tau_{n+1}}\} \\
&= \frac{1}{\lambda}E\{\mathrm{e}^{-\alpha\tau_m-\beta\tau_{\ell+1}-\lambda\tau_n}\}E\{1-\mathrm{e}^{-\lambda(\tau_{n+1}-\tau_n)}\} \\
&= \frac{1}{\lambda+c}E\{\mathrm{e}^{-\alpha\tau_m-\beta\tau_{\ell+1}-\lambda\tau_n}\} \\
&= \frac{1}{\lambda+c}E\{\mathrm{e}^{-(\lambda+\alpha+\beta)\tau_m}\}E\{\mathrm{e}^{-(\lambda+\beta)(\tau_{\ell+1}-\tau_m)}\}E\{\mathrm{e}^{-\lambda(\tau_n-\tau_{\ell+1})}\} \\
&= \frac{1}{\lambda+c}\left(\frac{c}{\lambda+\alpha+\beta+c}\right)^m\left(\frac{c}{\lambda+\beta+c}\right)^{\ell+1-m}\left(\frac{c}{\lambda+c}\right)^{n-\ell-1} \\
&= \frac{1}{\lambda+\beta+c}\left(\frac{\lambda+\beta+c}{\lambda+\alpha+\beta+c}\right)^m\left(\frac{\lambda+c}{\lambda+\beta+c}\right)^\ell\left(\frac{c}{\lambda+c}\right)^n,
\end{aligned}
$$

and similarly, for $0 \le m \le n$ with $n \ge 0$,

$$\int_0^\infty E(t;\ m,n,n)dt = \frac{1}{\lambda+\beta+c}\left(\frac{\lambda+\beta+c}{\lambda+\alpha+\beta+c}\right)^m \left(\frac{\lambda+c}{\lambda+\beta+c}\right)^\ell \left(\frac{c}{\lambda+c}\right)^n \quad \text{with } \ell = n\,. \tag{4.8}$$

From (4.5)–(4.8) we have

$$\begin{aligned}\int_0^\infty E\left\{\mathrm{e}^{-\lambda t + A(t)}\right\} dt &= \sum_{n=0}^\infty \sum_{0\le m\le \ell\le n} E\{\mathrm{e}^{A(m,\ell,n)};\ H_n^- = m,\ H_n^+ = \ell\} \\ &\quad \cdot \frac{1}{\lambda+\beta+c}\left(\frac{\lambda+\beta+c}{\lambda+\alpha+\beta+c}\right)^m \left(\frac{\lambda+c}{\lambda+\beta+c}\right)^\ell \left(\frac{c}{\lambda+c}\right)^n \\ &= \frac{1}{\lambda+\beta+c}\sum_{n=0}^\infty s^n E\{e^{A_n} r^{H_n^-} \rho^{H_n^+}\} \\ &\left(\text{with } s = \frac{c}{\lambda+c},\ r = \frac{\lambda+\beta+c}{\lambda+\alpha+\beta+c} \text{ and } \rho = \frac{\lambda+c}{\lambda+\beta+c}\right) \\ &= \frac{1}{\lambda+\beta+c}(1-s)^{-1}\exp\left(\sum_{n=1}^\infty \frac{B_n}{n}\right) \quad \text{(by (3.3))},\end{aligned} \tag{4.9}$$

where B_n is given by (3.2).

LEMMA 4.1. *For any $\theta > 0$ and for any $u = (\xi + i\xi', i\xi'')$, $v = (\eta + i\eta', i\eta'')$, $w = (\zeta + i\zeta', i\zeta'')$ with $\xi \le 0$, $\eta \ge 0$, $\xi', \eta', \zeta, \zeta' \in \mathbf{R}$ and $\xi'', \eta'', \zeta'' \in \mathbf{R}^{d-1}$, we have*

(i) $$\int_0^\infty \frac{1}{t}\mathrm{e}^{-\theta t} E\{\mathrm{e}^{u\cdot X(t)} - 1;\ X'(t) > 0\}dt = \sum_{n=1}^\infty \frac{1}{n}\left(\frac{c}{\theta+c}\right)^n E\{\mathrm{e}^{u\cdot S_n} - 1;\ S_n' > 0\}\,;$$

