On a forward-backward parabolic equation.

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Abstract - *Boundary value problems for the equation*

$$\operatorname{sgn}(x)\cdot u_y - u_{xx} + ku = f(x, y)$$

(where k is a positive constant and f is a given function) are investigated. The domain of the solutions will be the whole upper half-plane $y > 0$, or the half-plane $y > 0$ cut along the positive y-axis. We are interested in square integrable solutions u, with square integrable generalized derivatives u_y and u_{xx}.

Existence theorems are proved, with an integral equations technique. Thus a theory is developed of Wiener-Hopf integral equations of the first kind with solutions belonging to Sobolev spaces.

Introduction.

In this paper we consider the equation

$$\operatorname{sgn}(x)\cdot u_y - u_{xx} + ku = f \tag{1}$$

where k is a positive constant and f is a given complex valued function. Note that the equation (1) is forward parabolic in the half-plane $x > 0$ and backward parabolic in the half-plane $x < 0$. The equation (1) is an example of equations of the following form:

$$|x|^p \operatorname{sgn}(x)\cdot u_y - a(x, y)u_{xx} + (\text{lower order terms}) = f \tag{2}$$

where p is a nonnegative number and the coefficient $a(x, y)$ is bounded from below by a positive constant.

The equation (2) has been considered by GEVREY [3]. He has shown that, under suitable regularity assumptions, (2) is the canonical form of parabolic equations of the type $A(x, y)u_y - a(x, y)u_{xx} +$ (lower order terms) $= f$, where the coefficient $A(x, y)$ changes sign through the line $A(x, y) = 0$.

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Relevant equations of the form (2) are the following

$$
(3) \qquad \begin{aligned} &xu_y - \frac{1}{2} u_{xx} + xu_x = 0 \\ &xu_y - u_{xx} = 0 \end{aligned}
$$

Such equations are of interest in problems of kinetic theory and stochastic processes; see for instance [11], [2]. Equation (3) has been investigated by PAGANI [10], FLEMING [2] and, from the point of view of weak solutions, by BAOUENDI-GRISVARD [1].

An appropriate boundary value problem for the equation (1) is to look for a solution $u(x, y)$ which is defined in R^2_+, the whole upper half-plane $y > 0$, and which verifies the condition: $u(x, 0) = h(x)$ on the positive half-axis $0 < x < +\infty$, where h is a given function.

We are interested in square-integrable complex valued solutions u, whose generalized derivatives u_x, u_y and u_{xx} are square integrable. We shall call $W(G)$ the set of functions, defined in an open two-dimensional set G, with such a property. For a discussion of the properties of functions belonging to $W(G)$ see for instance NIKOL'SKIĬ [8], SLOBODECKIĬ [13], LIONS-MAGENES [6] chapter 4; in the NIKOL'SKIĬ notation: $W(G) = W^{(2,1)}_{(2,2)}(G)$. It is easy to prove the following theorem:

THEOREM 0.1. - *Let f and h be two given functions; we suppose that $f \in L^2(R^2_+)$ and $h \in H^1(0, +\infty)$. Let u be a solution of the equation (1) such that:*

$$
(4) \qquad u \in W(R^2_+)
$$

(5) $u(x, 0+)$, *the trace of u on the x-axis* $= h(x)$ *a.e. in* $0 < x < +\infty$.

Assertion: the following inequality holds:

$$
(6) \qquad \iint_{R^2_+} (2k|u_x|^2 + k^2|u|^2)\,dxdy + k\int_{-\infty}^{0} |u(x, 0+)|^2 dx \leq
$$

$$
\leq \iint_{R^2_+} |f|^2 dxdy + k\int_0^{+\infty} |h(x)|^2 dx .
$$

The (6) is an a priori estimate; it implies uniqueness for the problem (1)-(4)-(5).

With regard to the boundary condition (5), it should be remarked that $H^1(-\infty, +\infty)$ is exactly the space of the traces on the x-axis of the functions belonging to $W(R^2_+)$. We recall that $H^1(a, b)$ is the set of all absolutely continuous square integrable functions on the interval $]a, b[$ with square integrable first derivative.

PROOF OF THE THEOREM 0.1. - Clearly we can suppose that u is infinitely differentiable in the closure of R^2_+ and vanishes outside a bounded set; for the set of such functions is dense in $W(R^2_+)$, the norm in $W(R^2_+)$ being:

$$\|u\|_{W(R^2_+)} = \left[\iint_{R^2_+} (|u_{xx}|^2 + |u_y|^2 + |u|^2) dx dy\right]^{\frac{1}{2}}. \tag{7}$$

From the equation (1), squaring the absolute values of both members and neglecting the nonnegative sum $|u_y|^2 + |u_{xx}|^2 - 2 \, Re \operatorname{sgn}(x) u_y \bar{u}_{xx}$, we get:

$$-2k \, Re \, u \bar{u}_{xx} + k^2 |u|^2 + k \operatorname{sgn}(x) \frac{\partial}{\partial y} |u|^2 \leq |f|^2. \tag{8}$$

Integrating over R^2_+ both members of (8) we obtain easily (6).

We consider now another problem for the equation (1). Roughly speaking, this problem consists in looking for solutions of (1) which equal some given data on the positive x-axis and have prescribed discontinuities on the interface $x = 0$. More precisely, it is requested to find a solution u of (1) such that:

$$u \text{ belongs to } W(G_+) \text{ where } G_+ \text{ is } \{(x, y) : x \neq 0, \ y > 0\}, \tag{9}$$

the upper half-plane R^2_+ cut along the positive y-axis;

$$u(x, 0+) = h(x) \text{ a.e. in } 0 < x < +\infty, \tag{10}$$

where h is a given function and $u(\cdot, 0+)$ is, as before, the trace of u on the x-axis;

(11) u verifies either of the following conditions:

$$\begin{cases} u_x(0-, y) = u_x(0+, y) \\ u(0-, y) + u(0+, y) = 2k \displaystyle\int_y^{\infty} e^{+k(y-t)} u(0+, t) dt \\ \qquad \text{a.e. in } 0 < y < +\infty \end{cases} \tag{11'}$$

$$
(11'') \qquad \begin{cases} u(0-,\, y) = u(0+,\, y) \\ u_x(0-,\, y) + u_x(0+,\, y) = 2k \displaystyle\int_y^{\infty} e^{+k(y-t)} u_x(0-,\, t)\,dt \\ \qquad\qquad \text{a.e. in } 0 < y < +\infty \end{cases}
$$

Here $u(0+,\, \cdot)$ [or $u(0-,\, \cdot)$] is the trace on the y-axis of the restriction of u to the north-east [or to the north-west] quadrant; similarly for $u_x(0+,\, \cdot)$ and $u_x(0-,\, \cdot)$. It should be remembered that, if $u \in W(G_+)$, then $u(0+,\, \cdot)$ and $u(0-,\, \cdot)$ are HÖLDER continuous functions and belong to the SOBOLEV space $H^{3/4}(0,\, +\infty)$, while $u_x(0+,\, \cdot)$ and $u_x(0-,\, \cdot)$ belong to $H^{1/4}(0,\, +\infty)$; moreover the traces of u on the y-axis are connected with the trace of u on the x-axis by the equations:

$$
\lim_{0<y\to 0} u(0+,\, y) = \lim_{0<x\to 0} u(x,\, 0+), \quad \lim_{0<y\to 0} u(0-,\, y) = \lim_{0>x\to 0} u(x,\, 0+).
$$

For an exhaustive discussion of the SOBOLEV spaces of fractionary order see LIONS-MAGENES [6], chapter 1 and 4.

We have the following theorem, which gives uniqueness for the problem (1)-(9)-(10)-(11).

THEOREM 0.2. - *Let* $f \in L^2(R^2_+)$ *and* $h \in H^1(0,\, +\infty)$ *be given; moreover let* u *be a solution of* (1) *verifying* (9)-(10)-(11). *Then there exists an absolute constant* C *such that:*

$$
(12) \qquad \iint_{R^2_+} (|u_{xx}|^2 + |u_y|^2 + 2k|u_x|^2 + k^2|u|^2)\,dx\,dy \; +
$$

$$
+ \int_{-\infty}^{0} \left(\left| \frac{d}{dx} u(x,\, 0+) \right|^2 + k\,|u(x,\, 0+)|^2 \right) dx \le
$$

$$
\le C \left\{ \iint_{R^2_+} |f|^2\,dx\,dy + \int_0^{+\infty} (|h'(x)|^2 + k\,|h(x)|^2)\,dx \right\}.
$$

The theorem 0.2 is an easy consequence of the following formula:

$$
(13) \qquad \iint_{R^2_+} (|f|^2 - |u_{xx}|^2 - |u_y|^2 - 2k\,|u_x|^2 - k^2\,|u|^2)\,dx\,dy \; +
$$

$$+\int_{-\infty}^{+\infty}\left(\left|\frac{d}{dx}u(x,\ 0+)\right|^2+k\,|u(x,\ 0+)|^2\right)\operatorname{sgn}(x)dx=$$

$$=2Re<\left(\frac{d}{dy}-k\right)u(0-,\ \cdot)\,|\,u_x(0-,\ \cdot)>+$$

$$+\ 2Re<\left(\frac{d}{dy}+k\right)u(0+,\ \cdot)\,|\,u_x(0+,\ \cdot)>.$$

In this formula u is any function belonging to $W(G_+)$, where G_+ is defined in (9), and f is related to u by (1).

The symbol $<\,|\,>$ at the right of (13) denotes the pairing between $H^{-1/4}(0,\ +\infty)$ and $H^{1/4}(0,\ +\infty)$. We recall that $H^{-s}(0,\ +\infty)$, with $s>0$, is the set of all hermitian complex valued continuous functions on $H_0^s(0,\ +\infty)$ $=$ closure of $C_0^\infty(]0,\ +\infty[)$ in $H^s(0,\ +\infty)$. It must be borne in mind that $H_0^s(0,\ +\infty)=H^s(0,\ +\infty)$ if $0\leq s\leq\frac{1}{2}$. In other words, if φ is a function in $H^{1/4}(0,\ +\infty)$, a pairing $<\Phi\,|\,\varphi>$, between some element Φ in $H^{-1/4}(0,\ +\infty)$ and the function φ, is a limit of the form: $\lim\int_0^\infty\Phi_n\bar\varphi dy$, where Φ_n is some sequence of test functions such that: $\int_0^\infty(\Phi_m-\Phi_n)\psi dy\to 0$ uniformily as ψ runs on the balls of $H^{1/4}(0,\ +\infty)$.

The operators $(d/dy)\pm k$ at the right of (13) are taken in the sense of the distributions. As is well-known and easy to see, $(d/dy)\pm k$ are bounded operators from $H^s(0,\ +\infty)$ into $H^{s-1}(0,\ +\infty)$ if $s\neq\frac{1}{2}$.

It is a simple matter to show that the right-hand side of (13) vanishes if the function u verifies the conditions (11′): thus from (13) we see that (12) becomes an equality with $C=1$ if (11′) holds. If u verifies the conditions (11″), the right-hand side of (13) equals:

$$-4k\,Re\,[\lim_{0<y\to 0}u(0+,\ y)]\cdot\int_0^\infty e^{-kt}\overline{u_x(0-,\ t)}dt. \tag{14}$$

To see this we have only to use «integrations by parts» following the rule:

$$<\varphi'\,|\,\psi>=-\varphi(0+)\overline{\psi(0+)}-\int_0^\infty\varphi(y)\overline{\psi'(y)}dy$$

$$\varphi\in H^{3/4}(0,\ +\infty),\quad\psi\in H^1(0,\ +\infty).$$

The expression (14) can be easily majorized with:

$$2^{5/4}\left[\int_0^{\infty}\left(\left|\frac{d}{dx}u(x,\,0+)\right|^2+k\,|\,u(x,\,0+)\,|^2\right)dx\times\right.$$
$$\left.\int_{-\infty}^{0}dx\int_0^{+\infty}(\,|\,u_{xx}\,|^2+2k\,|\,u_x\,|^2)dy\right]^{\frac{1}{2}}$$

Thus (10)-(11″)-(13) imply an inequality of the form (12).

For the proof of (13) we can suppose, owing to density arguments, that the restrictions of u to the north quadrants coincide with the restrictions to the same quadrants of $C_0^{\infty}(R^2)$-functions. Then we obtain (13) by squaring the absolute values of both members of (1), integrating over G_+ and using integrations by parts. The details are straighforward.

In this paper we present some existence theorems for the problems (1)-(4)-(5) and (1)-(9)-(10)-(11), see sections 1 and 2. It should be pointed out that in one of such theorems compatibility conditions on the data must be imposed; instead no compatibility conditions occur if weak solutions are wanted, compare with [1].

The method for the existence proofs is a usual one and can be described in this way. We give a fictitious datum (belonging to suitable classes of data) on the y-axis. Then a solution v of the equation $-v_y-v_{xx}+kv=f$, which is of class W in the west half-plane $x<0$, is determined by the fictitious datum, and a solution w of the equation $w_y-w_{xx}+kw=f$, which is of class W in the north-east quadrant, is determined by the fictitious datum and the boundary condition on the x-axis. Hence we specialize the fictitious datum in such a way that the function u, defined by: $u=v$ if $x<0$ and $u=w$ if $x>0$, is the wanted solution. To do this, we have to solve a WIERNER-HOPF integral equation of the first kind. It should be remarked that our procedure gives solutions u which are of class W in the union of R_+^2 (or G_+) and the west half-plane $x<0$, and which verify a boundary condition on the lower half-axis $-\infty<y<0$.

As is well known, the WIENER-HOPF integral equations are of the following form:

$$\int_0^{+\infty}K(x-y)\varphi(y)dy=f(x)\qquad(0<x<+\infty)\tag{15}$$

$$\varphi(x)-\int_0^{+\infty}K(x-y)\varphi(y)dy=f(x)\quad(0<x<+\infty),\tag{16}$$

where K and f are given functions, φ is the unknown.

It should be pointed out that the equality between the left and the right member of (15) or (16) holds only on the half-line $0 < x < +\infty$. (15) is called of the first kind, (16) is of the second kind. For a formal approach to these equations, see NOBLE [9], MORSE-FESCHBACH [15]. Very comprehensive results are known for the WIERNER-HOPF equations of the second kind; see KREĬN [5], GOHBERG-KREĬN [4], see also ŠAHBAGJAN [12]. Instead it seems that the WIENER-HOPF equations of the first kind have not been exhaustively investigated.

In section 3 we present an existence theorem for solutions, belonging to SOBOLEV spaces $H^s(-\infty, +\infty)$, of such equations. The proof depends on some properties, which will be stated in lemmas 3.2-3.3-3.4, of CAUCHY integrals:

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\widehat{g}(t)}{t-\zeta}\,dt \qquad (\zeta = \xi + i\eta,\ \eta \neq 0) \tag{18}$$

where the density is the FOURIER transform of a distribution $g \in H^s(-\infty, +\infty)$.

The plan of the paper is the following:

1. - Results about the homogeneous equation.

2. - Solutions of the nonhomogeneous equation.

3. - A discussion of an integral equation of the WIENER-HOPF type (contents: statement of the problem, statement of the existence theorem, remarks, an example, lemmas on CAUCHY integrals, proof of the theorem proofs of the lemmas).

4. - Proofs of theorems 1.1 and 1.2.

APPENDIX. - Some proprieties of solutions of the heat equation.

We conclude the introductory remarks with a lemma on the WIENER-HOPF factorisation of a special kernel, which will be useful later and also for a more clear understanding of section 1.

LEMMA. - *Consider the kernel* $K(y) = (\pi|y|)^{-\frac{1}{2}}\exp(-k|y|)$ ($k =$ constant > 0) *and its Fourier transform* $\widehat{K}(\xi) = 2\,Re(k + i\xi)^{-\frac{1}{2}}$. *More explicitely:*

$$\widehat{K}(\xi) = \sqrt{2}(k^2+\xi^2)^{-\frac{1}{4}}\left[1 + k(k^2+\xi^2)^{-\frac{1}{2}}\right]^{\frac{1}{2}} \tag{19}$$

There exist two functions A and B such that:

(i) $\zeta = \xi + i\eta \to A(\zeta)$ *is holomorphic in the half-plane* $Im\,\zeta > 0$ *and*

Hölder continuous in the closed halp-plane $Im\,\zeta \geq 0$; $\zeta = \xi + i\eta \to B(\zeta)$ *is holomorphic in the half-plane* $Im\,\zeta < 0$ *and Hölder continuous in* $Im\,\zeta \leq 0$.

