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On a fractional Caputo–Hadamard problem with boundary value conditions via different orders of the Hadamard fractional operators

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Abstract

We investigate the existence of solutions for a Caputo–Hadamard fractional integro-differential equation with boundary value conditions involving the Hadamard fractional operators via different orders. By using the Krasnoselskii's fixed point theorem, the Leray–Schauder nonlinear alternative, and the Banach contraction principle, we prove our main results. Also, we provide three examples to illustrate our main results.

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1 Introduction

In recent decades, it has become clear to researchers that studying different types of fractional differential equations is of particular importance. This is a tool to complete our modeling information.

In fact, some practical instances done in the framework of the concepts and notions of the fractional calculus show us the power of this branch of mathematics in the modeling of different natural phenomena. In the meantime, fractional differential equations and inclusions of different types play an important role to reach desired practical goals. More precisely, in recent years, some researches invoked these fractional equations to model some processes and patterns via newly defined fractional operators (see, for example, [1–4]). The techniques used in these initial value problems are based on the analytical and the existence methods. In the following, some researchers designed new fractional models and investigated them via numerical techniques (see, for example, [5–14]). Therefore, the fractional calculus has been created a powerful tool for researchers to achieve more exact findings in other applied sciences. Also for further study, notice that a lot of works about different types of fractional integro-differential equations have been published (see, for example, [15–45]), q -difference equations (see, for example, [46–48]), integro-differential equations involving the Caputo–Fabrizio or the Caputo–Hadamard derivatives (see, for

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example, [49–51]), hybrid equations (see, for example, [52]), approximate solutions of different fractional equations (see, for example, [53, 54]), and modern models (see, for example, [55]).

In 2014, Ahmad et al. investigated the existence of solutions for the nonlinear fractional q -difference equation equipped with four-point nonlocal integral boundary conditions

$$\begin{cases} {}^c\mathcal{D}_q^\beta ({}^c\mathcal{D}_q^\gamma + \lambda)u(t) = f(t, u(t)), \\ u(0) = a\mathcal{I}_q^{\alpha-1}u(\eta), \quad u(1) = b\mathcal{I}_q^{\alpha-1}u(\sigma), \end{cases}$$

where $t \in [0, 1]$, $q \in (0, 1)$, $\lambda \in \mathbb{R}$, $0 < \eta, \sigma < 1$ and $\alpha > 2$; ${}^c\mathcal{D}_q^\vartheta$ denotes the Caputo q -fractional derivative of order $\vartheta \in \{\beta, \gamma : \beta, \gamma \in (0, 1]\}$, \mathcal{I}_q^α denotes the Riemann–Liouville q -fractional integral of order α , and $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function [46]. In 2016, Niyom et al. reviewed the problem

$$\begin{cases} (\lambda\mathcal{D}^\alpha + (1 - \lambda)\mathcal{D}^\beta)u(t) = f(t, u(t)), \\ u(0) = 0, \quad \mu\mathcal{D}^{\gamma_1}u(T) + (1 - \mu)\mathcal{D}^{\gamma_2}u(T) = \gamma_3, \end{cases}$$

where $T > 0$, $t \in [0, T]$, $1 < \alpha, \beta < 2$, $0 < \gamma_1, \gamma_2 < \alpha - \beta$, \mathcal{D}^ϕ is the Riemann–Liouville fractional derivative of order $\phi \in \{\alpha, \beta, \gamma_1, \gamma_2\}$, $0 < \lambda \leq 1$, $0 \leq \mu \leq 1$, $\gamma_3 \in \mathbb{R}$, and $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$ [56]. In the same year, Ahmad et al. extended the boundary value problem presented by Niyom to the sequential fractional integro-differential equation of the form

$$\begin{cases} ({}^c\mathcal{D}^q + k{}^c\mathcal{D}^{q-1})x(t) = f(t, x(t), {}^c\mathcal{D}^\beta x(t), \mathcal{I}^\gamma x(t)), \\ x(0) = 0, \quad x'(0) = 0, \quad \sum_{i=1}^m a_i x(\xi_i) = \lambda\mathcal{I}^\delta x(\eta), \end{cases}$$

where $t \in [0, 1]$, $\xi_i, \eta \in (0, 1)$, $q \in (2, 3]$, $\beta, \gamma \in (0, 1)$, $k, \delta > 0$, λ, a_i ($i = 1, \dots, m$) are real constants, ${}^c\mathcal{D}^{(\cdot)}$ denotes the Caputo derivative of the fractional order (\cdot) , and $f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function [57].