(ii) $$\int_0^\infty \frac{1}{t}\mathrm{e}^{-\theta t} E\{\mathrm{e}^{i\langle \zeta'', X''(t)\rangle} - 1;\ X'(t) = 0\}dt = \sum_{n=1}^\infty \frac{1}{n}\left(\frac{c}{\theta+c}\right)^n E\{\mathrm{e}^{i\langle\zeta'', S_n''\rangle} - 1;\ S_n' = 0\}\,;$$

(iii) $$\int_0^\infty \frac{1}{t}\mathrm{e}^{-\theta t} E\{\mathrm{e}^{v\cdot X(t)} - 1;\ X'(t) < 0\}dt$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{c}{\theta + c} \right)^n E\{e^{v \cdot S_n} - 1;\ S_n' < 0\};$$

(iv) $$\int_0^\infty \frac{1}{t} e^{-\theta t} P\left\{X'(t) > 0\right\} dt = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{c}{\theta + c} \right)^n P\{S_n' > 0\};$$

(v) $$\int_0^\infty \frac{1}{t} \left(e^{-\theta t} - e^{-\theta' t} \right) P\left\{X'(t) = 0\right\} dt$$

$$= \log \frac{\theta' + c}{\theta + c} + \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \left(\frac{c}{\theta + c} \right)^n - \left(\frac{c}{\theta' + c} \right)^n \right\} P\{S_n' = 0\}$$

(θ' is also positive).

PROOF. (i) The left hand side is equal to

$$\sum_{n=0}^{\infty} E\left\{ \int_{\tau_n}^{\tau_{n+1}} e^{-\theta t} (e^{u \cdot S_n} - 1) \frac{dt}{t};\ S_n' > 0 \right\}$$

$$= \sum_{n=1}^{\infty} E\left\{ \int_{\tau_n}^{\tau_{n+1}} e^{-\theta t} \frac{dt}{t} \right\} E\{e^{u \cdot S_n} - 1;\ S_n' > 0\},$$

which can be identified with the right hand side by virtue of a simple equality

$$E\left\{ \int_{\tau_n}^{\tau_{n+1}} e^{-\theta t} \frac{dt}{t} \right\} = \frac{1}{n} \left(\frac{c}{\theta + c} \right)^n. \tag{4.10}$$

The equalities (ii), (iii) and (iv) can be proved similarly. As for (v) the left hand side can be expressed as

$$E\left\{ \int_0^{\tau_1} (e^{-\theta t} - e^{-\theta' t}) \frac{dt}{t} \right\} + \sum_{n=1}^{\infty} E\left\{ \int_{\tau_n}^{\tau_{n+1}} (e^{-\theta t} - e^{-\theta' t}) \frac{dt}{t} \right\} P\{S_n' = 0\},$$

which is equal to the right hand side because of (4.10) and

$$E\left\{ \int_0^{\tau_1} (e^{-\theta t} - e^{-\theta' t}) \frac{dt}{t} \right\} = \log \frac{\theta' + c}{\theta + c}.$$

The proof of the lemma is finished.

We can now end this section. From Lemma 4.1, (4.9) and (3.2) with

$$s = \frac{c}{\lambda + c}, \quad r = \frac{\lambda + \beta + c}{\lambda + \alpha + \beta + c}, \quad \rho = \frac{\lambda + c}{\lambda + \beta + c},$$

we have

$$\begin{aligned}
&\int_0^\infty E\{e^{-\lambda t+A(t)}\}dt \\
&= \frac{1}{\lambda+\beta+c}\left(1-\frac{c}{\lambda+c}\right)^{-1}\exp\left\{-\log\frac{\lambda+c}{\lambda+\beta+c}\right\}\exp\left\{\int_0^\infty B(t)\frac{dt}{t}\right\} \\
&= \frac{1}{\lambda}\exp\left\{\int_0^\infty B(t)\frac{dt}{t}\right\},
\end{aligned}$$

where $B(t)$ is given by (2.2). This proves Theorem 1 when $X(t)$ is a compound Poisson prosess.