(ii) $2^{1/8} \leq (k^2 + |\zeta|^2)^{1/8} |A(\zeta)| \leq 2^{1/2}$ *if* $Im\,\zeta \geq 0$,

$2^{1/8} \leq (k^2 + |\zeta|^2)^{1/8} |B(\zeta)| \leq 2^{1/2}$ *if* $Im\,\zeta \leq 0$.

(iii) $\widehat{K}(\xi) = A(\xi) \cdot B(\xi)$ *for every real* ξ.

For ξ *real we have*:

$$A(\xi) = \tag{20}$$

$$= \frac{2^{1/4}}{(k - i\xi)^{1/4}} \left[1 + \frac{k}{(k^2 + \xi^2)^{\frac{1}{2}}}\right]^{\frac{1}{4}} \exp\left\{\frac{i}{2\pi} \int_{-\infty}^{+\infty} ln \left[1 + \frac{k}{(k^2 + t^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}} \frac{dt}{\xi - t}\right\}$$

$$B(\xi) = \frac{2^{1/4}}{(k + i\xi)^{1/4}} [\ \dots\ ]^{\frac{1}{4}} \exp\left\{-\frac{i}{2\pi} \dots\right\}, \tag{21}$$

where the integral is taken in the Cauchy principal value sense.

The proof is quite simple. We define A and B with the equations (20), (21) and the following:

$$A(\zeta) = 2^{1/4}(k - i\zeta)^{-1/4} \exp\{\Psi(\zeta)\} \quad \text{for } Im\,\zeta > 0 \tag{22}$$

$$B(\zeta) = 2^{1/4}(k + i\zeta)^{-1/4} \exp\{-\Psi(\zeta)\} \text{ for } Im\,\zeta < 0, \tag{23}$$

where

$$\Psi(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} ln \left[1 + \frac{k}{(k^2 + t^2)^{1/2}}\right]^{\frac{1}{2}} \frac{dt}{t - \zeta} \quad (Im\,\zeta \neq 0).$$

Then the well-known PLEMELJ formulas (see e.g. [7]) guarantee the continuity requested in (i); the (iii) is obvious; the (ii) follows from:

$$0 < Re\,\Psi(\zeta)\ \text{sgn}\ (Im\,\zeta) =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\eta|}{(t - \xi)^2 + \eta^2} ln \left[1 + \frac{k}{(k^2 + t^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}} dt < \frac{1}{4} ln\, 2.$$

1. - Results about the homogeneous equation.

In this section we consider the homogeneous equation:

$$\operatorname{sgn}(x)u_y - u_{xx} + ku = 0 \qquad (k = \text{constant} > 0). \tag{1.1}$$

THEOREM 1.1. - *Let h be a function belonging to $H^1(0, +\infty)$.*

A solution u of the equation (1.1), *such that:*

$$u \in W(R^2_+) \tag{1.2}$$

$$u(x, 0+) = h'(x) \text{ a.e. in } 0 < x < +\infty \tag{1.3}$$

exists provided that h verifies the following condition:

$$\int_0^\varepsilon \frac{1}{y} |g(y)|^2 dy < +\infty \text{ for some } \varepsilon > 0. \tag{1.4}$$

The function g is defined in this way:

$$\widehat{g}, \text{ the Fourier transform of } g = -\frac{l}{A}, \tag{1.5}$$

where l is any function such that:

$$\begin{cases} l \in H^{3/4}(-\infty, +\infty) \\ l(y) = \displaystyle\int_{-\infty}^{+\infty} \frac{\exp\left(-ky - \frac{t^2}{4y}\right)}{(4\pi y)^{\frac{1}{2}}} h(|t|)dt \quad \text{for } 0 < y < +\infty \end{cases} \tag{1.6}$$

and A is given by (19).

REMARKS. - (i) There exist functions l verifying (1.6). For the integral appearing in (1.6) is a function of the y-variable belonging to $H^{3/4}(0, +\infty)$. In fact such an integral is the trace on the positive y-axis of the function:

$$U(x, y) = \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-t)^2}{4y} - ky\right)}{(4\pi y)^{\frac{1}{2}}} h(|t|)dt \tag{1.7}$$

$$(-\infty < x < +\infty, \ y > 0).$$

As $h \in H^1(0, +\infty)$, the function $x \to h(|x|)$ is in $H^1(-\infty, +\infty)$. Thus, according to well-known properties of solutions of the heat equation (see the appendix), U is in $W(R^2_+)$. Incidentally, U is indefinitely differentiable and bounded. Note that U verifies the equation $U_y - U_{xx} + kU = 0$ and the initial condition: $U(x, 0+) = h(|x|)$ $(-\infty < x < +\infty)$.

(ii) The function g belongs to $H^{\frac{1}{2}}(-\infty, +\infty)$. This follows immediately from (1.5), the first condition (1.6) and the inequalities given in the introduction for the function A.

The condition (1.4) does not depend from the choice of the function l in the class defined by (1.6). For the restriction of g to the positive half-axis $[0, +\infty[$ is uniquely determined by the restriction of l to $[0, +\infty[$. For a proof of this assertion, see the remark (ii)-section 3.

Following LIONS-MAGENES [6] (chapter I, section 11.5), a function $g \in H^{\frac{1}{2}}(0, +\infty)$, verifying

$$\int_0^{+\infty} |g(y)|^2 \frac{dy}{y} < +\infty,$$

is said to be in $H^{\frac{1}{2}}_{00}(0, +\infty)$.

THEOREM 1.2. - *Let $h \in H^1(0, +\infty)$. There exists a solution u* of (1.1) *such that:*

$$u \in W(G_+), \text{ where } G_+ = \{(x, y) \in R^2 : x \neq 0,\ y > 0\}, \tag{1.8}$$

$$u(x, 0) = h(x) \text{ a.e. in } 0 < x < +\infty, \tag{1.9}$$

and verifying either of the following conditions:

$$\begin{cases} u_x(0-, y) = u_x(0+, y) \\ u(0-, y) + u(0+, y) = 2k \displaystyle\int_y^{+\infty} e^{k(y-t)} u(0+, t)\,dt \\ \text{a.e. in } 0 < y < +\infty \end{cases} \tag{1.10}$$

$$\begin{cases} u(0-, y) = u(0+, y) \\ u_x(0-, y) + u_x(0+, y) = 2k \displaystyle\int_y^{+\infty} e^{k(y-t)} u_x(0-, t)\,dt \\ \text{a.e. in } 0 < y < +\infty. \end{cases} \tag{1.11}$$

No compatibility conditions on h are imposed.

REMARK. - From the theorem 1.2 we may easily deduce an existence theorem for solutions of the equation:

$$\operatorname{sgn}(x)\cdot u_y - a(x, y)u_{xx} + ku = b(x, y) \tag{1.12}$$

where $k =$ constant > 0, $a(x, y)$ is a measurable function bounded from above and from below by positive constants, $b(x, y)$ is square integrable. More precisely we may prove that *there exists a unique solution* u *of* (1.12) *verifying* (1.8)-(1.9), *where* h *is any function in* $H^1(0, +\infty)$, *and* (1.10) *or* (1.11).

In view of the theorem 1.3 and the contents of the next section, this assertion follows, via standard principles of functional analysis, form a suitable a priori estimate. In order to obtain such an estimate, we may first derive form (1.12) an algebraic inequality similar to that known in the theory of elliptic equations as Bernstein's inequality, namely:

$$\begin{aligned} &|u_y|^2 + |u_{xx}|^2 + |ku|^2 + k\operatorname{sgn}(x)\frac{\partial}{\partial y}|u|^2 \leq \\ &\leq 2A\cdot Re\,(\operatorname{sgn}(x)u_y + ku)\bar{u}_{xx} + B|b|^2 \end{aligned} \tag{1.13}$$

where A is any constant greater than $\lambda = \frac{1}{2}\sup\left|a(x, y) + \frac{1}{a(x, y)}\right|$ and B is any constant not less than $(A^2 - 1)/(2Aa(x, y) - a(x, y)^2 - 1)$.

Arguing exactly in the same way as in the proof of the formula (13) and of the theorem 0.2, we deduce from (1.13):

$$\begin{aligned} &\iint_{R^2_+} (|u_y|^2 + |u_{xx}|^2 + 2Ak|u_x|^2 + |ku|^2)dxdy \\ &\leq C\left\{A\cdot\int_0^\infty (|h'|^2 + k|h|^2)dx + B\iint_{R^2_+} |b|^2dxdy\right\} \end{aligned} \tag{1.14}$$

where C is a constant depending only on λ and u is any solution of (1.12)-(1.8)-(1.9)-(1.10), or (1.12)-(1.8)-(1.9)-(1.11).

The (1.14) is the needed a priori inequality.

2. - Solutions of the nonhomogeneous equation.

In this section we consider the nonhomogeneous equation:

$$\operatorname{sgn}(x)u_y - u_{xx} + ku = f \qquad (k = \text{constant} > 0). \tag{2.1}$$

We shall prove the existence of solutions defined in the whole R^2, or solutions defined in $G = R^2 \setminus$(the y-axis) and verifying some conditions at $x=0$.

THEOREM 2.1. - *For every $f \in L^2(R^2)$ there exists a unique solution u of* (2.1), *belonging to $W(R^2)$. This solution verifies an inequality of the form:*

$$\iint_{R^2} (|u_y|^2 + |u_{xx}|^2 + 2k|u_x|^2 + k^2|u|^2)dxdy \leq C \iint_{R^2} |f|^2 dxdy \tag{2.2}$$

where C is an absolute constant.

THEOREM 2.2. - *Let p and q be temperate distributions; we suppose that the Fourier transforms $\widehat{p}$ and $\widehat{q}$ are complex valued measurable functions and:*

$$\inf_{-\infty<\xi<+\infty} \frac{|(k-i\xi)^{\frac{1}{2}}\widehat{p}(\xi) + (k+i\xi)^{\frac{1}{2}}\widehat{q}(\xi)|^2}{|k+i\xi|(1+|\widehat{p}(\xi)|^2)(1+|\widehat{q}(\xi)|^2)} = \frac{1}{A} > 0. \tag{2.3}$$

Then for every $f \in L^2(R^2)$ there exists a unique solution u of (2.1), *belonging to $W(G)$ and such that:*

$$u(0-, \cdot) = p * u(0+, \cdot), \quad u_x(0-, \cdot) = q * u_x(0+, \cdot). \tag{2.4}$$

This solution verifies the inequality:

$$\iint_{R^2} (|u_y|^2 + |u_{xx}|^2 + 2k|u_x|^2 + k^2|u|^2)dxdy \leq C \iint_{R^2} |f|^2 dxdy \tag{2.5}$$

where

$$C = 1 + 2A \cdot \sup_{-\infty<\xi<+\infty} \left| \widehat{p}(\xi)\overline{\widehat{q}(\xi)} - \frac{k+i\xi}{k-i\xi} \right| . \tag{2.6}$$

We recall that a temperate distribution p is a linear continuous functional on the space of all infinitely differentiable complex valued functions φ such that $\lim\limits_{t\to\pm\infty} |t|^m \varphi^{(n)}(t) = 0$, for all nonnegative integers m and n. We say that the FOURIER transform of p is a complex valued measurable function $\widehat{p}$ if :

$$<p\,|\,\widehat{\varphi}> = \int_{-\infty}^{+\infty} \widehat{p}(\xi)\varphi(\xi)d\xi \text{ for every } \varphi \in C_0^\infty(-\infty, +\infty),$$

where $<p|\widehat{\varphi}>$ is the value of p at $\widehat{\varphi}$.

REMARKS. - (i) The theorem 2.1 is a corollary of the theorem 2.2.

For the hypoteses of the theorem 2.2 are fulfilled, if we choose $p=q=\delta$, the Dirac mass. In particular (2.3) holds with $A=2$ when $p=q=\delta$. On the other hand, it is easy to see that if

$$u \in W(G),\ u(0-,\cdot)=u(0+,\cdot) \text{ and } u_x(0-,\cdot)=u_x(0+,\cdot),$$

then $u \in W(R^2)$. We recall that G denotes the R^2 cut along the y-axis.

(ii) The condition (2.3) holds with $A=2$ if p and q are related by the equation:

$$\hat{p}(\xi)\overline{\hat{q}(\xi)}=\frac{k+i\xi}{k-i\xi}, \tag{2.7}$$

whatever is the behaviour of $\hat{p}$. The proof is straighforward.

Obviously (2.7) implies $C=1$, where C is the constant of the estimate (2.5) given by (2.6). Note that (2.7) is true in the following cases, for instance:

$$p=\delta \qquad q=-\delta+2ke^{-k(\cdot)}Y$$

or:

$$p=-\delta+2ke^{k(\cdot)}(1-Y) \qquad q=\delta,$$

where Y is the Heaviside function. It is easy to see that in the first case the conditions (2.4) are equivalent to the following:

$$u(0-,y)=u(0+,y),\ u_x(0-,y)+u_x(0+,y)=2k\int_y^{+\infty}e^{k(y-t)}u_x(0-,t)dt$$

while in the second case the (2.4) become:

$$u_x(0-,y)=u_x(0+,y),\ u(0-,y)+u(0+,y)=2k\int_y^{+\infty}e^{k(y-t)}u(0-,t)dt;$$

compare with (1.9) and (1.14).

(iii) It is of some interest, e.g. in view of more general equations like (1.15), to estimate the smallest absolute constant C appearing in (2.2). We can prove that such a constant is strictly greater than 1. More precisely we have the following result:

Let C be a positive constant such that:

$$\begin{cases} \displaystyle\iint_{R^2}(|u_y|^2+|u_{xx}|^2)dxdy\le C\iint_{R^2}|\operatorname{sgn}(x)u_y-u_{xx}+ku|^2dxdy \\ \textit{for every } u\in C_0^\infty(R^2); \end{cases} \tag{2.10}$$

then:

$$(2.11)\qquad \begin{cases} \left| Re \displaystyle\iint_{R^2} \operatorname{sgn}(x) u_y \overline{u_{xx}}\, dxdy \right| \leq \\ \leq \left(1 - \dfrac{1}{C}\right) \left[\displaystyle\iint_{R^2} |u_y|^2 dxdy \iint_{R^2} |u_{xx}|^2 dxdy \right]^{\frac{1}{2}} \\ \textit{for every } u \in C_0^\infty(R^2). \end{cases}$$

Conversely, (2.11) *implies* (2.10).

Note that, if $u \in C_0^\infty(R^2)$:

$$Re \iint_{R^2} \operatorname{sgn}(x) u_y \overline{u_{xx}}\, dxdy = -2 Re \int_{-\infty}^{+\infty} u_y(0, y) \overline{u_x(0, y)}\, dy$$

Proof that (2.10) $\Rightarrow$ (2.11). - Integrations by parts and obvious arguments of dimensional analysis show that (2.10) is equivalent to:

$$(2.12)\qquad (C-1) \iint_{R^2} (t^2 |u_y|^2 + s^4 |u_{xx}|^2) dxdy +$$

$$+ C \iint_{R^2} (2ks^2 |u_x|^2 + k^2 |u|^2) dxdy - 2Cs^2 t Re \iint_{R^2} \operatorname{sgn}(x) u_y \overline{u_{xx}}\, dxdy \geq 0,$$

where s and t are arbitrary real parameters and $s > 0$. Setting $t \to +\infty$ we obtain at first $C - 1 \geq 0$. Hence minimizing with respect to t we obtain:

$$(2.13)\qquad C^2 s^4 \left(Re \iint_{R^2} \operatorname{sgn}(x) u_y \overline{u_{xx}}\, dxdy \right)^2 \leq$$

$$\leq (C-1) \iint_{R^2} |u_y|^2 dxdy \left[s^4 (C-1) \iint_{R^2} |u_{xx}|^2 dxdy + \right.$$

$$\left. + 2kCs^2 \iint_{R^2} |u_x|^2 dxdy + k^2 C \iint_{R^2} |u|^2 dxdy \right].$$

From (2.13) we obtain (2.11) setting $s \to +\infty$.