By using main ideas of the aforementioned articles, we investigate the Caputo–Hadamard fractional integro-differential equation of different orders:

$$[\kappa {}^{\text{CH}}\mathcal{D}_{1^+}^\varrho + (1 - \kappa) {}^{\text{CH}}\mathcal{D}_{1^+}^\varpi]w(t) = \alpha \psi(t, w(t)) + \beta {}^H\mathcal{I}_{1^+}^\mu \varphi(t, w(t)), \tag{1}$$

with mixed Hadamard and Caputo–Hadamard boundary value conditions

$$\begin{cases} w(1) = 0, \quad {}^{\text{CH}}\mathcal{D}_{1^+}^\delta w(e) = 0, \\ {}^{\text{CH}}\mathcal{D}_{1^+} w(1) = 0, \quad \frac{1}{\Gamma(\vartheta)} \int_1^e (\ln \frac{e}{s})^{\vartheta-1} w(s) \frac{ds}{s} = 0, \end{cases} \tag{2}$$

where $t \in [1, e]$, $\varrho, \varpi \in (3, 4]$, $\delta \in (1, 2]$, $\kappa \in (0, 1]$, $\mu, \vartheta > 0$ with $\delta + \vartheta \neq 0$ and also $\alpha, \beta \in \mathbb{R}^+$. The notation ${}^{\text{CH}}\mathcal{D}_{1^+}^\nu$ denotes the Caputo–Hadamard fractional derivative of order $\nu \in \{\varrho, \varpi\}$ and ${}^H\mathcal{I}^\mu$ is the Hadamard fractional integral of order μ . Moreover, functions $\psi, \varphi : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous. Note that the integro-differential equation (1) contains the Caputo–Hadamard derivatives of fractional orders ϱ and ϖ and a Hadamard integral of fractional order μ , while the Caputo–Hadamard derivative of order δ and Hadamard integral of order ϑ are involved in the boundary value conditions (2).

It should also be noted that boundary value conditions given in this paper are general and cover many different special cases. This new type of the modeling is an abstract idea and can include various existing natural processes in the future studies. Therefore, the main purpose of this manuscript is to focus on the existence results and provide some necessary conditions for the analytical investigation and so the practical aspects of boundary value problem (1)–(2) is not our main desire here. To reach our main aim, we apply three different fixed point theorems to establish the existence and uniqueness results. These analytical results guarantee the convergence of the numerical methods to desired solution with the least error, and so this can be a reliable criterion for modeling real processes.

The rest of the paper is arranged by follows. In the next section, we recall some basic notions and definitions which are necessary in the sequel. In Sect. 3, our main existence results are presented by three different analytical techniques such as the Krasnoselskii’s fixed point theorem, the Leray–Schauder nonlinear alternative, and the Banach contraction principle. In Sect. 4, we examine the validity of our theoretical findings by providing three illustrative examples. In Sect. 5, the conclusion is stated.

2 Preliminaries

In this section, we recall some important and basic definitions on the fractional operators.

Definition 1 ([58, 59]) Let $\varrho \geq 0$. The Hadamard fractional integral of a continuous function $w : (a, b) \rightarrow \mathbb{R}$ of order ϱ is defined by $({}^H\mathcal{I}_{a^+}^\varrho w)(t) = w(t)$ and

$$({}^H\mathcal{I}_{a^+}^\varrho w)(t) = \frac{1}{\Gamma(\varrho)} \int_a^t \left(\ln \frac{t}{s}\right)^{(\varrho-1)} w(s) \frac{ds}{s}$$

provided that the right-hand side integral exists.

Note that the semigroup property is satisfied by the Hadamard fractional integral as follows: ${}^H\mathcal{I}_{a^+}^\varpi {}^H\mathcal{I}_{a^+}^\varrho w(t) = {}^H\mathcal{I}_{a^+}^{\varpi+\varrho} w(t)$ for $\varrho, \varpi \in \mathbb{R}^+$. Also, we have

$${}^H\mathcal{I}_{a^+}^\varrho \left(\ln \frac{t}{a}\right)^\varpi = \frac{\Gamma(\varpi + 1)}{\Gamma(\varrho + \varpi + 1)} \left(\ln \frac{t}{a}\right)^{\varrho+\varpi}$$

for $\varrho, \varpi \geq 0$ and $t > a$ [58, 59]. It is clear that ${}^H\mathcal{I}_{a^+}^\varrho 1 = \frac{1}{\Gamma(\varrho+1)} (\ln \frac{t}{a})^\varrho$ for all $t > a$ by putting $\varpi = 0$ [59].