5. Proof of Theorem 1.

Any Lévy process can be approximated by a compound Poisson process. Our proof of Theorem 1 for general Lévy processes is based on this approximation. So suppose we are given an arbitrary Lévy process $\{X(t)\}$ in $\mathbf{R}^d$ with $X(0) = 0$ and let $\{X_\varepsilon(t)\}$ be an approximating compound Poisson process which is to be chosen suitably. We are going to give a considerably simpler proof to Theorem 1 by taking for $\{X_\varepsilon(t)\}$ a compound Poisson process which is subordinate to $\{X(t)\}$ with a suitable subordinator $\{T_\varepsilon(t)\}$.

Let $\tau'_k, k = 1, 2, \cdots$, be independent exponential random variables with mean 1, let $\{\pi(t)\}$ be a Poisson process with intensity 1 and assume that $\{X(t)\}$, $\{\tau'_k,\ k \geq 1\}$ and $\{\pi(t)\}$ are independent. Let $\tau_0 = 0$ and $\tau_n = \sum_{k=1}^n \tau'_k$ for $n \geq 1$. Then the process $\{T_\varepsilon(t)\}$, defined by $T_\varepsilon(t) = \varepsilon\tau_{\pi(t/\varepsilon)}$ for each fixed $\varepsilon > 0$, is a subordinator and we can easily show that

$$\text{for any fixed } t > 0,\ \sup_{0\leq s\leq t}|T_\varepsilon(s) - s| \to 0 \text{ as } \varepsilon \downarrow 0,\ \text{a.s.} \tag{5.1}$$

We also set

$$X_\varepsilon(t) = X(T_\varepsilon(t)). \tag{5.2}$$

Then $\{X_\varepsilon(t)\}$ is a compound Poisson process and it is not hard to prove that

$$X_\varepsilon(t) \to X(t) \text{ (in the Skorohod topology) as } \varepsilon \downarrow 0, \quad \text{a.s.} \tag{5.3}$$

Moreover, for each $n \geq 1$, the random variable $\varepsilon\tau_n$ has the Γ-distribution with density

$$\gamma_{1/\varepsilon,n}(t) = \left\{\varepsilon^n\Gamma(n)\right\}^{-1} t^{n-1}e^{-t/\varepsilon}, \quad t > 0. \tag{5.4}$$

As in §2 we write $X(t) = (X'(t), X''(t))$, $X_\varepsilon(t) = (X'_\varepsilon(t), X''_\varepsilon(t))$ and consider $A(t)$, $B(t)$, $A_\varepsilon(t)$, $B_\varepsilon(t)$ defined for $X(t)$ and $X_\varepsilon(t)$, respectively (see (2.1) and (2.2)). Then an application of Theorem 1 for the compound Poisson process $\{X_\varepsilon(t)\}$ implies

$$\int_0^\infty e^{-\lambda t}E\{e^{A_\varepsilon(t)}\}dt = \frac{1}{\lambda}\exp\left\{\int_0^\infty B_\varepsilon(t)\frac{dt}{t}\right\}, \tag{5.5}$$

under the condition (2.4). In the following lemma we set

$$b_\varepsilon(t;\theta) = t^{-1}\left\{\exp\left(-\frac{\theta t}{1+\theta\varepsilon}\right) - \exp\left(-\frac{t}{\varepsilon}\right)\right\},$$
$$c_\varepsilon(t;\theta) = t^{-1}\left\{\exp\left(-\frac{\theta t}{1+\theta\varepsilon}\right) - \exp\left(-\frac{\lambda t}{1+\lambda\varepsilon}\right)\right\}.$$

LEMMA 5.1. *Under the condition* (2.4) *we have*

$$\begin{aligned}(5.6)\qquad \int_0^\infty B_\varepsilon(t)\frac{dt}{t} &= \int_0^\infty b_\varepsilon(t;\lambda+\alpha+\beta)E\{\mathrm{e}^{u\cdot X(t)}-1;\, X'(t)>0\}dt\\ &\quad+\int_0^\infty b_\varepsilon(t;\lambda+\beta)E\{\mathrm{e}^{i\langle\zeta'' X''(t)\rangle}-1;\; X'(t)=0\}dt\\ &\quad+\int_0^\infty b_\varepsilon(t;\lambda)E\{\mathrm{e}^{v\cdot X(t)}-1;\; X'(t)<0\}dt\\ &\quad+\int_0^\infty c_\varepsilon(t;\lambda+\alpha+\beta)P\{X'(t)>0\}dt\\ &\quad+\log\frac{1+\lambda\varepsilon}{1+(\lambda+\beta)\varepsilon}+\int_0^\infty c_\varepsilon(t;\lambda+\beta)P\{X'(t)=0\}dt\,.\end{aligned}$$