Proof that (2.11) $\Rightarrow$ (2.10). - Integrating by parts and neglecting some non negative terms we see that:

$$(2.14)\qquad \iint_{R^2} \left| \operatorname{sgn}(x) u_y - u_{xx} + ku \right|^2 dxdy \geq$$

$$\geq \iint_{2} (|u_y|^2 + |u_{xx}|^2) dxdy - 2Re \iint_{R^2} \operatorname{sgn}(x) u_y \overline{u_{xx}} dxdy;$$

for every $u \in C_0^\infty(R^2)$. Coupling (2.14) with (2.11) and the inequality $a^2 + b^2 - 2\left(1 - \frac{1}{C}\right) ab \geq \frac{1}{C}(a^2 + b^2)$. we get (2.10).

PROOF OF THEOREM 2.2. - Let f be any function in $L^2(R^2)$.

We shall call g the FOURIER transform of f with respect to the y-variable; g can be defined by the equation $g =$ limit in $L^2(R^2)$ of g_n, where

$$g_n(x, \xi) = \int_{-\infty}^{+\infty} e^{-iy\xi} f_n(x, y) dy$$

and f_n is any sequence of infinitely differentiable compactly supported functions converging to f in $L^2(R^2)$. From Parseval's formula and Fubini's theorem we have:

$$\iint_{R^2} |g(x, \xi)|^2 dxd\xi = \frac{1}{2\pi} \iint_{R^2} |f(x, y)|^2 dxdy.$$

Suppose that u is a $W(G)$-solution of (2.1)-(2.4). It is easy to see that v, the FOURIER transform of u with respect to the y-variable, has the following properties:

(2.20)
$$\begin{cases} \displaystyle\iint_{R^2} |v|^2 dxd\xi \text{ and } \iint_{R^2} |i\xi v|^2 dxd\xi \text{ are finite;} \\ v \text{ belongs to the NIKOL'SKII space } W^{(2,0)}_{(2,2)}(G), \text{ that is the function} \\ (0 \neq x, \xi) \to v(x, \xi) \text{ has square integrable generalized derivatives} \\ \text{up to the second order with respect to the } x\text{-variable;} \\ -v_{xx}(x, \xi) + (k + i\xi \operatorname{sgn}(x)) v(x, \xi) = g(x, \xi); \\ v(0-, \xi) = p(\xi) v(0+, \xi) \text{ and } v_x(0-, \xi) = q(\xi) v_x(0+, \xi). \end{cases}$$

In the last equations $v(0\,-,\ \xi)$ etc. denote traces, as usually.
Moreover:

$$\iint_{R^2} (|v_{xx}|^2 + 2k|v_x|^2 + (k^2+\xi^2)|v|^2)dxd\xi = \tag{2.21}$$

$$= \frac{1}{2\pi} \iint_{R^2} (|u_y|^2 + |u_{xx}|^2 + 2k|u_x|^2 + k^2|u|^2)dxdy.$$

Conversely if a function v has the properties (2.20) then the inverse FOURIER transform (with respect to the ξ-variable) u of v is a $W(G)$-solution of (2.1)-(2.4), and the equation (2.21) holds.

We shall prove that, under the hypothesis (2.3), the problem (2.20) has a unique solution v and this solution verifies:

$$\iint_{R^2} (|v_{xx}|^2 + 2k|v_x|^2 + (k^2+\xi^2)|v|^2)dxd\xi \leq \tag{2.22}$$

$$\leq C \iint_{R^2} |g|^2 dxd\xi$$

where C is the constant (2.6).

In fact, elementary calculations show that if a solution v of (2.20) exixts then the following representation holds:

$$v(x,\xi) = \begin{cases} -\dfrac{\exp\,[-x(k+i\xi)^{\frac{1}{2}}]}{(k+i\xi)^{\frac{1}{2}}} v_x(0+,\xi) + \displaystyle\int_{-\infty}^{+\infty} \dfrac{\exp\,[-|x-t|(k+i\xi)^{\frac{1}{2}}]}{2(k+i\xi)^{\frac{1}{2}}} g(|t|,\ \xi)dt & \text{if } x>0 \\ \dfrac{\exp\,[x(k-i\xi)^{\frac{1}{2}}]}{(k-i\xi)^{\frac{1}{2}}} v_x(0\,-,\ \xi) + \displaystyle\int_{-\infty}^{+\infty} \dfrac{\exp\,[-|x-t|(k-i)^{\frac{1}{2}}|}{2(k-i\xi)^{\frac{1}{2}}} g(-|t|,\ \xi)dt & \text{if } x>0 \end{cases} \tag{2.23}$$

and $v_x(0+,\ \xi)$, $v_x(0\,-,\ y)$ are given by:

$$\begin{cases} v_x(0+,\ \xi) = \dfrac{(k^2+\xi)^{\frac{1}{2}}}{(k-i\xi)^{\frac{1}{2}}\widehat{p}(\xi) + (k+i\xi)^{\frac{1}{2}}\widehat{q}(\xi)} h(\xi) \\ v_x(0\,-,\ \xi) = \widehat{q}(\xi) v_x(0+,\ \xi) \end{cases} \tag{2.24}$$

where:

$$h(\xi)=\left(\widehat{p}(\xi)\int\limits_0^{+\infty}\frac{\exp\left[-t(k+i\xi)^{\frac{1}{2}}\right]}{(k+i\xi)^{\frac{1}{2}}}-\int\limits_{-\infty}^{0}\frac{\exp\left[t(k-i\xi)^{\frac{1}{2}}\right]}{(k-i\xi)^{\frac{1}{2}}}\right)g(t,\ \xi)dt.$$

We point out that in this paper $(k+i\xi)^{\frac{1}{2}}$ is defined as that branch of the square root of $k+i\xi$ with positive real part. Therefore:

$$Re(k+i\xi)^{\frac{1}{2}}=\frac{1}{\sqrt{2}}(k^2+\xi^2)^{\frac{1}{2}}\left[1+k(k^2+\xi^2)^{-\frac{1}{2}}\right]^{\frac{1}{2}},$$

$$Re(k+i\xi)^{\frac{1}{2}}\geq\frac{1}{\sqrt{2}}(k^2+\xi^2)^{\frac{1}{2}}.$$

Actually (2.23)-(2.4) is a solution of (2.20). This fact can be easily verified with a straightforward inspection, using the Young inequality on convolutions and the estimate:

$$(2.26)\qquad (k^2+\xi^2)^{1/4}\left\{\begin{array}{l}|v_x(0+,\ \xi)|^2\\ \\ |v_x(0-,\ \xi)|^2\end{array}\right.\leq\frac{A}{\sqrt{2}}\int\limits_{-\infty}^{+\infty}|g(t,\ \xi)|^2dt,$$

where A is the constant given by (2.3).

To obtain (2.26) we apply the hypothesis (2.3) and the equations (2.24), hence we estimate $h(\xi)$ with the Schawartz inequality as follows:

$$|h(\xi)|^2\leq\frac{1+|\widehat{p}(\xi)|^2}{\sqrt{2}(k^2+\xi^2)^{3/4}}\int\limits_{-\infty}^{+\infty}|g(t,\ \xi)|^2dt.$$

With an analogous procedure we may find:

$$(2.28)\qquad (k^2+\xi^2)^{3/4}\left\{\begin{array}{l}|v(0+,\ \xi)|^2\\ \\ |v(0-,\ \xi)|^2\end{array}\right.\leq\sqrt{2}\,A\int\limits_{-\infty}^{+\infty}|g(t,\ \xi)|^2dt.$$

Finally, squaring both members of the differential equation $-v_{xx}+$ $+(k+i\xi\ \mathrm{sgn}\ (x))v=g$ and integrating over G, we get:

$$(2.29)\qquad \iint\limits_{R^2}(|v_{xx}|^2+2k|v_x|^2+(k^2+\xi^2)|v|^2)\,dxdy=\iint\limits_{R^2}|g|^2dxd\xi+$$

$$+2Re\int\limits_{-\infty}^{+\infty}\left[(k-i\xi)v(0-,\ \xi)\overline{v_x(0-,\ \xi)}-(k+i\xi)v(0+,\ \xi)\overline{v_x(0+,\ \xi)}\right]d\xi$$

hence (2.22) follows, in virtue of the conditions: $v(0-,\xi)=\widehat{p}(\xi)v(0+,\xi)$, $v_x(0-,\xi)=\widehat{q}(\xi)v_x(0+,\xi)$ and the estimate (2.26)-(2.28). In the derivation of (2.29) we have used the rule:

$$\int_0^\infty dx \int_{-\infty}^{+\infty} v\overline{v_{xx}}d\xi = -\int_{-\infty}^{+\infty} v(0+,\xi)\overline{v_x(0+,\xi)}d\xi - \int_0^\infty dx \int_{-\infty}^{+\infty} |v_x|^2 d\xi .$$

The proof is complete.

3. - A discussion of an integral equation of the the Wiener-Hopf type.

3.1. - In this section we consider an integral equation of the form:

$$\int_0^{+\infty} K(x-y)\varphi(y)dy = f(x) \qquad (0<x<+\infty). \tag{3.1}$$

Here K and f are given, φ is the unknown; we exphasize that the equation holds only in the half-line $0<x<+\infty$.

The following hypothesis are made on the kernel K. K is an absolutely integrable complex valued function; the Fourier transform of K can be factorized according to the formula:

$$\widehat{K}(\xi)=K_+(\xi)\cdot K_-(\xi) \text{ for every real } \xi, \tag{3.2}$$

where the functions K_+ and K_- have these properties:

(i) $\zeta=\xi+i\eta\to K_+(\zeta)$ is holomorphic in the upper half-plane $Im\zeta>0$ and continuous in the closed half-plane $Im\zeta\geq 0$, $\zeta=\xi+i\eta\to K_-(\zeta)$ is holomorphic in the lower half-plane $Im\zeta<0$ and continuous in $Im\zeta\leq 0$; (ii) some real constants p and q exist such that:

$$\begin{cases} 0<C_+=\text{const.}\leq (1+|\zeta|^2)^{\frac{p}{2}}|K_+(\zeta)|\leq \text{constant} \\ \text{for every } \zeta \text{ in } Im\zeta\geq 0 \end{cases} \tag{3.3}$$

$$\begin{cases} 0<C_-=\text{const.}\leq (1+|\zeta|^2)^{\frac{q}{2}}|K_-(\zeta)|\leq \text{constant} \\ \text{for every } \zeta \text{ in } Im\zeta\leq 0. \end{cases} \tag{3.4}$$

Clearly, the numbers p and q are restricted by $p+q>0$, since $\widehat{K}$ is bounded and tends to zero at the infinity.

The right-hand side f and the unknown φ we take in consideration belong to SOBOLEV spaces $H^s(-\infty, +\infty)$; they are also allowed to be distributions belonging to spaces $H^s(-\infty, +\infty)$ with negative indexes. In the latter case the precise meaning of the equation (3.1) is the following: spt φ, the support of φ, is contained in $[0, +\infty[$ and the support of $K * \varphi - f$ is contained in $]-\infty, 0]$.

THEOREM 3.1. - *Let the aforesaid hypotheses on* K *be fulfilled; moreover let us suppose that* $f \in H^s(-\infty, +\infty)$ *with* $s > p - \dfrac{1}{2}$.

Then a solution φ *of* (3.1) *such that:*

$$\varphi \in H^{s-r}(-\infty, +\infty) \qquad (r = p + q) \tag{3.5}$$

$$\text{spt } \varphi \subset [0, +\infty[\tag{3.6}$$

exists provided that one of the following conditions holds:

(i) $p - \dfrac{1}{2} < s < p + \dfrac{1}{2}$;

(ii) *a positive integer* n *exists such that* $n + p - \frac{1}{2} < s < n + p + \frac{1}{2}$, *and:*

$$g^{(k)}(0) = 0 \qquad (k = 0, 1, \ldots, n-1); \tag{3.7}$$

(iii) $s = p + \dfrac{1}{2}$, *and:*

$$\int_0^\varepsilon |g(x)|^2 \frac{dx}{x} < +\infty \text{ for some } \varepsilon > 0; \tag{3.8}$$

(iv) $s = p + \dfrac{1}{2} + n$ *(where* n *is a positive integer), the equations* (3.7) *are verified, and:*

$$\int_0^\varepsilon |g^{(n)}(x)|^2 \frac{dx}{x} < +\infty \text{ for some } \varepsilon > 0. \tag{3.9}$$

Here g *is the distribution defined by:*

$$\widehat{g} = -\frac{\widehat{f}}{K_+}. \tag{3.10}$$

The solution we shall find verifies the following estimates.

If $s-p-\frac{1}{2}$ *is not an integer, and either of the conditions* (i)-(ii) *holds:*

(3.11)
$$\|\varphi\|_{H^{s-r}(-\infty,+\infty)} \leq$$
$$\leq \frac{C}{C_+ C_-} \inf_{u\in C_0^\infty(]-\infty,0[)} \|f-u\|_{H^s(-\infty,+\infty)},$$

where $r=p+q$, *C is a constant depending on s only,* C_+ *and* C_- *are defined by* (3.3)-(3.4). *If* $s-p-\frac{1}{2}$ *is an integer* n, *and either of the conditions* (iii)-(iv) *holds:*

(3.12)
$$\|\varphi\|_{H^{s-r}(-\infty,+\infty)} \leq$$
$$\leq \frac{C}{C_-}\left[\frac{1}{C_+} \inf_{u\in C_0^\infty(]-\infty,0[)} \|f-u\|_{H^s(-\infty,+0)} + \right.$$
$$\left. + \left(\int_0^+ |g^{(n)}(x)|^2 \frac{dx}{x}\right)^{\frac{1}{2}}\right],$$

where C is an absolute constant.

REMARKS. - (i) The distribution g belongs to $H^{s-p}(-\infty,+\infty)$.

This fact follows immediately from (3.10) and (3.3). With regard to the conditions (3.7) and (3.9), it should be remembered that a distribution belonging to $H^t(-\infty,+\infty)$, where $t>\frac{1}{2}+m$ and $m=$ a nonnegative integer, is a m — fold continuously differentiable function. In accordance with LIONS-MAGENES [6], the condition (iv), coupled with: $g\in H^{s-p}(-\infty,+\infty)$, means that the restriction of g to $[0,+\infty[$ belongs to $H_{00}^{n+\frac{1}{2}}(0,+\infty)$.

(ii) If spt $f\subseteq\,]-\infty,0]$, then also spt $g\subseteq\,]-\infty,0]$.

This means that the restriction of g to the half-axis $]0,+\infty[$ is uniquely determined by the restriction of f to $]0,+\infty[$.

Thus the conditions (3.7)-(3.9), and the integral at the right of (3.12), do not depend on the ristriction of f to $]-\infty,0]$.

The prof is quite simple.

Let us suppose that spt $f\subseteq\,]-\infty,0]$ and consider $\langle g|u\rangle$ with a $u\in C_0^\infty(]0,+\infty])$. Here $\langle g|u\rangle$ is the value of the distribution g at u, $\int_{-\infty}^{+\infty} g(x)\overline{u(x)}dx$ if g is a square integrable function.

We have: $<g|u>=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{g}(\xi)\overline{\widehat{u}(\xi)}d\xi$, hence from (3.10) we get $<g|u>$ $=<f|u>$ where v is given by: $\widehat{v}(\xi)=\frac{\widehat{u}(\xi)}{\overline{K_+(\xi)}}$.

Clearly $v\in H^t(-\infty, +\infty)$ for every t, in consequence of (3.3).