Definition 2 ([58, 59]) Let $n = [\varrho] + 1$ and $n - 1 < \varrho \leq n$. The Hadamard fractional derivative of order ϱ for a continuous function $w : (a, b) \rightarrow \mathbb{R}$ is defined by

$$({}^H\mathcal{D}_{a^+}^\varrho w)(t) = \frac{1}{\Gamma(n - \varrho)} \left(t \frac{dt}{t}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{(n-\varrho-1)} w(s) \frac{ds}{s}$$

provided that the right-hand side integral exists.

Definition 3 ([51, 58]) Let $AC_\theta^n[a, b] = \{w : [a, b] \rightarrow \mathbb{R} : \theta^{n-1}w(t) \in AC[a, b], \theta = t \frac{d}{dt}\}$. The Caputo–Hadamard fractional derivative of order ϱ for an absolutely continuous function

$w \in AC_{\theta}^n([a, b], \mathbb{R})$ is defined by

$$({}^{\text{CH}}\mathcal{D}_{a^+}^{\varrho} w)(t) = \frac{1}{\Gamma(n - \varrho)} \int_a^t \left(\ln \frac{t}{s}\right)^{(n-\varrho-1)} \left(\frac{dt}{t}\right)^n w(s) \frac{ds}{s}$$

whenever the right-hand side integral exists.

Assume that $w \in AC_{\theta}^n([a, b], \mathbb{R})$ and $n - 1 < \varrho \leq n$. It has been proved that the solution of the Caputo–Hadamard fractional differential equation $({}^{\text{CH}}\mathcal{D}_{a^+}^{\varrho} w)(t) = 0$ is in the form $w(t) = \sum_{k=0}^{n-1} c_k (\ln \frac{t}{a})^k$, and we have

$${}^H\mathcal{I}_{a^+}^{\varrho} {}^{\text{CH}}\mathcal{D}_{a^+}^{\varrho} w(t) = w(t) + c_0 + c_1 \left(\ln \frac{t}{a}\right) + c_2 \left(\ln \frac{t}{a}\right)^2 + \dots + c_{n-1} \left(\ln \frac{t}{a}\right)^{n-1}$$

for all $t > a$ [58, 59]. We need the following results.

Lemma 4 (Krasnoselskii’s, [60]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space \mathcal{E} . Consider two operators Υ_1 and Υ_2 from M into \mathcal{E} such that*

- (i) $\Upsilon_1 w_1 + \Upsilon_2 w_2 \in M$ for all $w_1, w_2 \in M$,
- (ii) Υ_1 is compact and continuous,
- (iii) Υ_2 is a contraction map.

Then there exists $z \in M$ such that $z = \Upsilon_1 z + \Upsilon_2 z$.

Lemma 5 ([61]) *Let \mathcal{E} be a Banach space, C a closed, convex subset of \mathcal{E} , \mathcal{U} an open subset of C , and $0 \in \mathcal{U}$. Suppose that $\Upsilon : \overline{\mathcal{U}} \rightarrow C$ is a continuous and compact map (that is, $\Upsilon(\overline{\mathcal{U}})$ is a relatively compact subset of C). Then Υ has a fixed point in $\overline{\mathcal{U}}$ or there is a $w \in \partial\mathcal{U}$ (the boundary of \mathcal{U} in C) and $\lambda \in (0, 1)$ with $w = \lambda\Upsilon(w)$.*

Lemma 6 ([62]) *Let \mathcal{E} be a Banach space and M a closed subset of \mathcal{E} . Suppose that $\Upsilon : M \rightarrow M$ is a contraction. Then Υ has a unique fixed point in M .*

3 Main results

Here, we are ready to prove our main results. We first characterize the structure of the solutions of the problem (1)–(2). Consider the Banach space $\mathcal{E} = \{w : w(t) \in C([1, e], \mathbb{R})\}$ with the norm $\|w\|_{\mathcal{E}} = \sup_{t \in [1, e]} |w(t)|$. We first provide our key lemma.