PROOF. We make use of the expression

$$(5.7)\qquad B_\varepsilon(t) = \mathrm{e}^{-(\lambda+\alpha+\beta)t}E\{\mathrm{e}^{u\cdot X_\varepsilon(t)}-1;\; X'_\varepsilon(t)>0\} + \text{four similar terms.}$$

With $\theta = \lambda+\alpha+\beta$ we have

$$\begin{aligned}&\int_0^\infty \frac{1}{t}\mathrm{e}^{-(\lambda+\alpha+\beta)t}E\{\mathrm{e}^{u\cdot X_\varepsilon(t)}-1;\; X'_\varepsilon(t)>0\}dt\\ &\quad=\sum_{n=1}^\infty\int_0^\infty\frac{1}{t}\mathrm{e}^{-\theta t}\cdot\mathrm{e}^{-t/\varepsilon}\cdot\frac{(t/\varepsilon)^n}{n!}E\{\mathrm{e}^{u\cdot X(\varepsilon\tau_n)}-1;\; X'(\varepsilon\tau_n)>0\}dt\\ &\quad=\sum_{n=1}^\infty n^{-1}(1+\theta\varepsilon)^{-n}E\{\mathrm{e}^{u\cdot X(\varepsilon\tau_n)}-1;\; X'(\varepsilon\tau_n)>0\}\\ &\quad=\sum_{n=1}^\infty n^{-1}(1+\theta\varepsilon)^{-n}\int_0^\infty E\{\mathrm{e}^{u\cdot X(t)}-1;\; X'(t)>0\}\gamma_{1/\varepsilon,n}(t)dt\\ &\quad=\int_0^\infty b_\varepsilon(t;\lambda+\alpha+\beta)E\{\mathrm{e}^{u\cdot X(t)}-1;\; X'(t)>0\}dt\,,\end{aligned}$$

where we used the following elementary equalities:

$$(5.8)\qquad \begin{cases} \displaystyle\int_0^{\infty} \frac{1}{t} e^{-\theta t - t/\varepsilon} \frac{(t/\varepsilon)^n}{n!} dt = n^{-1}(1+\theta\varepsilon)^{-n}\,, \\ \displaystyle\sum_{n=1}^{\infty} n^{-1}(1+\theta\varepsilon)^{-n}\gamma_{1/\varepsilon,n}(t) = b_\varepsilon(t;\theta)\,. \end{cases}$$

The integrals corresponding to the other four terms in the expression (5.7) can be computed similarly. The proof of the lemma is finished.

From Lemma 5.1 we can obtain a *formal* proof of Theorem 1 by letting $\varepsilon \downarrow 0$ in (5.4), but for a *rigorous* proof we need some lemmas.

LEMMA 5.2 ([10]). *Suppose that $\{X'(t)\}$ is not a compound Poisson process and that $X'(\cdot) \not\equiv 0$. Then, for any $t > 0$, $P\left\{H^-(t) = H^+(t)\right\} = 1$.*

For the proof see Lemma 49.4 of Sato [10, p. 370].

Next we remark that $X''(H^-(t)*)$ and $X''(H^+(t)*)$ are well defined a.s. for each fixed t. When $\{X(t)\}$ is a compound Poisson process the well-definedness was clear from (4.4). In a general case the well-definedness is supported by the following lemma.