We observe that the function $\zeta=\xi+i\eta \rightarrow 1/\overline{K_+(\xi-i\eta)}$ is holomorphic in the lower half-plane $Im\zeta<0$, continuous in $Im\zeta\leq 0$, and does not increases faster than a polynomial as $|\zeta|\rightarrow+\infty$. On the other hand, the FOURIER transform $\widehat{u}$ is an entire holomorphic function of $\zeta=\xi+i\eta$, growing less than: $(1+|\zeta|)^{-k}\exp(b\eta^+-a\eta^-)$, where k is arbitrary and $[a, b]$ is any interval containing spt u. As usually: $2\eta^+=|\eta|+\eta$, $2\eta^-=|\eta|-\eta$. Therefore $\widehat{v}$ can be continued with a function of $\zeta=\xi+i\eta$, holomorphic in $Im\zeta<0$ and continuous in $Im\zeta\leq 0$, estimated by $(1+|\zeta|)^{-k}\exp(-a|Im\zeta|)$ with an arbitrary k.

From a theorem of PALEY-WIENER, it follows; spt $v\subseteq[a, +\infty[$.

Then $<f|v>=0$, hence $<g|u>=0$.

(iii) The term:

$$\inf_{u\in C_0^\infty(]-\infty,0[)}\|f-u\|_{H^s(-\infty,+\infty)} \tag{3.13}$$

appearing in the estimates (3.11)-(3.12), is the distance of f from the closed subspace $\{u\in H^s(-\infty, +\infty)$: spt $u\subseteq]-\infty, 0]\}$. *Thus* (3.13) *does not depends on the restriction of* f *to* $]-\infty, 0]$.

For it is easy to see that $\{u\in H^s(-\infty, +\infty)$: spt $u\subseteq]-\infty, 0]\}$ is the closure in $H^s(-\infty, +\infty)$ of $C_0^\infty(]-\infty, 0[)$.

(iv) The condition $s>p-\frac{1}{2}$ and the conditions (ii) ... (iv), appearing in the statement of the theorem 3.1, guarantee that the distribution $(g\cdot Y)$, where g is defined by (3.10) and Y is the Heaviside function, belongs to $H^{s-p}(-\infty, +\infty)$. See the previous remark (i), the remark after lemma 3.7, and the lemma 3.11.

We point out that if $(g\cdot Y)$ does not belongs to $H^{s-p}(-\infty, +\infty)$ then the equation (3.1) may not have a solution φ such that: spt $\varphi\subseteq[0, +\infty[$ and $\varphi\in H^{s-r}(-\infty, +\infty)$.

Let us consider for example the equation:

$$\int_x^{+\infty} e^{k(x-y)}\varphi(y)dy=f(x) \qquad (0<x<+\infty), \tag{3.14}$$

where k is a positive constant and f is in some space $H^s(-\infty, +\infty)$. Here the kernel K is given by: $K(x) = e^{kx}$ if $x < 0$, $K(x) = 0$ if $x > 0$; we can readily see that $g = f' - kf$, where the derivative is taken in the sense of the distributions.

Let us suppose that (3.15) has a solution φ such that: $\varphi \in H^{s-1}(-\infty, +\infty)$ and spt $\varphi \subseteq [0, +\infty[$. By definition, we have: $< \varphi | e^{-k(\cdot)} Y) * u > = < f | u >$ for every $u \in C_0^\infty(]0, +\infty[)$.

It follows $< \varphi | v > = < -f' + kf | v >$ for every $v \in C_0^\infty(]0, +\infty[)$, as we can see taking $u = v' + k\, v$ in the previous equation.

Hence the restriction of $-g = -f' + kf$ to $C_0^\infty(]0, +\infty[)$ can be extended with a distribution belonging to $H^{s-1}(-\infty, +\infty)$ and supported by $[0, +\infty[$.

3.2 EXAMPLE. - We sketch now some considerations about a mixed problem for the HELMOLTZ equation, wich can be solved with the aid of the theorem 3.1.

Let $a \in H^{3/2}(-\infty, +\infty)$ and $b \in H^{1/2}(-\infty, +\infty)$ be two given functions. Assertion: a solution u of the problem

$$(3.16) \qquad \begin{cases} u_{xx} + u_{yy} - k^2 u = 0 \\ u(x, 0+) = a(x) \text{ if } 0 < x < +\infty \\ u_y(x, 0+) = b(x) \text{ if } -\infty < x < 0 \\ u \in W^{2,2}(R^2_+) \end{cases}$$

exists if and only if the following equation holds:

$$(3.18) \qquad \int_0^{+\infty} \frac{e^{-kx}}{\sqrt{|x|}} [a'(x) - ka(x)] dx = \int_{-\infty}^{0} \frac{e^{kx}}{\sqrt{|x|}} b(x) dx .$$

Here k is a positive constant, $W^{2,2}(R^2_+)$ indicates the space of all $L^2(R^2_+)$ — functions with second derivatives in $L^2(R^2_+)$.

Proof of the «if». - Let φ be a solution of the following problem:

$$(3.19) \qquad \begin{cases} \varphi \in H^{1/2}(-\infty, +\infty), \text{ spt } \varphi \subseteq [0, +\infty[, \\ \displaystyle\int_0^{+\infty} K(x - y) \varphi(y) dy = f(x) \text{ if } 0 < x < +\infty; \end{cases}$$

where:

$$\begin{cases} f(x) = -a(x) - \displaystyle\int_{-\infty}^{+\infty} K(x-y)b(y)dy, \\ \widehat{K}(\xi) = (k^2+\xi^2)^{-\frac{1}{2}}. \end{cases}$$

Incidentally, K can be expressed with a BESSEL function of the third kind, or:

$$K(x) = \frac{1}{2\pi} \int_0^{+\infty} \exp\left(-k^2 t - \frac{x^2}{4t}\right)\frac{dt}{t}.$$

It is not hardy to see that the function:

$$u(x, y) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\,\frac{\exp\left[-y(k^2+\xi^2)^{\frac{1}{2}}\right]}{(k^2+\xi^2)^{\frac{1}{2}}}\,[\widehat{b}(\xi)+\widehat{\varphi}(\xi)]d\xi$$

$$(-\infty < x < +\infty,\ y > 0)$$

is a solution of (3.16).

Claim: in consequence of the theorem 3.1, a solution of (3.19) exists provided that (3.18) holds. In fact here is $s = \frac{3}{2}$, $p = \frac{1}{2}$,

$$g(x) = \int_{-\infty}^{x} \frac{e^{k(t-x)}}{[\pi(x-t)]^{\frac{1}{2}}}\, b(t)dt - \int_{x}^{+\infty} \frac{e^{k(x-t)}}{[\pi(t-x)]^{\frac{1}{2}}}\,[a'(t) - ka(t)]dt.$$

Therefore (3.18) means exactly $g(0) = 0$.

Proof of the «only if». - Let u be a solution of (3.16) and consider the traces $\varphi = u_y(\cdot, 0+)$ and $\psi = u(\cdot, 0+)$.

From well-known theorems on SOBOLEV spaces we know that $\varphi \in H^{\frac{1}{2}}(-\infty, +\infty)$ and $\psi \in H^{3/2}(-\infty, +\infty)$. It is easy to see the formula:

$$u(x, y) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\,\frac{\exp\left[-y(k^2+\xi^2)^{\frac{1}{2}}\right]}{(k^2+\xi^2)^{\frac{1}{2}}}\,\widehat{\varphi}(\xi)d\xi$$

$$(-\infty < x < +\infty,\ y > 0).$$

Hence we have:

$$\psi(x) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}(k^2+\xi^2)^{-\frac{1}{2}}\widehat{\varphi}(\xi)d\xi,$$

or:

$$\psi(\xi) = -(k^2 + \xi^2)^{-\frac{1}{2}}\widehat{\varphi}(\xi).$$

Rewrite this formula in the form:

$$\frac{(i\xi - k)\widehat{\psi}(\xi)}{(k - i\xi)^{\frac{1}{2}}} = \frac{\widehat{\varphi}(\xi)}{(k + i\xi)^{\frac{1}{2}}}$$

and integrate over $-\infty < \xi < +\infty$ both members of the last equation.

We find:

$$\int_0^{+\infty} x^{-\frac{1}{2}} e^{-kx}[\psi'(x) - k\psi(x)]dx = \int_{-\infty}^{0} |x|^{-\frac{1}{2}} e^{kx}\varphi(x)dx$$

that is, the (3.18).

3.3. - For the proof of theorem 3.1 we need some lemmas on Cauchy integrals:

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\widehat{g}(t)}{t - \zeta}\,dt \qquad (\zeta = \xi + i\eta,\ \eta \neq 0) \tag{3.25}$$

where the density is the FOURIER transform of a distribution $g \in H^s(-\infty, +\infty)$. Clearly we shall assume $s > -\frac{1}{2}$, for otherwise the integral (3.25) woudl be divergent.

LEMMA 3.2. - *Suppose that* $g \in H^s(-\infty, +\infty)$ *and that either of the following conditions holds:* (i) $-\frac{1}{2} < s < \frac{1}{2}$; (ii) *a positive integer* n *exists such such that* $n - \frac{1}{2} < s < n + \frac{1}{2}$ *and* $g^{(k)}(0) = 0$ $(k = 0, 1, \ldots, n-1)$. *Then there exists a constant* C, *depending only on* s, *such that:*

$$|\Phi(\zeta)| \leq C\|g\|_{H^s(-\infty,+\infty)}|Im\zeta|^{-1/2}(1 + |\zeta|)^{-s} \qquad (Im\zeta \neq 0). \tag{3.26}$$

Here Φ *is given by* (3.25). *It can be proved that an absolute constant* C *exists such that:*

$$|\Phi(\zeta)| \leq C\|g\|_{H^{n+1/2}(-\infty,+\infty)} \frac{(1 + ln^+|\zeta|)^{1/2}}{|Im\zeta|^{1/2}(1+|\zeta|)^{n+1/2}} \qquad (Im\zeta \neq 0) \tag{3.27}$$

provided that $g \in H^{n+1/2}(-\infty, +\infty)$ *and either* (iii) $n = 0$, *or* (iv) n *is a positive integer and* $g^{(k)}(0) = 0$ $(k = 0, 1, \ldots, n-1)$.

Here ln^+ *denotes the positive part of the logarithm.*

LEMMA 3.3. - *Let* $g \in H^s(-\infty, +\infty)$ *and suppose that either of the conditions* (i), (ii) *of the previous lemma holds.*

Then there exists a constant C, *depending only on* s, *such that:*

$$\int_{-\infty}^{+\infty} (1 + \xi^2)^s |\Phi(\xi + i\eta)|^2 \, d\xi \leq C \|g\|^2_{H^s(-\infty, +\infty)} \qquad (\eta \neq 0); \tag{3.28}$$

moreover $\xi \to \Phi(\xi + i\eta)$, *the restriction of* Φ *to a straight line* $Im\zeta = constant \neq 0$, *converges in mean if* $0 < \eta \to 0$ *or* $0 > \eta \to 0$. *More precisely, there exist two measurable complex valued functions* $]-\infty, +\infty[\ni \xi \to \Phi_+(\xi)$ *and* $]-\infty, +\infty[\ni \xi \to \Phi_-(\xi)$, *the upper and the lower trace of* Φ *on the real axis, which have the following properties:*

$$\text{(i)} \quad \int_{-\infty}^{+\infty} (1 + \xi^2)^s |\Phi_+(\xi)|^2 d\xi \leq C \inf_{u \in C_0^\infty(]0, +\infty[)} \|g - u\|^2_{H^s(-\infty, +\infty)}$$

$$\int_{-\infty}^{+\infty} (1 + \xi^2)^s |\Phi_-(\xi)|^2 d\xi \leq C \inf_{u \in C_0^\infty(]-\infty, 0[)} \|g - u\|^2_{H^s(-\infty, +\infty)}$$

where C *is the same constant as in* (3.28);

$$\text{(ii)} \quad \int_{-\infty}^{+\infty} (1 + \xi^2)^s |\Phi(\xi + i\eta) - \Phi_\pm(\xi)|^2 d\xi \to 0 \ \textit{according as}$$

$0 < \eta \to 0$ or $0 > \eta \to 0$.

We have the equation:

$$\Phi_+ - \Phi_- = \hat{g}. \tag{3.29}$$

LEMMA 3.4. - *Let* $g \in H^{n+\frac{1}{2}}(-\infty, +\infty)$; *suppose that:*

$$\int_{-\infty}^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{|x|} < +\infty$$

and that either: (i) $n = 0$ *or* (ii) n *is a positive integer and* $g^{(k)}(0) = 0$ $(k = 0, \ldots, n-1)$.

Then there exists an absolute constant C such that:

$$\int_{-\infty}^{+\infty} (1+\xi^2)^{n+\frac{1}{2}} |\Phi(\xi+i\eta)|^2 d\xi \leq \tag{3.30}$$

$$\leq C\left(\|g\|^2_{H^{n+1/2}(-\infty,+\infty)} + \int_{-\infty}^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{|x|}\right) (\eta \neq 0);$$

moreover the limits $\Phi_+ = \underset{0<\eta\to 0}{\text{l.i.m.}}\ \Phi(\cdot + i\eta)$ *and* $\Phi_- = \underset{0>\eta\to 0}{\text{l.i.m.}}\ \Phi(\cdot + i\eta)$ *exist in* $L^2(-\infty. +\infty;\ (1+\xi^2)^{n+\frac{1}{2}} d\xi)$.

We have the estimates:

$$\int_{-\infty}^{+\infty} (1+\xi^2)^{n+\frac{1}{2}} |\Phi_+(\xi)|^2 d\xi \leq C\left[\inf_{u\in C_0^\infty(]0,+\infty[)} \|g-u\|^2_{H^{n+1/2}(-\infty,+\infty)} + \int_{-\infty}^{0} |g^{(n)}(x)|^2 \frac{dx}{|x|}\right], \tag{3.31}$$

$$\int_{-\infty}^{+\infty} (1+\xi^2)^{n+\frac{1}{2}} |\Phi_-(\xi)|^2 d\xi \leq C\left[\inf_{u\in C_0^\infty(]-\infty,0[)} \|g-u\|^2_{H^{n+1/2}(-\infty,+\infty)} + \int_{0}^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{x}\right], \tag{3.32}$$

where C is the same constant as in (3.30).

The equation holds:

$$\Phi_+ - \Phi_- = \widehat{g}. \tag{3.33}$$

REMARK. - For the application of the lemma 3.4 to the proof of the theorem 3.1 it is useful to bear in mind the following fact. If a function g is in $H^{n+\frac{1}{2}}(-\infty, +\infty)$, $g^{(k)}(0) = 0$ $(k = 0, \dots, n-1)$ and $\int_0^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{x} < +\infty$, then $\int_{-\infty}^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{|x|} < +\infty$.

This property will result clear after the reading of the lemma 3.11 and its proof.

Proof of the theorem 3.1. - Consider the Cauchy integral:

$$\Phi(\zeta)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\widehat{g}(t)}{t-\zeta}\,dt \qquad (Im\zeta \neq 0). \tag{3.34}$$

where g is defined in (3.10). Let us define two distibutions φ and ψ with the formulas:

$$\widehat{\varphi}=\frac{1}{K_-}\Phi_-, \qquad \widehat{\psi}=K_+\Phi_+ . \tag{3.35}$$

Here K_+ and K_- come from (3.2), Φ_+ and Φ_- are the traces of (3.34) on the real axis.

From the inequalities (3.3)-(3.4), the hypotheses of the theorem 3.1, the remark (i) and the lemmas 3.3-3.4, we get:

$$\begin{cases} \varphi \in H^{s-r}(-\infty, +\infty) \\ \psi \in H^{s}(-\infty, +\infty) \end{cases} \qquad (r=p+q). \tag{3.36}$$

Note that φ verifies the estimate (3.11) or (3.12). In fact we can apply the definition (3.35), the inequality (3.4) and the estimates of the lemmas 3.3-3.4. Moreover we have:

$$\inf_{v\in C_0^\infty(]-\infty,0[)}\|g-v\|_{H^{s-p}(-\infty,+\infty)} \leq \tag{3.37}$$

$$\leq \frac{1}{C_+}\inf_{u\in C_0^\infty(]-\infty,0[)}\|f-u\|_{H^{s}(-\infty,+\infty)}$$

where C_+ is given by (3.3), f and g are related by (3.10).