Lemma 7 *Let $\phi(t) \in \mathcal{E}$. Then w_0 is a solution for the Caputo–Hadamard problem*

$$\begin{cases} [\kappa {}^{\text{CH}}\mathcal{D}_{1^+}^{\varrho} + (1 - \kappa) {}^{\text{CH}}\mathcal{D}_{1^+}^{\varpi}] w(t) = \phi(t) & (t \in [1, e]), \\ w(1) = 0, & {}^{\text{CH}}\mathcal{D}_{1^+}^{\delta} w(e) = 0, \\ {}^{\text{CH}}\mathcal{D}_{1^+} w(1) = 0, & \frac{1}{\Gamma(\vartheta)} \int_1^e (\ln \frac{e}{s})^{\vartheta-1} w(s) \frac{ds}{s} = 0 \end{cases} \tag{3}$$

if and only if w_0 is a solution for the fractional integral equation

$$\begin{aligned} w(t) = & \frac{(\kappa - 1)}{\kappa \Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-\varpi-1} w(s) \frac{ds}{s} + \frac{1}{\kappa \Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-1} \phi(s) \frac{ds}{s} \\ & + \frac{(1 - \kappa)[3\Gamma(4 + \vartheta)(\ln t)^2 + (\delta - 3)\Gamma(4 - \vartheta)(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(\varrho - \varpi + \vartheta)} \end{aligned}$$

$$\begin{aligned}
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1} w(s) \frac{ds}{s} \\
 & + \frac{(1-\kappa)\Gamma(4-\delta)[\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(3+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\varpi-\delta-1} w(s) \frac{ds}{s} \\
 & + \frac{(3-\delta)\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(4+\vartheta)(\ln t)^2}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\vartheta-1} \phi(s) \frac{ds}{s} \\
 & + \frac{\Gamma(4-\delta)[3\Gamma(3+\vartheta)(\ln t)^2 - \Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\delta-1} \phi(s) \frac{ds}{s}. \tag{4}
 \end{aligned}$$

Proof Let w_0 be a solution for the Caputo–Hadamard problem (3). Then, we have

$$\kappa \text{CH}\mathcal{D}_{1^+}^{\varrho} w_0(t) + (1-\kappa) \text{CH}\mathcal{D}_{1^+}^{\varpi} w_0(t) = \phi(t)$$

and so $\text{CH}\mathcal{D}_{1^+}^{\varrho} w_0(t) = \frac{\kappa-1}{\kappa} \text{CH}\mathcal{D}_{1^+}^{\varpi} w_0(t) + \frac{1}{\kappa} \phi(t)$. By using the Hadamard fractional integral of order ϱ , we obtain

$$\begin{aligned}
 w_0(t) &= \frac{\kappa-1}{\kappa} {}^H\mathcal{I}_{1^+}^{\varrho} \text{CH}\mathcal{D}_{1^+}^{\varpi} w_0(t) + \frac{1}{\kappa} {}^H\mathcal{I}_{1^+}^{\varrho} \phi(t) \\
 &+ b_0 + b_1(\ln t) + b_2(\ln t)^2 + b_3(\ln t)^3,
 \end{aligned}$$

where $b_0, b_1, b_2,$ and b_3 are some real constants. Hence,

$$\begin{aligned}
 w_0(t) &= \frac{\kappa-1}{\kappa\Gamma(\varrho-\varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-\varpi-1} w_0(s) \frac{ds}{s} \\
 &+ \frac{1}{\kappa\Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-1} \phi(s) \frac{ds}{s} \\
 &+ b_0 + b_1(\ln t) + b_2(\ln t)^2 + b_3(\ln t)^3. \tag{5}
 \end{aligned}$$