LEMMA 5.3. *For each fixed $t > 0$, $X''(\cdot)$ is continuous at $H^-(t)$ almost surely on the event that $X'(\cdot)$ is continuous at $H^-(t)$. The same assertion holds when $H^-(t)$ is replaced by $H^+(t)$.*

PROOF. Since it is clear that $X''(\cdot)$ is continuous at $H^-(t)$ (resp. $H^+(t)$) almost surely on the event $\{H^-(t) = t\}$ (resp. $\{H^+(t) = t\}$), we prove the formula

$$(5.9)\qquad P\left\{\begin{matrix} H^-(t) < t \text{ and } X'(\cdot) \text{ is continuous at } H^-(t) \\ \text{but } X''(\cdot) \text{ is not continuous at } H^-(t) \end{matrix}\right\} = 0$$

and the same one for $H^+(t)$ hold. Firstly we consider the case where $\sup_{0<s<\varepsilon} X'(s) > 0$ for any $\varepsilon > 0$ almost surely. For each $n \geq 1$, let $\theta_{n,k},\ k = 1, 2, \cdots$, be the stopping times defined by

$$\theta_{n,1} = \inf\left\{s > 0 :\ \left|X''(s) - X''(s-)\right| > \frac{1}{n}\right\}\,,$$
$$\theta_{n,k} = \inf\left\{s > \theta_{n,k-1} :\ \left|X''(s) - X''(s-)\right| > \frac{1}{n}\right\},\quad k \geq 2\,.$$

Then the left hand side in (5.9) is dominated by

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P\{H^-(t)=\theta_{n,k}<t \text{ and } X'(\cdot) \text{ is continuous at } H^-(t)\} \\
&\le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P\{\theta_{n,k}<t \text{ and } X'(\cdot) \text{ takes a local maximum at } \theta_{n,k}\} \\
&\le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P\left\{\theta_{n,k}<t \text{ and } \sup_{0<s<\varepsilon} X'(s+\theta_{n,k}) \le X'(\theta_{n,k}) \text{ for some } \varepsilon>0\right\} \\
&\le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} E\left[P\left\{\sup_{0<s<\varepsilon} X'(s) \le 0 \text{ for some } \varepsilon>0\right\};\ \theta_{n,k}<t\right]=0\,,
\end{aligned}$$

and hence (5.9) holds. Since $H^-(t)=H^+(t)$ (a.s.) by Lemma 5.2, (5.9) also holds with the replacement of $H^-(t)$ by $H^+(t)$. Secondly we consider the case where $\inf_{0<s<\varepsilon} X'(s)<0$ for any $\varepsilon>0$ almost surely. We set $Y(s)=X((t-s)-)-X(t-)$ for $0\le s<t$. Since the process $\{Y(s),\ 0\le s<t\}$ is identical in law to $\{-X(s),\ 0\le s<t\}$, $\sup_{0<s<\varepsilon} Y'(s)$ is also identical in law to $-\inf_{0<s<\varepsilon} X'(s)$ which is strictly positive a.s. Therefore, by a similar argument applied to $\{Y(s),\ 0\le s<t\}$ we can prove (5.9) and the same one for $H^+(t)$. Thirdly it remains to treat the case where $X'(\cdot)$ is a compound Poisson process, but in this case the assertion of the lemma follows immediately from

$$\begin{cases} P\{H^-(t)>0 \text{ and } X'(\cdot) \text{ is continuous at } H^-(t)\}=0\,, \\ P\{H^+(t)<t \text{ and } X'(\cdot) \text{ is continuous at } H^+(t)\}=0\,. \end{cases} \tag{5.10}$$

The proof of the lemma is finished.

LEMMA 5.4. *For each fixed* $t>0$, $H_\varepsilon^-(t)$, $H_\varepsilon^+(t)$, $M_\varepsilon^-(t)$ *and* $M_\varepsilon^+(t)$ *tend to* $H^-(t)$, $H^+(t)$, $M^-(t)$ *and* $M^+(t)$, *respectively, as* $\varepsilon\downarrow 0$ *almost surely.*

PROOF. The proof of $\lim_{\varepsilon\downarrow0} X''_\varepsilon(H^-(t)*)=X''(H^-(t)*)$ and $\lim_{\varepsilon\downarrow0} X''_\varepsilon(H^+(t)*)=X''(H^+(t)*)$ is the most crucial part. To clarify the essential point of our proof we take up the event

$$\Lambda=\{0<H^-(t)<t \quad \text{and} \quad X'(H^-(t)-)=M'(t)>X'(H^-(t))\}$$

and prove that

$$\begin{cases} H_\varepsilon^-(t),\ M'_\varepsilon(t) \text{ and } X''_\varepsilon(H_\varepsilon^-(t)*) \text{ tend to } H^-(t),\ M'(t) \text{ and} \\ X''(H^-(t)*),\ \text{respectively, as } \varepsilon\downarrow 0 \text{ almost surely on } \Lambda\,. \end{cases} \tag{5.11}$$