To prove (3.37) we fix any $u \in C_0^\infty(]-\infty, 0[)$ and start from the obvious inequality:

$$\|f-u\|_{H^{s}(-\infty,+\infty)} \geq C_+\|g-w\|_{H^{s-p}(-\infty,+\infty)}$$

where w is given by $\widehat{w}=-\widehat{u}/K_+$. Arguing as in the remark (ii) we see that $w \in H^t(-\infty, +\infty)$ and spt w is strictly contained in $]-\infty, 0[$. Since such a w can be approximated closely as please in the metric of $H^{s-p}(-\infty, +\infty)$ with $C_0^\infty(]-\infty, 0[)$-functions, we have:

$$\|g-w\|_{H^{s-p}(-\infty,+\infty)} \geq \inf_{v\in C_0^\infty(]-\infty,0])}\|g-v\|_{H^{s-p}(-\infty,+\infty)} .$$

Thus (3.37) follows.

From (3.35) and the PLEMELJ formulas (3.29) or (3.33) we get, remembering (3.2):

$$K * \varphi - f = \psi. \tag{3.33}$$

It remains to show that:

$$\operatorname{spt} \varphi \subseteq [0, +\infty[\qquad \operatorname{spt} \psi \subseteq]-\infty, 0]. \tag{3.39}$$

We shall prove the first inclusion, the proof of the second being quite similar. Let us fix any $u \in C_0^\infty(]-\infty, 0[)$ and consider $< \varphi | u >$, the value of distibution φ at u. If $s - r \geq 0$ we have $< \varphi | u > = \int_{-\infty}^{+\infty} \varphi(x)\overline{u(x)}dx$; in any case:

$$< \varphi | u > = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{\varphi}(\xi)\overline{\widehat{u}(\xi)}d\xi.$$

Hence, remembering (3.35):

$$< \varphi | u > = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_-(\xi) \frac{\overline{\widehat{u}(\xi)}}{K_-(\xi)} d\xi. \tag{3.40}$$

As is well-known, the FOURIER transform of an indefinitely differentiable compactly supported function u is an entire holomorphic function such that:

$$|\widehat{u}(\zeta)| \leq \frac{\exp\left[-a(Im\zeta)^- + b(Im\zeta)^+\right]}{1 + |\zeta|^k} \int_{-\infty}^{+\infty} (|u| + |u^{(k)}|)\, dx$$

where k is any nonnegative integer and $[a, b]$ is any interval containing spt u. Therefore, from the hypotheses on K_- made at the beginning, the function:

$$\zeta = \xi + i\eta \rightarrow U(\zeta) = \frac{\overline{\widehat{u}(\xi - i\eta)}}{K_-(\xi + i\eta)} \tag{3.41}$$

is holomorphic in the half-plane $Im\zeta < 0$, continuous im $Im\zeta \leq 0$, and verifies the estimate $U(\zeta) = 0(|\zeta|^{-k+q}e^{-b\, Im\zeta})$ for $k = 0, 1, 2, \ldots$, $Im\zeta \leq 0$ and $|\zeta| \rightarrow +\infty$. Here u is the same function appearing in (3.40); since the support of such a u is contained in $]-\infty, 0[$, we have actually:

$$U(\zeta) = 0(|\zeta|^{-k+q}) \qquad (k = 0, 1, \ldots; \; Im\zeta \leq 0, \; |\zeta| \rightarrow +\infty). \tag{3.42}$$

With an application of the Cauchy integral theorem we obtain:

$$(3.34)\qquad \int_{-(R^2-\eta^2)^{1/2}}^{+(R^2-\eta^2)^{1/2}} \Phi(\xi+i\eta)U(\xi+i\eta)d\xi =$$

$$= \int_{-\pi+\text{arcsen}(\eta/R)}^{\text{arcsen}(\eta/R)} \Phi(Re^{i\theta})U(Re^{i\theta})Re^{i\theta}\, i d\theta,$$

where R is any positive constant and $0>\eta>-R$.

Using (ii) of lemma 3.3, or the lemma 3.4, it is a simple matter to show that:

$$(3.44)\qquad \lim_{0>\eta\to 0} \int_{-(R^2-\eta^2)^{1/2}}^{+(R^2-\eta^2)^{1/2}} \Phi(\xi+i\eta)U(\xi+i\eta)d\xi =$$

$$= \int_{-R}^{+R} \Phi_-(\xi)\, \frac{\overline{\widehat{u}(\xi)}}{K_-(\xi)}\, d\xi .$$

From (3.40)-(3.43)-(3.44) we infer:

$$(3.45)\qquad <\varphi\,|\,u> = \frac{1}{2\pi}\lim_{R\to+\infty} \int_{-\pi}^{0} \Phi(Re^{i\theta})U(Re^{i\theta})Re^{i\theta}id\theta .$$

But $|U(Re^{i\theta})| \leq (\text{const.})\, R^{-k}$ for every $k>0$ and $0\geq\theta\geq-\pi$, in consequence of (3.42); moreover:

$$|\Phi(Re^{i\theta})| \leq (\text{constant})\ |\text{sen}^\theta|^{-1/2} R^{-s+p-1/2}(1+ln^+R)^{1/2}$$

in virtue of lemma 3.2. Thus the right-hand side of (3.45) vanishes, so $<\varphi\,|\,u> = 0$. For the arbitraryness of u, we conclude spt $\varphi \subseteq [0, +\infty[$.

The proof is complete.

3.4. **Proof of the lemma 3.2.** - Let g be in $H^s(-\infty, +\infty)$ and consider the integral (3.25).

From the SHWARTZ inequality we get:

$$(3.47)\qquad |\Phi(\zeta)| \leq \|g\|_{H^s(-\infty,+\infty)} \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^s\,|t-\zeta|^2}\right]^{\frac{1}{2}} .$$

We estimate the right-hand side of (3.47) at first in the case $-\frac{1}{2}<s<\frac{1}{2}$. For convenience we shall write:

$$\zeta=\xi+i\eta \ (\eta \neq 0) \text{ or } \zeta=re^{i\theta}(r>0, \ \frac{\theta}{\pi} \text{ not integer}).$$

Suppose that $0 \leq s < \frac{1}{2}$. Then we fix k, $0<k<1$, and write:

$$\int_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^s|t-\zeta|^2}=\left(\int_{-\infty}^{-kr}+\int_{kr}^{+\infty}\right) \dots dt + \tag{3.48}$$

$$+\int_{-kr}^{0}\frac{(1+t^2)^{-s}dt}{(t+r)^2-4rt\cos^2\frac{\theta}{2}}+\int_{0}^{kr}\frac{(1+t^2)^{-s}dt}{(t-r)^2+4rt\operatorname{sen}^2\frac{\theta}{2}} \leq$$

$$\leq (1+k^2r^2)^{-s}\int_{-\infty}^{+\infty}\frac{dt}{|t-\zeta|^2}+2\int^{kr}\frac{(1+t^2)^{-s}}{(r-t)^2}dt=$$

$$=(1+k^2r^2)^{-s}\frac{\pi}{|\eta|}+2\int_0^{kr}\frac{1}{t^{2s}(r-t)^2}\left[\frac{t^2}{1+t^2}\right]^s dt \leq$$

$$\leq \dots + 2\left[\frac{k^2r^2}{1+k^2r^2}\right]^s\int_0^{kr}\frac{dt}{t^{2s}(r-t)^2}=$$

$$=\frac{1}{|\eta|}(1+k^2r^2)^{-s}\left[\pi+2k^{2s}|\operatorname{sen}\theta|\int_0^k\frac{dt}{t^{2s}(1-t)^2}\right] \leq$$

$$\leq (\text{constant})\frac{1}{|\eta|}(1+k^2r^2)^{-s} \qquad \left(0 \leq s<\frac{1}{2}, \ 0<k<1\right).$$

Suppose that $-\frac{1}{2}<s\leq 0$. Then we fix $k>1$ and write:

$$\int_{-\infty}^{+\infty}\frac{dt}{(1+t^2)^s|t-\zeta|^2}=\int_{-kr}^{+kr}\dots dt + \tag{3.49}$$

$$+\int_{-\infty}^{-kr}\frac{(1+t^2)^{-s}dt}{(t+r)^2-4rt\cos^2\frac{\theta}{2}}+\int_{kr}^{+\infty}\frac{(1+t^2)^{-s}dt}{(t-r)^2+4rt\operatorname{sen}^2\frac{\theta}{2}} \leq$$

$$\leq (1+k^2r^2)^{-s} \int_{-\infty}^{+\infty} \frac{dt}{|t-\zeta|^2} + 2 \int_{kr}^{+\infty} \frac{(1-t^2)^{-s}}{(t-r)^2}\, dt =$$

$$= (1+k^2r^2)^{-s} \frac{\pi}{|\eta|} + 2 \int_{kr}^{+\infty} \frac{1}{t^{2s}(t-r)^2}\left(1+\frac{1}{t^2}\right)^{-s} dt \leq$$

$$\leq \ \ldots + 2\left(1+\frac{1}{k^2r^2}\right)^{-s} \int_{kr}^{+\infty} \frac{dt}{t^{2s}(t-r)^2} =$$

$$= \frac{1}{|\eta|}(1+k^2r^2)^{-s}\left[\pi + 2k^{2s}|\operatorname{sen}\theta| \int_{k}^{+\infty} \frac{dt}{t^{2s}(t-1)^2}\right] \leq$$

$$\leq (\text{constant}) \frac{1}{|\eta|}(1+k^2r^2)^{-s}\left(-\frac{1}{2} < s \leq 0,\ k > 1\right).$$

From (3.47)-(3.48)-(3.49) the (3.26) follows, at least if $-\frac{1}{2} < s < \frac{1}{2}$.

We estimate now the right-hand side of (3.47) in the case $s = \frac{1}{2}$.

With the aid of the change of variable $t \to t + (t^2+1)^{\frac{1}{2}}$, we find:

$$(3.50) \qquad \int_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^{\frac{1}{2}}|t-\zeta|^2} =$$

$$= \frac{2}{Im\zeta}\, Im \int_0^{+\infty} \frac{dt}{t^2 - 2\zeta t - 1} =$$

$$= \frac{2}{Im\zeta}\, Im\left[(1+\zeta^2)^{-\frac{1}{2}}\left(\log\frac{1}{|w|} + i \operatorname{arctg}\frac{Re\, w}{Im\, w}\right)\right]$$

$$(Im\zeta \neq 0),$$

where $w = \zeta + (1+\zeta^2)^{\frac{1}{2}}$, and $(1+\zeta^2)^{\frac{1}{2}}$ is the square root of $1+\zeta^2$ with positive real part.

From (3.47) and (3.50) we obtain the (3.27), at least if $n = 0$.

Thus we have proved (3.26) in the case $-\frac{1}{2} < s < \frac{1}{2}$ and (3.27) in the case $n = 0$. To conclude the proof we can apply the following lemma.

LEMMA 3.5. - *Suppose that* $g \in H^s(-\infty, +\infty)$ *and* $s > n - \frac{1}{2}$, *where* n *is a positive integer; moreover let* $g^{(k)}(0) = 0$ *if* $k = 0, 1, \ldots, n-1$. *Then:*

$$(i\zeta)^n \Phi(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{1}{t-\zeta} \left(\frac{d^n}{dx^n} g\right)^{\frown}(t)dt$$

when Φ *is given by* (3.25).

PROOF.

$$(i\zeta)^n \Phi(\zeta) - \int_{-\infty}^{+\infty} (it)^n \widehat{g}(t)/2\pi i(t-\zeta)dt =$$

$$= \frac{1}{2\pi} \sum_{k=0}^{n-1} (i\zeta)^{n-k-1} \int_{-\infty}^{+\infty} (it)^k \widehat{g}(t)dt =$$

$$= \sum_{k=0}^{n-1} (i\zeta)^{n-k-1} g^{(k)}(0).$$

3.5. - For the proof of lemmas 3.3 and 3.4 we need some other lemmas.

LEMMA 3.6. - *Let* u *be any function in* $L^2(-\infty, +\infty)$ *and consider the Cauchy integral* $\Psi(z) = \int_{-\infty}^{+\infty} u(t)dt/2\pi i(t-z)$, *where* $z = x + iy$ *and* $y \neq 0$. *The following properties hold:*

$$\text{(i)} \quad \Psi(x+iy) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{|y|}{(x-t)^2 + y^2} \left[\frac{1}{2} \operatorname{sgn}(y) u(t) + \frac{i}{2} (Hu)(t)\right] dt$$

for every $y \neq 0$, *where* H *denotes Hilbert's transform.* Note that the right-hand side is the Poisson integral of the function in square brackets.

$$\text{(ii)} \quad \Psi(z) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{+\infty} e^{iz\xi} \widehat{u}(\xi) d\xi & \textit{if} \ \ Imz > 0 \\[2ex] -\dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{0} e^{iz\xi} \widehat{u}(\xi) d\xi & \textit{if} \ \ Imz > 0 \end{cases}$$

$$\text{(iii)} \quad \int_{-\infty}^{+\infty} |\Psi(x+iy)|^2 dx \leq \int_{-\infty}^{+\infty} |u(t)|^2 dt \ \ \textit{for every} \ \ y \neq 0$$

(iv) $\int_{-\infty}^{+\infty} \left| \Psi(x+iy) - \left[\frac{1}{2} \operatorname{sgn}(y) u(t) + \frac{i}{2} (Hu)(t)\right]\right|^2 dt \to 0$ *if*

$0 < y \to 0$ *or* $0 > y \to 0$.

REMARK. - As is well-known, the HILBERT transformation is defined by:

$$(Hu)(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u(t)}{x-t}\, dt, \tag{3.51}$$

where the integral is taken in the CAUCHY principal value sense.

Suppose that u is a *locally* square integrable function and

$$\left(\int_{-\infty}^{-k} + \int_{k}^{+\infty} \right) |x|^{2s} |u(x)|^2 \, dx < +\infty \text{ for every } k > 0$$

and for some $s > -\frac{1}{2}$. Then the precise meaning of the equation (3.51) is the following:

$$\begin{cases} \varphi(x)(Hu)(x) = \underset{0<r\to 0}{\text{l.i.m.}}\ \varphi(x) \left(\int_{-\infty}^{x-r} + \int_{x+r}^{+\infty} \right) \dfrac{u(t)}{\pi(x-t)}\, dt \\ \text{for every bounded compactly supported function } \varphi. \end{cases} \tag{3.53}$$

where l.i.m. stands for limit in $L^2(-\infty, +\infty)$. We recall that if $u \in L^2(-\infty, +\infty)$ the following equations hold:

$$(Hu)(x) = \underset{0<r\to 0}{\text{l.i.m.}} \left(\int_{-\infty}^{x-r} + \int_{x+r}^{+\infty} \right) \frac{u(t)}{\pi(x-t)}\, dt, \tag{3.53}$$

$$\int_{-\infty}^{+\infty} |(Hu(x)|^2 \, dx = \int_{-\infty}^{+\infty} |u(x)|^2 \, dx \tag{3.54}$$

$$(Hu)^{\frown}(\xi) = -i \operatorname{sgn}(\xi) \cdot \widehat{u}(\xi). \tag{3.55}$$

Proof of the lemma 3.6. - The (i) is a consequence of the equations:

$$\frac{1}{i(t-z)} = \frac{y}{(x-t)^2+y^2} + i\frac{x-t}{(x-t)^2+y^2}$$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x-t}{(x-t)^2+y^2}u(t)dt = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{|y|}{(x-t)^2+y^2}(Hu)(t)dt$$

$$(y \neq 0).$$

The last equation follows the fact that, for every fixed $y \neq 0$ the function $x \to x/\pi(x^2+y^2)$ is the HILBERT transform of the POISSON kernel $x \to |y|/\pi(x^2+y^2)$; the symmetry property of the HILBERT transformation:

$$\int_{-\infty}^{+\infty}(Hu)(x)v(x)dx = -\int_{-\infty}^{+\infty}u(x)(Hv)(x)dx$$

$$u,\ v \in L^2(-\infty, +\infty)$$

should also be used.