Now by using the boundary value conditions and properties of the Hadamard and Caputo–Hadamard fractional operators, we get

$$\begin{aligned}
 \text{CH}\mathcal{D}_{1^+} w_0(t) &= \frac{\kappa-1}{\kappa\Gamma(\varrho-\varpi-1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-\varpi-2} w_0(s) \frac{ds}{s} \\
 &+ \frac{1}{\kappa\Gamma(\varrho-1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-2} \phi(s) \frac{ds}{s} + b_1 + 2b_2(\ln t) + 3b_3(\ln t)^2, \\
 \text{CH}\mathcal{D}_{1^+}^{\delta} w_0(t) &= \frac{\kappa-1}{\kappa\Gamma(\varrho-\varpi-\delta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-\varpi-\delta-1} w_0(s) \frac{ds}{s} \\
 &+ \frac{1}{\kappa\Gamma(\varrho-\delta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-\delta-1} \phi(s) \frac{ds}{s} + b_2 \frac{2}{\Gamma(3-\delta)} (\ln t)^{2-\delta} \\
 &+ b_3 \frac{6}{\Gamma(4-\delta)} (\ln t)^{3-\delta},
 \end{aligned}$$

and

$$\begin{aligned}
 {}^H\mathcal{I}_{1^+}^\vartheta w_0(t) &= \frac{\kappa - 1}{\kappa \Gamma(\varrho - \varpi + \vartheta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi + \vartheta - 1} w_0(s) \frac{ds}{s} \\
 &+ \frac{1}{\kappa \Gamma(\varrho + \vartheta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho + \vartheta - 1} \phi(s) \frac{ds}{s} + b_0 \frac{1}{\Gamma(1 + \vartheta)} (\ln t)^\vartheta \\
 &+ b_1 \frac{1}{\Gamma(2 + \vartheta)} (\ln t)^{1 + \vartheta} \\
 &+ b_2 \frac{2}{\Gamma(3 + \vartheta)} (\ln t)^{2 + \vartheta} + b_3 \frac{6}{\Gamma(4 + \vartheta)} (\ln t)^{3 + \vartheta}.
 \end{aligned}$$

By using two first boundary conditions, we obtain $b_0 = b_1 = 0$. By using two other boundary conditions, we obtain

$$\begin{aligned}
 b_2 &= \frac{(1 - \kappa) \Gamma(4 + \vartheta)}{2\kappa(\delta + \vartheta) \Gamma(\varrho - \varpi + \vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} w_0(s) \frac{ds}{s} \\
 &+ \frac{(\kappa - 1) \Gamma(4 - \delta)}{2\kappa(\delta + \vartheta) \Gamma(\varrho - \varpi - \delta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} w_0(s) \frac{ds}{s} \\
 &- \frac{\Gamma(4 + \vartheta)}{2\kappa(\delta + \vartheta) \Gamma(\varrho + \vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} \phi(s) \frac{ds}{s} \\
 &+ \frac{\Gamma(4 - \delta)}{2\kappa(\delta + \vartheta) \Gamma(\varrho - \delta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} \phi(s) \frac{ds}{s}
 \end{aligned}$$

and

$$\begin{aligned}
 b_3 &= \frac{(1 - \kappa)(\delta - 3) \Gamma(4 - \vartheta)}{6\kappa(\delta + \vartheta) \Gamma(\varrho - \varpi + \vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} w_0(s) \frac{ds}{s} \\
 &+ \frac{(1 - \kappa) \Gamma(4 - \delta) \Gamma(4 - \vartheta)}{6\kappa(\delta + \vartheta) \Gamma(3 + \vartheta) \Gamma(\varrho - \varpi - \delta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} w_0(s) \frac{ds}{s} \\
 &+ \frac{(3 - \delta) \Gamma(4 - \vartheta)}{6\kappa(\delta + \vartheta) \Gamma(\varrho + \vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} \phi(s) \frac{ds}{s} \\
 &- \frac{\Gamma(4 - \delta) \Gamma(4 - \vartheta)}{6\kappa(\delta + \vartheta) \Gamma(3 + \vartheta) \Gamma(\varrho - \delta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} \phi(s) \frac{ds}{s}.
 \end{aligned}$$

Now by substituting the values for b_0, b_1, b_2, b_3 in equation (5), we see that w_0 is a solution for the integral equation. For the converse part, by using some direct calculations, one can see that w_0 is a solution for the Caputo–Hadamard problem (3) whenever w_0 is a solution for the integral equation (4). This completes the proof. □