The other cases can be treated in the same spirit. In the following argument the adverb phrases "for all sufficiently small $\varepsilon>0$" and "almost surely on Λ" are needed frequently but often omitted for simplicity. For instance, the assertion (5.12) below should be read as "for all sufficiently small $\varepsilon>0$ there exists $n'\le n(\varepsilon,t)$ such that $M'_\varepsilon(t)=X'(\varepsilon\tau_{n'})$, almost sure on Λ." Since $\{X(\cdot)\}$ and $\{\tau_n,\ n \geq 0\}$ are independent, it is clear that, for each $\varepsilon > 0$ and for any $n \geq 0$, $H^-(t) \neq \varepsilon\tau_n$ a.s. on Λ. Therefore there exists $n = n(\varepsilon, t)$ such that $\varepsilon\tau_n < H^-(t) < \varepsilon\tau_{n+1}$ (a.s. on Λ). Taking into account that the subordinator $\{T_\varepsilon(\cdot)\}$ takes values in the (random) set $\{\varepsilon\tau_n,\ n \geq 0\}$ and also noting that $H^-(t) = H^+(t)$ a.s. on Λ (when $X'(\cdot)$ is a compound Poisson process, $P(\Lambda) = 0$), we can easily see that

(5.12) $$\text{there exists } m \leq n(\varepsilon, t) \text{ such that } M'_\varepsilon(t) = X'(\varepsilon\tau_m)\,.$$

Denoting by $m(\varepsilon, t)$ the smallest m in (5.12), we now claim that

(5.13) $$\varepsilon\tau_{m(\varepsilon,t)} \to H^-(t) \quad \text{as } \varepsilon \downarrow 0,$$

(5.14) $$H^-_\varepsilon(t) \to H^-(t) \quad \text{as } \varepsilon \downarrow 0\,, \quad T_\varepsilon(H^-_\varepsilon(t)) = \varepsilon\tau_{m(\varepsilon,t)} < t\,.$$

In fact, the proof of (5.13) is easy. To prove (5.14) let $0 < \sigma_1 < \sigma_2 < \cdots$ be the jumping times of Poisson process $\{\pi(\cdot)\}$. Then it is easy to see that (5.13) implies

$$\lim_{\varepsilon\downarrow 0} \varepsilon\sigma_{m(\varepsilon,t)} = \lim_{\varepsilon\downarrow 0} \varepsilon\tau_{m(\varepsilon,t)} \cdot \frac{m(\varepsilon,t)}{\tau_{m(\varepsilon,t)}} \cdot \frac{\sigma_{m(\varepsilon,t)}}{m(\varepsilon,t)} = H^-(t)\,,$$

and hence $\varepsilon\sigma_{m(\varepsilon,t)} < t$ for all sufficiently small $\varepsilon > 0$. Therefore we have $H^-_\varepsilon(t) = \varepsilon\sigma_{m(\varepsilon,t)} \to H^-(t)$ as $\varepsilon \downarrow 0$ and

$$T_\varepsilon(H^-_\varepsilon(t)) = \varepsilon\tau[\pi(H^-_\varepsilon(t)/\varepsilon)] = \varepsilon\tau[\pi(\sigma_{m(\varepsilon,t)})] = \varepsilon\tau_{m(\varepsilon,t)}\,,$$

which implies (5.14). From (5.13) and (5.14) we see that $T_\varepsilon(H^-_\varepsilon(t))$ is less than t and tends to $H^-(t)$ as $\varepsilon \downarrow 0$, a.s. on Λ. Consequently

$$\lim_{\varepsilon\downarrow 0} X''_\varepsilon(H^-_\varepsilon(t)*) = \lim_{\varepsilon\downarrow 0} X''(T_\varepsilon(H^-_\varepsilon(t))) = X''(H^-(t)-) = X''(H^-(t)*)\,,$$

and this proves (5.11). The proof of the lemma is finished.