The (ii) is a consequence of the formulas:

$$\frac{1}{i(t-z)} = \begin{cases} \displaystyle\int_0^{+} e^{-i\xi(t-z)}d\xi & \text{if } Im z > 0 \\ \displaystyle -\int_{-\infty}^{0} e^{-i\xi(t-z)}d\xi & \text{if } Im z < 0 \end{cases}$$

and the « transfert theorem » on FOURIER transforms.

(iii) follows from (ii) and Perseval's theorem.

(iv) follows form (i) and well known properties of the POISSON integral.

LEMMA 3.7. - *Let u be a measurable function such that:*

$$\int_{-\infty}^{+\infty}(1+x^2)^s|u(x)|^2dx < +\infty \quad \textit{for some real constant } s.$$

Assertion: *if* $-\frac{1}{2} < s < +\frac{1}{2}$, *the Hilbert transform Hu of u verifies the inequality*:

$$\int_{-\infty}^{+\infty}(1+x^2)^s|Hu(x)|^2dx \leq (1+C)^2\int_{-\infty}^{+\infty}(1+x^2)^s|u(x)|^2dx$$

where:

$$C = 4 \int_0^1 \frac{1 - t^{4|s|}}{1 - t^4} t^{-2|s|} dt. \tag{3.56}$$

REMARK. - Lemma 3.7. enables us to prove very easily the following theorem on SOBOLEV spaces. *If* $-\frac{1}{2} < s < +\frac{1}{2}$, *the multiplication by the Heaviside function* Y *is a bounded operator in* $H^s(-\infty, +\infty)$, *whose norm not exceedes* $1 + \frac{C}{2}$. Here C is the constant indicated by the last formula; recall that $Y(x) = 1$ if $x > 0$, $Y(x) = 0$ if $x < 0$. In fact, let $u \in C_0^\infty(-\infty, +\infty)$. In consequence of (3.55) we have: $(uY)^\frown = \frac{1}{2}\widehat{u} - \frac{i}{2}\widehat{Hu}$, for $\frac{1}{2} + \frac{1}{2}\operatorname{sgn}(x) = Y(x)$; then from MINKOWSKI's inequality and lemma 3.7 we get:

$$\|uY\|_{H^s(-\infty,+\infty)} = \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} (1+\xi^2)^s \left|\frac{1}{2}\widehat{u}(\xi) - \frac{i}{2}(\widehat{Hu})(\xi)\right|^2 d\xi\right]^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{2}\|u\|_{H^s(-\infty,+\infty)} + \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}(1+\xi^2)^s|(\widehat{Hu})(\xi)|^2 d\xi\right]^{\frac{1}{2}} \leq$$

$$\leq \left(1 + \frac{C}{2}\right)\|u\|_{H^s(-\infty,+\infty)}.$$

Proof of the lemma 3.7. - We have

$$(1+x^2)^{s/2}|(Hu)(x)| \leq \left|\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(1+t^2)^{s/2}u(t)}{x-t}dt\right| +$$

$$+ \frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\frac{(1+x^2)^{s/2} - (1+t^2)^{s/2}}{x-t}\right| |u(t)|\, dt.$$

The first term at the right is the HILBERT transform of a square integrable function, so it can be estimated with the equation (3.54).

The second term can be estimated using the following lemmas.

LEMMA 3.8. - *Let s, t, x be real numbers. The following inequalities hold:*

$$|(1+x^2)^{s/2} - (1+t^2)^{s/2}| \leq \left|1 - \left|\frac{x}{t}\right|^s\right| (1+t^2)^{s/2} \tag{3.57}$$

$$(1+x^2)^{s/2} \leq \left(\frac{1}{2}|x-t| + \sqrt{1 + \frac{1}{4}(x-t)^2}\right)^{|s|}(1+t^2)^{s/2}. \tag{3.58}$$

LEMMA 3.9. - *Let $u \in L^2(-\infty, +\infty)$, u nonnegative, and consider:*

$$v(x) = \int_{-\infty}^{+\infty} \left| \frac{1 - \left|\frac{x}{t}\right|^s}{x - t} \right| u(t)dt$$

where $-\frac{1}{2} < s < \frac{1}{2}$. The following inequality holds:

$$\int_{-\infty}^{+\infty} v(x)^2 dx \leq C^2 \int_{-\infty}^{+\infty} u(x)^2 dx ,$$

where C is given by (3.56).

Proof of the lemma 3.8. - The (3.58) is a refinement of PEETRE's inequality, which will be useful later. For sake of brevity we prove here the (3.57) only. Consider the function:

$$f(\lambda) = 1 - \left(\frac{1 + a\lambda}{1 + \lambda}\right)^{s/2}$$

where $a = \text{constant} \geq 0$ and the variable λ runs in $0 \leq \lambda < +\infty$.

It is easy to see that $f(\lambda)$ is monotone increasing if $s \geq 0$ and $0 \leq a \leq 1$ or if $s \leq 0$ and $a \geq 1$; $f(\lambda)$ is monotone decreasing if $s \geq 0$ and $a \geq 1$ or if $s \leq 0$ and $0 \leq a \leq 1$.

As $f(0) = 0$ and $f(+\infty) = 1 - a^{s/2}$, we have then:

$$|f(\lambda)| \leq |1 - a^{s/2}| \quad \text{for every} \quad \lambda \geq 0.$$

Putting in this inequality $\lambda = t^2$, $a = x^2/t^2$ we obtain (3.57).

Proof of the lemma 3.9. - Let as consider the kernel:

$$N(x, t) = \left| \frac{1 - \left|\frac{x}{t}\right|^s}{x - t} \right| .$$

With some manipulations one can see that:

$$\int_{-\infty}^{+\infty} N(x, 1)\, |x|^{-\frac{1}{2}} dx = \int_{-\infty}^{+\infty} N(1, t)\, |t|^{-\frac{1}{2}} dt = C$$

where C is the constant (3.56). As N is positively homogeneous of degre -1, it follows:

$$\int_{-\infty}^{+\infty} N(x,\ t)\,|x|^{-\frac{1}{2}}\,dx = C\,|t|^{-\frac{1}{2}} \qquad (t \neq 0)$$

$$\int_{-\infty}^{+\infty} N(x,\ t)\,|t|^{-\frac{1}{2}}\,dt = C\,|x|^{-\frac{1}{2}} \qquad (x \neq 0).$$

From SCHWARTZ inequality, and the previous formulas, we get:

$$v(x)^2 \leq \int_{-\infty}^{+\infty} N(x,\ t)\,|t|^{-\frac{1}{2}}\,dt \cdot \int_{-\infty}^{+\infty} N(x,\ t)\,|t|^{\frac{1}{2}}\,u(t)^2 dt =$$

$$= C \int_{-\infty}^{+\infty} N(x,\ t)\,|x|^{-\frac{1}{2}}\,|t|^{\frac{1}{2}}\,|u(t)|^2\,dt.$$

Hence:

$$\int_{-\infty}^{+\infty} v(x)^2\,dx \leq C \int_{-\infty}^{+\infty} |t|^{\frac{1}{2}}\,u(t)^2 dt \int_{-\infty}^{+\infty} N(x,\ t)\,|x|^{-\frac{1}{2}}\,dx =$$

$$= C^2 \int_{-\infty}^{+\infty} u(t)^2\,dt.$$

This proof has been leaded from [13], theorem 3.19.

LEMMA 3.10. - *Let P be a nonnegative integrable function such that:*

$$\int_{-\infty}^{+\infty} P(x)dx = 1,\ A = \int_{-\infty}^{+\infty} (1 + x^2)^{|s|/2}\,P(x)dx < +\infty;$$

where s is some real constant. Consider the convolution:

$$v_r(x) = \int_{-\infty}^{+\infty} \frac{1}{r}\,P\Big(\frac{x-t}{r}\Big)\,u(t)\,dt$$

where r *is a positive parameter and* u *is a measurable function such that:*

$$\int_{-\infty}^{+\infty} (1+x^2)^s |u(x)|^2 \, dx < +\infty .$$

The following properties are true:

$$\text{(i)} \quad \int_{-\infty}^{+\infty} (1+x^2)^s |v_r(x)|^2 \, dx \le A^2 (r^2+4)^{|s|} \int_{-\infty}^{+\infty} (1+x^2)^s |u(x)|^2 \, dx$$

$$\text{(ii)} \quad \int_{-\infty}^{+\infty} (1+x^2)^s |v_r(x) - u(x)|^2 \, dx \to 0 \quad \textit{if} \quad 0 < r \to 0.$$

This lemma is a very slight variant of standard theorems on convolutions. The (i) follows immediately from the Peetre inequality (3.58) and from Young's theorem. The (ii) is trivial if P and u have compact supports, for in this case also v_r has compact support. In the other cases the (ii) can be easily proved by approximating P and u with compactly supported functions.

Proof of the lemma 3.3. - Let g be in $H^s(-\infty, +\infty)$ and consider the Cauchy integral (3.25).

Suppose at first $-\dfrac{1}{2} < s < \dfrac{1}{2}$.

We can write:

$$(1+\xi^2)^{s/2}\Phi(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{(1+t^2)^{s/2}\hat{g}(t)}{t-\zeta} \, dt + \\ + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{(1+\xi^2)^{s/2} - (1+t^2)^{s/2}}{t-\zeta} \hat{g}(t) dt,$$

then, applying (3.57) and the obvious inequality $|t-\zeta| \ge |t-\xi|$, we obtain:

$$(1+\xi^2)^{s/2} |\Phi(\zeta)| \le \left| \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{(1+t^2)^{s/2}\hat{g}(t)}{t-\zeta} \, dt \right| + \\ + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{1-|\xi/t|^s}{\xi - t} \right| (1+t^2)^{s/2} |\hat{g}(t)| \, dt.$$

The first term at the right can be estimated with the (iii) of lemma 3.6, the second with the lemma 3.9. Then:

$$\int_{-\infty}^{+\infty} (1+\xi^2)^s |\Phi(\xi+i\eta)|^2 d\xi \leq (1+C)^2 \int_{-\infty}^{+\infty} (1+t^2)^s |\widehat{g}(t)|^2 dt$$

where C is given by (3.56).

The (3.28) is so proved, at least if $-\frac{1}{2} < s < \frac{1}{2}$.

We define:

$$\begin{matrix}\Phi_+ \\ \\ \Phi_-\end{matrix} = \left\{\begin{matrix} +\frac{1}{2}\widehat{g} & \\ & +\frac{i}{2}H\widehat{g} \\ -\frac{1}{2}\widehat{g} & \end{matrix}\right. \tag{3.50}$$

where H denotes HILBERT transform, compare with (3.52).

Note that, taking into account the remark after lemma 3.7, we have:

$$\Phi_- = -(gY)^\frown \qquad \Phi_+ = (g - gY)^\frown, \tag{3.59$'$}$$

where Y is the Heaviside function.

Obviously;

$$\Phi_+ - \Phi_- = \widehat{g}.$$

From (3.59)$'$ we deduce:

$$\left\{\begin{aligned} &\Phi_+ = \frac{1}{2}(g-u)^\frown + \frac{i}{2}H(g-u)^\frown \\ &\text{for every } u \in C_0^\infty(]0, +\infty[) \end{aligned}\right. \tag{3.60}$$

$$\left\{\begin{aligned} &\Phi_- = -\frac{1}{2}(g-u)^\frown + \frac{i}{2}H(g-u)^\frown \\ &\text{for every } u \in C_0^\infty(]-\infty, 0[). \end{aligned}\right.$$

From (3.60) and lemma 3.7 we get:

$$\int_{-\infty}^{+\infty} (1+\xi^2)^s |\Phi_+(\xi)|^2 d\xi \leq \left(1+\frac{C}{2}\right)^2 \inf_{u \in C_0^\infty(]0, +\infty[)} \|g-u\|^2_{H^s(-\infty, +\infty)}$$

(3.61)

$$\int_{-\infty}^{+\infty}(1+\xi^2)^s\,|\Phi_-(\xi)|^2 d\xi \leq \left(1+\frac{C}{2}\right)^2 \inf_{u\in C_0^\infty(]-\infty,\,0[)} \|g-u\|^2_{H^s(-\infty,\,+\infty)}$$

where C is the constant (3.56).

Let us admit for a moment the formula:

$$\Phi(\xi+i\eta)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{|\eta|}{(\xi-t)^2+\eta^2}\begin{cases}\Phi_+(t) & \text{if } \eta>0\\ & dt\\ \Phi_-(t) & \text{if } \eta<0.\end{cases} \tag{3.62}$$

The from (3.61), (3.62) and the lemma 3.10 we get:

$$\int_{-\infty}^{+\infty}(1+\xi^2)^s\left|\Phi(\xi+i\eta)-\begin{matrix}\Phi_+(\xi)\\ \\ \Phi_-(\xi)\end{matrix}\right|^2 d\xi\to 0\quad\begin{matrix}\text{if } 0<\eta\to 0\\ \\ \text{if } 0>\eta\to 0.\end{matrix}$$

So the properties asserted in the statement of the lemma 3.3 are proved, at least if $-\frac{1}{2}<s<+\frac{1}{2}$.

The formula (3.62) follows immediately from lemma 3.6 if $s\geq 0$, that is if $\widehat{g}$ is square integrable. In any case, if $-\frac{1}{2}<s<\frac{1}{2}$ the formula can be established with the following argument. Let g_n be any sequance of infinitely differentiable compactly supported functions converging to g in $H^s(-\infty,+\infty)$, and consider

$$\Phi_n(\xi)=\int_{-\infty}^{+\infty}\widehat{g}_n(t)dt/2\pi i(t-\zeta).$$

The lemma 3.6, or even the classical PLEMELJ formulas, gives to us:

$$\Phi_n(\xi+i\eta)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{|\eta|}{(\xi-t)^2+\eta^2}\begin{cases}(\Phi_n)_+(t) & \text{if } \eta>0\\ & dt\\ (\Phi_n)_-(t) & \text{if } \eta<0\end{cases} \tag{3.63}$$

where $(\Phi_n)_\pm$ are defined as in (3.59). In virtue of lemma 3.2, $\Phi_n(\zeta)$ converges to $\Phi(\zeta)$ for every fixed ζ with $Im\zeta\neq 0$.

Using the SCHWARTZ inequality, it is easy to see that the right-hand side of (3.63) converges to the rigth-hand side of (3.62) for every fixed non real ζ, because:

$$\int_{-\infty}^{+\infty} (1+\xi^2)^s \,|\, \Phi_{\pm}(\xi) - (\Phi_n)_{\pm}(\xi)\,|^2 \, d\xi \leq$$

$$\leq \left(1+\frac{C}{2}\right)^2 \|g - g_n\|^2_{H^s(-\infty,\,+\infty)} \to 0$$

in virtue of lemma 3.7. Then (3.63) implies (3.62).

Suppose now that $n - \frac{1}{2} < s < n + \frac{1}{2}$ and $g^{(k)}(0) = 0$ $(k = 0, \ldots, n-1)$, where n is a positive integer.

It is convenient to write:

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{(g-u)^{\frown}(t)}{t-\zeta}\, dt, \tag{3.64}$$

where u is any $C_0^\infty(-\infty, +\infty)$-function whose support is contained in $]-\infty, 0[$ or in $]0, +\infty[$ according as $Im\zeta < 0$ or $Im\zeta > 0$. The (3.64) follows from:

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\widehat{u}(t)}{t-\zeta}\, dt = \begin{cases} \displaystyle\int_{-\infty}^{0} e^{-ix\zeta} u(x)dx & \text{if } \; Im\zeta > 0 \\ \displaystyle -\int_{0}^{+\infty} e^{-ix\zeta} u(x)dx & \text{if } \; Im\zeta < 0, \end{cases}$$

a consequence of (ii), lemma 3.6.