Now, consider the operator $\Upsilon : \mathcal{E} \rightarrow \mathcal{E}$ defined by

$$\begin{aligned}
 (\Upsilon w)(t) &= \frac{(\kappa - 1)}{\kappa \Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi - 1} w(s) \frac{ds}{s} \\
 &+ \frac{\alpha}{\kappa \Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - 1} \psi(s, w(s)) \frac{ds}{s} + \frac{\beta}{\kappa \Gamma(\varrho + \mu)}
 \end{aligned}$$

$$\begin{aligned}
 & \times \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho+\mu-1} \varphi(s, w(s)) \frac{ds}{s} \\
 & + \frac{(1-\kappa)[3\Gamma(4+\vartheta)(\ln t)^2 + (\delta-3)\Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1} w(s) \frac{ds}{s} \\
 & + \frac{(1-\kappa)\Gamma(4-\delta)[\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(3+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\varpi-\delta-1} w(s) \frac{ds}{s} \\
 & + \frac{\alpha[(3-\delta)\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(4+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\vartheta-1} \psi(s, w(s)) \frac{ds}{s} \\
 & + \frac{\beta[(3-\delta)\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(4+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\vartheta+\mu-1} \varphi(s, w(s)) \frac{ds}{s} \\
 & + \frac{\alpha\Gamma(4-\delta)[3\Gamma(3+\vartheta)(\ln t)^2 - \Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\delta-1} \psi(s, w(s)) \frac{ds}{s} \\
 & + \frac{\beta\Gamma(4-\delta)[3\Gamma(3+\vartheta)(\ln t)^2 - \Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\mu-\delta-1} \varphi(s, w(s)) \frac{ds}{s}, \tag{6}
 \end{aligned}$$

where $w \in \mathcal{E}$ and $t \in [1, e]$. Put

$$\begin{aligned}
 \mathcal{K}_0^* & := \frac{|\kappa-1|}{\kappa\Gamma(\varrho-\varpi+1)} + \frac{(1-\kappa)[|3\Gamma(4+\vartheta)| + |(\delta-3)\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta+1)} \\
 & + \frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)| + 3|\Gamma(3+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta+1)}, \\
 \mathcal{K}_1^* & := \frac{\alpha}{\kappa\Gamma(\varrho+1)} + \frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)| + 3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+1)} \\
 & + \frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)| + |\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta+1)}, \\
 \mathcal{K}_2^* & := \frac{\beta}{\kappa\Gamma(\varrho+\mu+1)} + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)| + 3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)} \\
 & + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)| + |\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)}. \tag{7}
 \end{aligned}$$

Theorem 8 Suppose that $\psi, \varphi : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions such that (\mathcal{N}_1) there is $L > 0$ so that $|\psi(t, w_1) - \psi(t, w_2)| \leq L|w_1 - w_2|$ for all $w_1, w_2 \in \mathbb{R}$ and $t \in [1, e]$,

(\mathcal{N}_2) there exists a real-valued continuous function σ on $[1, e]$ such that $|\varphi(t, w)| \leq \sigma(t)$ for all $w \in \mathbb{R}$ and $t \in [1, e]$.

If $\mathcal{K}_0^* + L\mathcal{K}_1^* < 1$, then the Caputo–Hadamard boundary value problem (1)–(2) has at least one solution, where \mathcal{K}_0^* and \mathcal{K}_1^* are given by (7).

Proof Let $\|\sigma\| := \sup_{t \in [1, e]} |\sigma(t)|$ and $O := \sup_{t \in [1, e]} |\psi(t, 0)|$. Consider the operator $\mathcal{Y} : \mathcal{E} \rightarrow \mathcal{E}$ and the set $\mathcal{V}_r := \{w \in \mathcal{E} : \|w\|_{\mathcal{E}} \leq r\}$ which is a closed, convex, and bounded nonempty subset of Banach space \mathcal{E} , where $r \geq \frac{\|\sigma\| \mathcal{K}_2^* + O\mathcal{K}_1^*}{1 - (\mathcal{K}_0^* + L\mathcal{K}_1^*)}$ and \mathcal{K}_1^* and \mathcal{K}_2^* are given by (7). Note that each fixed point of \mathcal{Y} is a solution for the Caputo–Hadamard problem (1)–(2). Let $t \in [1, e]$ be given. Then, we have