LEMMA 5.5. (i) *For any* $\lambda > 0$ *and* $u = (i\xi', i\xi'')$ *with* $\xi' \in \mathbf{R}$ *and* $\xi'' \in \mathbf{R}^{d-1}$

(5.15) $$\int_0^\infty \frac{1}{t} \mathrm{e}^{-\lambda t} E\{|\mathrm{e}^{u\cdot X(t)} - 1|\}dt < \infty\,.$$

(ii) *For any* $\lambda > 0$ *and* $\theta \geq 0$

(5.16) $$\int_0^\infty \frac{1}{t} \mathrm{e}^{-\lambda t} E\{1 - \mathrm{e}^{-\theta|X'(t)|}\}dt < \infty\,.$$

PROOF. (i) Making use of the fact that a Lévy process has finite absolute moments of all positive orders if the support of the Lévy measure is a bounded set (e.g. see Sato [10, p. 159]), we can express the Lévy process $Y(t) := -iu \cdot X(t)$ as $Y(t) = Y_1(t) + Y_2(t)$ where $\{Y_1(t)\}$ is a Lévy process with $E\{|Y_1(t)|^2\} < \infty$ and $\{Y_2(t)\}$ is a compound

Poisson process independent of $\{Y_1(t)\}$. Let T be the first jumping time of $\{Y_2(t)\}$. Then $P\{T \le t\} = 1 - \mathrm{e}^{-ct}$, $t > 0$, with some constant $c \ge 0$. Therefore

$$(5.17) \qquad E\{|\mathrm{e}^{u\cdot X(t)} - 1|\} = E\{|\mathrm{e}^{iY_1(t)} - 1|;\ T > t\} + E\{|\mathrm{e}^{iY(t)} - 1|;\ T \le t\}$$
$$\le E\{|\mathrm{e}^{iY_1(t)} - 1|\} + 2P\{T \le t\}\,.$$

Using the elementary inequality $|\mathrm{e}^{ix} - 1| \le |x|$ (for any real x), we have

$$E\{|\mathrm{e}^{iY_1(t)} - 1|\} \le E\,\{|Y_1(t)|\} \le \text{const.}\, t^{1/2} \quad \text{for } 0 \le t \le 1\,,$$

and hence (5.17) yields

$$E\{|\mathrm{e}^{u\cdot X(t)} - 1|\} \le \text{const.}\, t^{1/2}, \quad 0 \le t \le 1\,.$$

This implies (5.15).

(ii) As in (i), we make use of a decomposition $X'(t) = X'_1(t) + X'_2(t)$ where $\{X'_1(t)\}$ is a Lévy process with $E\{|X'_1(t)|^2\} < \infty$ and $\{X'_2(t)\}$ is a compound Poisson process independent of $\{X'_1(t)\}$. Let T be the first jumping time of $\{X'_2(t)\}$. Then

$$E\{1 - \mathrm{e}^{-\theta|X'(t)|}\} \le E\{1 - \mathrm{e}^{-\theta|X'_1(t)|};\ T > t\} + P\,\{T \le t\}$$
$$\le E\,\left\{\theta|X'_1(t)|\right\} + P\,\{T \le t\} \le \text{const.}\, t^{1/2} \quad \text{for } 0 \le t \le 1\,,$$

from which (5.16) follows.

We are now able to complete the proof of Theorem 1 by letting $\varepsilon \downarrow 0$ in (5.5). By Lemma 5.4 the left hand side in (5.5) tends to $\int_0^\infty \mathrm{e}^{-\lambda t} E\{\mathrm{e}^{A(t)}\}dt$ as $\varepsilon \downarrow 0$. As for the right hand side in (5.5), first we rewrite it using Lemma 5.1 and then apply Lebesgue's dominated convergence theorem, which is guaranteed by Lemma 5.5. Thus we see that the right hand side tends to $\lambda^{-1} \exp\{\int_0^\infty B(t)\frac{dt}{t}\}$ as $\varepsilon \downarrow 0$. This complete the proof of Theorem 1.

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