Consider the analytic function:

$$\Psi(\zeta) = (i\zeta - \operatorname{sgn}(Im\zeta))^n \Phi(\zeta) \qquad (Im\zeta \neq 0).$$

From (3.64) and lemma 3.5 we infer:

$$\Psi(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{1}{t-\zeta}\begin{cases} h_+(t) & \text{if } Im\zeta > 0 \\ h_-(t) & \text{if } Im\zeta < 0 \end{cases} dt$$

where:

$$h_+ = \left(\frac{d}{dx} - 1\right)^n (g - u_+), \quad h_- = \left(\frac{d}{dx} + 1\right)^n (g - u_-)$$

and u_+, u_- are arbitrary functions belonging to $C_0^\infty(]0, +\infty[)$ and $C_0^\infty(]-\infty, 0[)$ respectively.

Clearly:

$$h_\pm \in H^{s-n}(-\infty, +\infty), \; \|h_\pm\|_{H^{s-n}(-\infty,+\infty)} =$$

$$= \|g - u_\pm\|_{H^s(-\infty,+\infty)}.$$

Note that:

$$-\frac{1}{2} < s - n < +\frac{1}{2}.$$

Applying to the function Ψ the arguments developed before, we can easily obtain the desired properties of Φ.

The proof is complete.

3.6. - An easy proof of the lemma 3.4 is based on the following lemma.

Lemma 3.11. - *Let* $g \in H^{n+1/2}(-\infty, +\infty)$ *and suppose that either of the following conditions holds:* (i) $n = 0$ *and* $\int_0^\varepsilon |g(x)|^2 \frac{dx}{x} < +\infty$ *for some* $\varepsilon > 0$; (i) *n is a positive integer,* $g^{(k)}(0) = 0$ $(k = 0, \ldots, n-1)$ *and* $\int_0^\varepsilon |g^{(n)}(x)|^2 \frac{dx}{x} < +\infty$ *for some* $\varepsilon > 0$.

Assertion: $(gY) \in H^{n+1/2}(-\infty, +\infty)$ *and the estimate holds:*

$$\|g(Y)\|_{H^{n+1/2}(-\infty,+\infty)} \leq \tag{3.70}$$

$$\leq C\left[\inf_{u \in C_0^\infty(]-\infty,0[)} \|g - u\|^2_{H^{n+1/2}(-\infty,+\infty)} + \right.$$

$$\left. + \int_0^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{x}\right]^{1/2}$$

where C is an absolute constant. Here Y is the Heaviside function.

Proof. - Consider for istance the case (ii). Clearly $(gY) \in H^n(-\infty, +\infty)$, in virtue of the conditions: $g(0) = \ldots g^{(n-1)}(0) = 0$.

Using a representation of the $H^{1/2}$ — norm involving the differential quotient, we can verify that the conditions $g^{(n)} \in H^{1/2}(-\infty, +\infty)$ and

$$\int_0^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{x} < +\infty \quad \text{imply} \quad g^{(n)} \cdot Y \in H^{1/2}(-\infty, +\infty).$$

For more details see Lions-Magenes [6], chapter 1, sections 11.2 — 11.3.

Proof of lemma 3.4. - Let g be in $H^{n+\frac{1}{2}}(-\infty, +\infty)$ and consider the CAUCHY integral (3.25). We suppose that either of the conditions (i) and (ii) of the previous lemma holds.

Let us define: $\Phi_- = \frac{-1}{2}\widehat{g} + \frac{i}{2}H\widehat{g}$, $\Phi_+ = \frac{1}{2}\widehat{g} + \frac{i}{2}H\widehat{g}$, where H denotes the HILBERT's transform. Equivalently:

$$\Phi_- = -(gY)^\frown \qquad \Phi_+ = (g - gY)^\frown, \tag{3.71}$$

for $\frac{1}{2} + \frac{1}{2}\operatorname{sgn}(x) = Y(x)$. Here Y is the Heaviside function and the formula (3.55) should be used.

From the lemma 3.11 we infer:

$$\int_{-\infty}^{+\infty} (1 + \xi^2)^{n+\frac{1}{2}} |\Phi_\pm(\xi)|^2 d\xi < +\infty,$$

more precisely

$$\begin{aligned} \int_{-\infty}^{+\infty} (1 + \xi^2)^{n+\frac{1}{2}} |\Phi_-(\xi)|^2 d\xi &\leq C \inf_{u \in C_0^\infty(]-\infty, 0[)} \|g - u\|^2_{H^{n+\frac{1}{2}}(-\infty, +\infty)} \\ &+ C \int_{-\infty}^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{x} \end{aligned} \tag{3.72}$$

$$\begin{aligned} \int_{-\infty}^{+\infty} (1 + \xi^2)^{n+\frac{1}{2}} |\Phi_+(\xi)|^2 d\xi &\leq C \inf_{u \in C_0^\infty(]0, +\infty[)} \|g - u\|^2_{H^{n+\frac{1}{2}}(-\infty, +\infty)} \\ &+ C \int_{-\infty}^{0} |g^{(n)}(x)|^2 \frac{dx}{|x|} \end{aligned}$$

where C is an absolute constant.

Obviously:

$$\Phi_+ - \Phi_- = \widehat{g}.$$

We show later that

$$\left\{ \begin{matrix} (1 + i\zeta)^{n+\frac{1}{2}} \\ \\ (1 - i\zeta)^{n+\frac{1}{2}} \end{matrix} \right. \Phi(\zeta) = \tag{3.73}$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{|\eta|}{(\xi - t)^2 + \eta^2} \left\{ \begin{matrix} (1 + it)^{n+\frac{1}{2}} \Phi_-(t) \\ \\ (1 - it)^{n+\frac{1}{2}} \Phi_+(t) \end{matrix} \right. dt \qquad \begin{matrix} \zeta = \xi + i\eta \text{ and } \eta < 0 \\ \\ \zeta = \xi + i\eta \text{ and } \eta > 0 \end{matrix}$$

From (3.72), (3.73) and the obvious inequality $|1-i\zeta \operatorname{sgn}(Im\zeta)|^2 \geq 1+(Re\zeta)^2$, we obtain for every $\eta \neq 0$:

$$(3.74)\qquad \int_{-\infty}^{+\infty} (1+\xi^2)^{n+\frac{1}{2}} |\Phi(\xi+i\eta)|^2 d\xi \leq C\left[\|g\|^2_{H^{n+1/2}(-\infty,+\infty)} + \int_{-\infty}^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{|x|}\right]$$

via well-known properties of the POISSON integral (compare with lemma 3.10); moreover:

$$(3.75)\qquad \begin{aligned} &\int_{-\infty}^{+\infty} |(1+i\xi-\eta)^{n+\frac{1}{2}}\Phi(\xi+i\eta) - (1+i\xi)^{n+\frac{1}{2}}\Phi_-(\xi)|^2 d\xi \to 0 \qquad \text{if } 0>\eta\to 0\\ &\int_{-\infty}^{+\infty} |(1-i\xi+\eta)^{n+\frac{1}{2}}\Phi(\xi+i\eta) - (1-i\xi)^{n+\frac{1}{2}}\Phi_+(\xi)|^2 d\xi \to 0 \qquad \text{if } 0<\eta\to 0.\end{aligned}$$

From (3.74) and (3.75) we can easily deduce:

$$(3.76)\qquad \int_{-\infty}^{+\infty} (1+\xi^2)^{n+\frac{1}{2}} \left| \Phi(\xi+i\eta) - \begin{matrix}\Phi_-(\xi)\\ \Phi_+(\xi)\end{matrix}\right|^2 d\xi \to 0 \qquad \begin{matrix}\text{if } 0>\eta\to 0\\ \text{if } 0<\eta\to 0\end{matrix}$$

owing to the inequality:

$$\begin{aligned} &|(1\pm i\xi \mp \eta)^{n+\frac{1}{2}} - (1\pm i\xi)^{n+\frac{1}{2}}| \leq \\ &\leq (\text{constant})\,(1+\xi^2)^{\frac{1}{2}\left(n+\frac{1}{2}\right)} |\eta|\,(1+|\eta|)^{n-\frac{1}{2}}\end{aligned}$$

Thus the properties asserted in the statement of the lemma 3.4 are proved.

We prove now the formula (3.73). For semplicity we restrict ourselves to the case $Im\zeta < 0$.

Let u_k be a sequence of $C_0^\infty(]0,+\infty[)$ — functions coverging to $-(gY)$ in $H^{n+\frac{1}{2}}(-\infty,+\infty)$. Such sequence exists, for $(gY)\in H^{n+\frac{1}{2}}(-\infty,+\infty)$ in reason of lemma 3.11 and spt $(gY)\subseteq [0,+\infty[$.

As already remarked, the FOURIER transforms $\widehat{u}_k$ are entire holomorphic functions such that: $|\widehat{u}_k(\zeta)| \leq (\text{constant})(1+|\zeta|)^{-m}$ for every integer m and every ζ in the lower half plane $Im\zeta \leq 0$.

Then, as $\zeta \to (1+i\zeta)^{n+\frac{1}{2}}$ is holomorphic in the half plane $Im\zeta < 1$, from the Cauchy integral theorem we get

$$(3.77)\qquad (1+i\zeta)^{n+\frac{1}{2}}\widehat{u}_k(\zeta) = -\int_{-\infty}^{+\infty} \frac{(1+it)^{n+\frac{1}{2}}\widehat{u}_k(t)}{2\pi i(t-\zeta)}\,dt \qquad (Im\zeta<0).$$

In reason of (i), lemma 3.6, the right side of (3.77) equals the POISSON integral of $+\frac{1}{2}\widehat{v}_k - \frac{i}{2}H\widehat{v}_k = (v_k Y)^\frown$, where v_k is the function defined by: $\widehat{v}_k(\xi) = (1+i\xi)^{n+1/2}\widehat{u}_k(\xi)$.

Applying a theorem of PALEY-WIENER, we see that v_k is an infinitely differentiable function whose support is strictly contained in $]0, +\infty[$, so the right side of (3.77) is exactly the POISSON integral of $\widehat{v}_k$. Thus (3.78) can be rewritten in this form:

$$(3.78) \qquad (1+i\zeta)^{n+\frac{1}{2}}\widehat{u}_k(\zeta) = \\ = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{|\eta|}{(\xi - t)^2 + \eta^2}(1+it)^{n+\frac{1}{2}}\widehat{u}_k(t)dt \\ (\xi + i\eta = \zeta,\ \eta < 0).$$

Clearly $\widehat{u}_k(\zeta) \to \Phi(\zeta)$ for every ζ such that $Im\zeta < 0$. In fact by (ii), lemma 3.6, we have:

$$\Phi(\zeta) = -\int_0^{+\infty} e^{-ix\zeta} g(x)dx \qquad (Im\zeta < 0).$$

hence:

$$|\widehat{u}_k(\zeta) - \Phi(\zeta)| = \left|\int_0^{+\infty} e^{-ix\zeta}(u_k(x) + g(x))dx\right| \leq$$

$$\leq |2Im\zeta|^{-1/2}\|u_k + (gY)\|_{L^2(-\infty,+\infty)} \to 0 \ (Im\zeta < 0).$$

On the other hand, in reason of the SCHWARTZ inequality, the right side of (3.78) couverges for every ξ and $\eta \neq 0$ to the POISSON integral of $\xi \to (1+i\xi)^{n+\frac{1}{2}}\Phi_-(\xi)$, since $\Phi_- = -(gY)^\frown$ and u_k tends to $-gY$ in $H^{n+\frac{1}{2}}(-\infty, +\infty)$.

Thus (3.78) implies (3.73) in the case $Im\zeta < 0$. The proof of (3.73) in the case $Im\zeta > 0$ is similar.

The lemma 3.4 is completely proved.

4. - Proof of the theorems 1.1 and 1.2.

We look for solutions u represented in this way:

$$u(x, y) = \begin{cases} \displaystyle\int_{-\infty}^{y} \frac{\exp\left(-\frac{x^2}{4(y-t)} - k(y-t)\right)}{\sqrt{\pi(y-t)}} \varphi(t)dt + U(x, y) & \text{if } x > 0 \text{ and } y > 0; \\ \displaystyle\int_{y}^{+\infty} \frac{\exp\left(-\frac{x^2}{4(t-y)} - k(t-y)\right)}{\sqrt{\pi(t-y)}} \psi(t)dt & \text{if } x < 0 \text{ and } y > 0. \end{cases} \tag{4.1}$$

Here φ and ψ are functions, to be determined, such that:

$$\begin{cases} \varphi \in H^{1/4}(-\infty, +\infty) \quad \operatorname{spt} \varphi \subseteq [0, +\infty[, \\ \psi \in H^{1/4}(-\infty, +\infty); \end{cases} \tag{4.2}$$

here U is given by (1.7), that is:

$$U(x, y) = \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-t)^2}{4y} - ky\right)}{(4\pi y)^{\frac{1}{2}}} h(|t|)\,dt \qquad (-\infty < x < +\infty,\ y > 0).$$

As already remarked, U is an infinitely differentiable bounded function belonging to $W(R^2_+)$, and verifies: $U_y - U_{xx} + kU = 0$, $U(x, 0+) = h(|x|)$ $(-\infty < x < +\infty)$. An easy inspection shows that:

$$U_x(0, y) = 0 \quad \text{for every} \quad y > 0. \tag{4.3}$$

For the following assertions one should to bear in mind the theorems listed in the appendix.

The function u is given by (4.1) verifies (1.1) and (1.8), namely:

$$u \in W(G_+)$$

$$\operatorname{sgn}(x)u_y - u_{xx} + ku = 0\,.$$

By the second condition (4.2) we have $u(x, 0+) = U(x, 0+)$ if $x > 0$, thus the condition $(1.3) = (1.9)$ holds true:

$$u(x, 0+) = h(x) \quad \text{if} \quad x > 0.$$

The following equations hold:

$$(4.4)\qquad \begin{cases} u_x(0+,\,y)=\varphi(y) \\ \qquad\qquad\qquad\qquad \text{if } y>0 \\ u_x(0-,\,y)=\psi(y) \end{cases}$$

the first being a consequence of (4.3).

Moreover:

$$(4.5)\qquad \begin{cases} u(0+,\;y)=-\displaystyle\int_{-\infty}^{y}\frac{e^{-k(y-t)}}{\sqrt{\pi(y-t)}}\,\varphi(t)dt+U(0,\;y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } y>0 \\ u(0-,\;y)=\displaystyle\int_{y}^{+\infty}\frac{e^{k(y-t)}}{\sqrt{\pi|x-t|}}\,\psi(t)dt \end{cases}$$

We point out that a function $u\in W(G_+)$ belongs actually to $W(R^2_+)$ if and only if $u(0-,\;\cdot)=u(0+,\;\cdot)$ and $u_x(0-,\;\cdot)=u_x(0+,\;\cdot)$.

Thus, by (4.4) and (4.5), the function u given by (4.1) verifies one of the conditions: (1.2), (1.10), (1.11) if and only if the pair φ, ψ solves respectively the system: (4.6), (4.7), (4.8).

$$(4.6)\qquad \begin{cases} \varphi(y)=\psi(y), \\ \displaystyle\int_{-\infty}^{y}\frac{e^{-k(y-t)}}{\sqrt{\pi(y-t)}}\,\varphi(t)dt+\int_{y}^{+\infty}\frac{e^{k(y-t)}}{\sqrt{\pi|y-t|}}\,\psi(t)dt=U(0,\;y), \\ 0<y<+\infty. \end{cases}$$

$$(4.7)\qquad \begin{cases} \varphi(y)=\psi(y), \\ -\displaystyle\int_{-\infty}^{y}\frac{e^{-k(y-t)}}{[\pi(y-t)]^{1/2}}\,\varphi(t)dt+2k\int_{y}^{+\infty}e^{k(y-t)}\,dt\int_{-\infty}^{t}\frac{e^{-k(t-\lambda)}}{[\pi(t-\lambda)]^{1/2}}\,\varphi(\lambda)d\lambda \\ \qquad\qquad +\displaystyle\int_{y}^{+\infty}\frac{e^{k(y-t)}}{[\pi|y-t|]^{1/2}}\,\varphi(t)dt= \\ =-U(0,\;y)+2k\displaystyle\int_{y}^{+\infty}e^{k(y-t)}U(0,\;t)dt, \\ 0<y<+\infty. \end{cases}$$

$$
(4.8)\qquad \begin{cases} \varphi(y) + \psi(y) = 2k \displaystyle\int_y^{+\infty} e^{k(y-t)}\,\psi(t)dt, \\ \displaystyle\int_{-\infty}^{y} \frac{e^{-k(y-t)}}{[\pi(y-t)]^{1/2}}\,\varphi(t)dt + \int_y^{+\infty} \frac{e^{k(y-t)}}{[\pi\,|\,y-t\,|\,]^{1/2}}\,\psi(t)dt = U(0,\ y), \\ 0 < y < +\infty . \end{cases}
$$

As is easy to see by eliminating ψ, a solution verifying (4.2) of each of the previous systems can be obtained in this way: φ is a solution, such that

$$
(4.9)\qquad \varphi \in H^{1/4}(-\infty,\ +\infty), \quad \operatorname{spt} \quad \varphi \subseteq [0,\ +\infty[
$$

of the integral equation:

$$
(4.10)\qquad \int_0^{+\infty} K(y-t)\varphi(t)dt = f(y) \qquad (0 < y < +\infty);
$$

the kernel K, the right-hand side f the and ψ are given by the following tables respectively.

In the tables we have indicated besides: the FOURIER transform $\widehat{K}$ of the kernel and its factorisation, the function g and the parameters appearing in the statement of theorem 3.1 (the exponents p, q connected with the factorisation of $\widehat{K}$; the order s of the SOBOLEV space containing f; the number $s-p$, which determines the compatibility conditions).

TABLE I

connected with the system (4.6) and the problem (1.1)-(1.2)-(1.3)

$K(y)$	$(\pi\|y\|)^{-1/2}\, e^{-k\|y\|}$
$f(y)$	$l(y)$ defined by (1.6)
$\psi(y)$	$\varphi(y)$ for every $y > 0$
$\widehat{K}(\xi)$ $K_+(\xi)$ $K_-(\xi)$	$2Re(k + i\xi)^{-1/2}$ $\left.\begin{array}{l} A(\xi) \\ B(\xi) \end{array}\right\}$ defined by (20)-(21)
s p q	$\frac{3}{4}$ $\frac{1}{4}$ $\frac{1}{4}$
$\widehat{g}(\xi)$	$-\,\widehat{l}(\xi)/A(\xi)$
$s-p$	$\frac{1}{2}$

TABLE III

connected with the system (4.7) and the problem (1.1)-(1.8)-(1.9)-(1.10)

$K(y)$	$-(\pi\|y\|)^{-1/2} e^{-k\|y\|} \operatorname{sgn}(y) + 2k \int_0^{+\infty} e^{-k\|y-t\|} Y(-y+t)(\pi t)^{-1/2} e^{-kt} dt$
$f(y)$	$-l(y) + 2k \int_y^{+\infty} e^{k(y-t)} l(t) dt$, l defined by (1.6)
$\psi(y)$	$\varphi(y)$ for every $y > 0$
$\widehat{K}(\xi)$ $K_+(\xi)$ $K_-(\xi)$	$(k+i\xi)^{1/2}(k-i\xi)^{-1/2}\, 2Re(k+i\xi)^{-1/2}$ $(k-i\xi)^{-1/2} A(\xi)$ $(k+i\xi)^{1/2} B(\xi)$ $A(\zeta)$, $B(\zeta)$ defined by (20) ...... (23)
s p q	$\frac{3}{4}$ $\frac{3}{4}$ $-\frac{1}{4}$
$\widehat{g}(\xi)$	$-\dfrac{(k+i\xi)\widehat{l}(\xi)}{(k-i\xi)^{1/2} A(\xi)}$
$s-p$	0

TABLE III

connected with the system (4.8) and the problem (1.1)-(1.8)-(1.9)-(1.11)

$K(y)$	$(\pi\|y\|)^{-1/2}e^{-k\|y\|}\operatorname{sgn}(y)+(2k)^{1/2}e^{-k\|y\|}Y(y)+$ $+2k\,e^{-k\|y\|}(1-Y(y))\int_0^{+\infty}\frac{e^{-2kt}}{[\pi(t+\|y\|)]^{1/2}}dt$
$f(y)$	$l(y)-\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{iy\xi}\frac{(k-i\xi)^{1/2}}{k+i\xi}\widehat{\chi}(\xi)d\xi$ l defined by (1.6) χ any function belonging to $C_0^\infty(]-\infty,0[)$
$\psi(y)$	$-\varphi(y)-\chi(y)+2k\int_{-\infty}^{y}e^{-k(y-t)}(\varphi(t)+\chi(t))dt$
$\widehat{K}(\xi)$ $K_+(\xi)$ $K_-(\xi)$	$(k+i\xi)^{-1/2}(k-i\xi)^{1/2}2Re(k+i\xi)^{-1/2}$ $(k-i\xi)^{1/2}A(\xi)$ $(k+i\xi)^{-1/2}B(\xi)$ $A(\zeta)$, $B(\zeta)$ defined by (20) ...... (23)
s p q	$\frac{3}{4}$ $-\frac{1}{4}$ $\frac{3}{4}$
$\widehat{g}(\xi)$	$-\frac{\widehat{l}(\xi)}{(k-i\xi)^{1/2}A(\xi)}+\frac{\widehat{\chi}(\xi)}{(k+i\xi)A(\xi)}$
$s-p$	1

According to theorem 3.1, the following compatibility conditions guarantee the existence of a solution of (4.9)-(4.10).

	compatibility conditions
case of table I	$\int_0^\varepsilon \lvert g(y)\rvert^2 \frac{dy}{y} < +\infty$ for some $\varepsilon > 0$.
case of table II	none
case of table III	$g(0) = 0$.

The first is exactly the condition (1.4). The last can be verified choosing the function χ in such a way that:

$$\int_{-\infty}^{0} e^{ky}\chi(y)dy = \frac{1}{2\pi} A(ik) \int_{-\infty}^{+\infty} \frac{\widehat{l}(\xi)}{(k - i\xi)^{1/2} A(\xi)} d\xi.$$

In fact, from the table III we deduce:

$$2\pi g(0) = -\int_{-\infty}^{+\infty} \frac{\widehat{l}(\xi)}{(k - i\xi)^{1/2} A(\xi)} d\xi + 2\pi \frac{\widehat{\chi}(ik)}{A(ik)}$$

for the residues theorems shows that:

$$\int_{-\infty}^{+\infty} \frac{\widehat{\chi}(\xi)}{(k + i\xi) A(\xi)} d\xi = 2\pi \frac{\widetilde{\chi}(ik)}{A(ik)}$$

as $\chi \in C_0^\infty(]-\infty, 0[)$.

In conclusion, by the hypotheses of theorems 1.1 and 1.2, a solution φ of (4.9)-(4.10) exists. Theorefore, (4.1) gives the wanted solution.

Appendix: properties of solutions of the heat equation.

LEMMA A.1. - Consider:

$$u(x, y) = \int_{-\infty}^{+\infty} f(t) \frac{\exp\left(-\dfrac{(x-t)^2}{4y} - ky\right)}{(4\pi y)^{1/2}} dt \quad (-\infty < x < +\infty,\ y > 0) \tag{A.1}$$

where $f \in C_0^\infty(-\infty, +\infty)$ and k is a positive constant; equivalently we can write:

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{-y(k+\xi^2)+ix\xi} d\xi \tag{A.2}$$

for $\xi \to \exp(-y\xi^2)$, with $y > 0$, is the FOURIER transform of $x \to (4\pi y)^{-1/2} \exp\left(-\dfrac{x^2}{4y}\right)$.

Clearly $u \in C^\infty(\overline{R^2_+})$, $u(\cdot, 0) = f$. The following estimates hold:

(i) $$|u(x, y)|^2 \le e^{-2ky}\left(\int_{-\infty}^{+\infty} |f|^2 dx \int_{-\infty}^{+\infty} |f'|^2 dx\right)^{1/2}$$

(ii) $$\iint_{R^2_+} |u|^2 dx dy = \frac{1}{2} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{x} e^{-\sqrt{k}(x-t)} f(t) dt \right|^2 dx$$

(iii) $$\iint_{R^2_+} (|u_x|^2 + k|u|^2) dx dy = \frac{1}{2} \int_{-\infty}^{+\infty} |f|^2 dx$$

(iv) $$\iint_{R^2_+} (|u_y|^2 + |u_{xx}|^2 + 2k|u_x|^2 + k^2|u|^2) dx dy = \frac{1}{2} \int_{-\infty}^{+\infty} (|f'|^2 + k|f|^2) dx$$

(v) $$\int_{-\infty}^{+\infty} |u(x, y)|^2 dx \le e^{-ky} \int_{-\infty}^{+\infty} |f|^2 dx$$

$$\int_{\infty}^{+\infty} |u(x, y) - f(x)|^2 dx \le \frac{1}{2} y \int_{-\infty}^{+\infty} (|f'|^2 + |f|^2) dx$$

for every $y > 0$.

LEMMA A.2. - Consider

$$u(x, y) = \tag{A.3}$$

$$= -\int_{-\infty}^{y} f(t) \frac{\exp\left(-\dfrac{x^2}{4(y-t)} - k(y-t)\right)}{[\pi(y-t)]^{1/2}} dt \quad (x \geq 0, -\infty < y < +\infty)$$

where $f \in C_0^\infty(-\infty, +\infty)$ and k is a positive constant; equivalently we can write:

$$u(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) \frac{e^{-x(k+i\xi)^{1/2}}}{(k+i\xi)^{1/2}} e^{iy\xi} d\xi \tag{A.4}$$

for $\xi \to (k+i\xi)^{-1/2} \exp[-x(k+i\xi)^{1/2}]$, with $x > 0$, is the FOURIER transform of $y \to (\pi y)^{-1/2} Y(y) \exp\left(-\dfrac{x^2}{4y} - ky\right)$. Here $(k+i\xi)^{1/2}$ is the square root of $k+i\xi$ with positive real part; hence $Re(k+i\xi)^{1/2} \geq \dfrac{1}{\sqrt{2}}(k^2+\xi^2)^{1/4}$.

Clearly u is infinitely differentiable in the half-plane $x \geq 0$ and $u_x(0, \cdot) = f$. The following estimates hold:

(i) $$|u(x, y)|^2 \leq \frac{\Gamma(1/4)^2}{2\pi\sqrt{\pi k}} e^{-2\sqrt{kx}} \int_{-\infty}^{+\infty} |D^{1/4}f|^2 dy$$

where the fractional derivative $D^{1/4}f$ is defined by: $(D^{1/4}f)^\frown(\xi) = |\xi|^{1/4} \exp\left(i\dfrac{\pi}{8} \operatorname{sgn} \xi\right) \hat{f}(\xi)$ or:

$$(D^{1/4}f)(y) = \frac{1}{4\Gamma(3/4)} \int_0^{+\infty} \frac{f(y) - f(y-t)}{t^{5/4}} dt$$

(ii) $$\int_0^{+\infty} dx \int_{-\infty}^{+\infty} |u|^2 dy \leq \frac{1}{\sqrt{2}\; 2\pi} \int_{-\infty}^{+\infty} (k^2+\xi^2)^{-3/4} |\hat{f}(\xi)|^2 d\xi$$

$$\int_0^{+\infty} dx \int_{-\infty}^{+\infty} |u_x|^2 dy \leq \frac{1}{\sqrt{2}\; 2\pi} \int_{-\infty}^{+\infty} (k^2+\xi^2)^{-1/4} |\hat{f}(\xi)|^2 d\xi$$

(iii) $$\int_0^{+\infty} dx \int_{-\infty}^{+\infty} (|u_y|^2 + |u_{xx}|^2 + 2k|u_x|^2 + k^2|u|^2) dy =$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} Re(k + i\xi)^{1/2} |\widehat{f}(\xi)|^2 \, d\xi$$

(iv) $$\int_{-\infty}^{+\infty} |u_x(x, y)|^2 dy \leq e^{-2\sqrt{kx}} \int_{-\infty}^{+\infty} |f|^2 \, dy$$

$$\int_{-\infty}^{+\infty} |u_x(x, y) - f(y)|^2 \, dy \leq \frac{x}{\sqrt{2}\ 2\pi} \int_{-\infty}^{+\infty} (k^2 + \xi^2)^{1/4} |\widehat{f}(y)|^2 \, d\xi$$

for every $x > 0$.

LEMMA A.3. - Consider the trace $v(y) = u(0, y)$ of the function (A.1) on the positive y-axis; from (A.2) we have:

(A.5) $$v(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{f}(\xi) e^{-y(k+\xi^2)} \, d\xi \qquad (y > 0).$$

We have the following estimates:

(i) $$\int_0^{+\infty} |v^{(s)}(y)|^2 \, dy \leq \frac{1}{4\pi} \int_0^{+\infty} (k + \xi^2)^{2s-1/2} |\widehat{f}(\xi)|^2 \, d\xi$$

if $s = 0, 1, 2, 3, \ldots$

(ii) $$\int_0^{+} t^{-1-2s} dt \int_0^{+\infty} |v(y + t) - v(y)|^2 \, dy \leq$$

$$\leq \frac{2^{2s-1} - 1}{s(2s - 1)} \Gamma(2 - 2s) \frac{1}{4\pi} \int_{-\infty}^{+\infty} (k + \xi^2)^{2s-1/2} |\widehat{f}(\xi)|^2 \, dy$$

if $0 < s < 1$

(iii) $$\left| \int_0^{+\infty} \varphi(y)\, v(y) \, dy \right|^2 \leq$$

$$\leq \int_{-\infty}^{+\infty} |D^{-s}\varphi|^2 dy \frac{1}{4\pi} \int_{-\infty}^{+\infty} (k + \xi^2)^{2s-1/2} |\widehat{f}(\xi)|^2 \, d\xi$$

if $s < 0$. Here φ is any function in $C_0^\infty(]0, +\infty[)$ and the fractional derivative $D^{-s}\varphi$ is defined by:

$$(D^r\varphi)(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\xi|^r \exp\left(i\frac{r\pi}{2}\operatorname{sgn}\xi\right)\widehat{\varphi}(\xi)d\xi \qquad (r > 0).$$

Lemma A.4. - Consider the trace $v(x) = u(x, 0)$ of the function (A.3) on the positive x-axis; from (A.4) we have:

$$v(x) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{f}(\xi)\frac{e^{-x(k+i\xi)^{1/2}}}{(k+i\xi)^{1/2}}d\xi \qquad (x > 0). \tag{A.6}$$

The following estimates hold:

(i) $$\int_0^{+\infty}|v^{(s)}(x)|^2dx \leq \frac{\sqrt{2}}{\pi}\int_{-\infty}^{+\infty}(k^2+\xi^2)^{\frac{1}{2}\left(s-\frac{1}{2}\right)}|\widehat{f}(\xi)|^2d\xi$$

if $s = 0, 1, 2, \ldots$

(ii) $$\int_0^{+\infty}t^{-1-2s}dt\int_0^{+\infty}|v(x+t)-v(x)|^2dx \leq$$

$$\leq \frac{2^{3/2}}{\pi}\frac{2^{2s-1}-1}{s(2s-1)}\Gamma(2-2s)\int_{-\infty}^{+\infty}(k^2+\xi^2)^{\frac{1}{2}\left(s-\frac{1}{2}\right)}|\widehat{f}(\xi)|^2d\xi$$

if $0 < s < 1$

(iii) $$\left|\int_0^{+\infty}v(x)\varphi(x)dx\right|^2 \leq$$

$$\leq \frac{\sqrt{2}}{\pi}\int_{-\infty}^{+\infty}|D^{-s}\varphi|^2dx\int_{-\infty}^{+\infty}(k^2+\xi^2)^{\frac{1}{2}\left(s-\frac{1}{2}\right)}|\widehat{f}(\xi)|^2d\xi$$

if $s < 0$ and $\varphi \in C_0^\infty(]0, +\infty[)$.

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