$$\begin{aligned} (\mathcal{Y}_1 w)(t) &= \frac{(\kappa - 1)}{\kappa \Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi - 1} w(s) \frac{ds}{s} \\ &\quad + \frac{\alpha}{\kappa \Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - 1} \psi(s, w(s)) \frac{ds}{s} \\ &\quad + \frac{(1 - \kappa)[3\Gamma(4 + \vartheta)(\ln t)^2 + (\delta - 3)\Gamma(4 - \vartheta)(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(\varrho - \varpi + \vartheta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} w(s) \frac{ds}{s} \\ &\quad + \frac{(1 - \kappa)\Gamma(4 - \delta)[\Gamma(4 - \vartheta)(\ln t)^3 - 3\Gamma(3 + \vartheta)(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \varpi - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} w(s) \frac{ds}{s} \\ &\quad + \frac{\alpha[(3 - \delta)\Gamma(4 - \vartheta)(\ln t)^3 - 3\Gamma(4 + \vartheta)(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} \psi(s, w(s)) \frac{ds}{s} \\ &\quad + \frac{\alpha\Gamma(4 - \delta)[3\Gamma(3 + \vartheta)(\ln t)^2 - \Gamma(4 - \vartheta)(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} \psi(s, w(s)) \frac{ds}{s} \end{aligned}$$

and

$$\begin{aligned} (\mathcal{Y}_2 w)(t) &= \frac{\beta}{\kappa \Gamma(\varrho + \mu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho + \mu - 1} \varphi(s, w(s)) \frac{ds}{s} \\ &\quad + \frac{\beta[(3 - \delta)\Gamma(4 - \vartheta)(\ln t)^3 - 3\Gamma(4 + \vartheta)(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta + \mu - 1} \varphi(s, w(s)) \frac{ds}{s} \\ &\quad + \frac{\beta\Gamma(4 - \delta)[3\Gamma(3 + \vartheta)(\ln t)^2 - \Gamma(4 - \vartheta)(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \mu - \delta - 1} \varphi(s, w(s)) \frac{ds}{s}. \end{aligned}$$

Thus, we have

$$\begin{aligned}
 & |(\mathcal{Y}_1 w_1)(t) + (\mathcal{Y}_2 w_2)(t)| \\
 & \leq \frac{|\kappa - 1|}{\kappa \Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi - 1} |w_1(s)| \frac{ds}{s} \\
 & \quad + \frac{\alpha}{\kappa \Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - 1} (|\psi(s, w_1(s)) - \psi(s, 0)| + |\psi(s, 0)|) \frac{ds}{s} \\
 & \quad + \frac{\beta}{\kappa \Gamma(\varrho + \mu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho + \mu - 1} |\varphi(s, w_2(s))| \frac{ds}{s} \\
 & \quad + \frac{(1 - \kappa)[|3\Gamma(4 + \vartheta)|(\ln t)^2 + |(\delta - 3)\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(\varrho - \varpi + \vartheta)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} |w_1(s)| \frac{ds}{s} \\
 & \quad + \frac{(1 - \kappa)|\Gamma(4 - \delta)|[|\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(3 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \varpi - \delta)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} |w_1(s)| \frac{ds}{s} \\
 & \quad + \frac{\alpha[|(3 - \delta)\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(4 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} (|\psi(s, w_1(s)) - \psi(s, 0)| + |\psi(s, 0)|) \frac{ds}{s} \\
 & \quad + \frac{\beta[|(3 - \delta)\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(4 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta + \mu - 1} |\varphi(s, w_2(s))| \frac{ds}{s} \\
 & \quad + \frac{\alpha|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)|(\ln t)^2 + |\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \delta)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} (|\psi(s, w_1(s)) - \psi(s, 0)| + |\psi(s, 0)|) \frac{ds}{s} \\
 & \quad + \frac{\beta|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)|(\ln t)^2 + |\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \mu - \delta - 1} |\varphi(s, w_2(s))| \frac{ds}{s} \\
 & \leq \frac{|\kappa - 1|}{\kappa \Gamma(\varrho - \varpi + 1)} \|w_1\| + \frac{\alpha}{\kappa \Gamma(\varrho + 1)} (L\|w_1\| + O) + \frac{\beta}{\kappa \Gamma(\varrho + \mu + 1)} \|\sigma\| \\
 & \quad + \frac{(1 - \kappa)[|3\Gamma(4 + \vartheta)| + |(\delta - 3)\Gamma(4 - \vartheta)|]}{6\kappa(\delta + \vartheta)\Gamma(\varrho - \varpi + \vartheta + 1)} \|w_1\| \\
 & \quad + \frac{(1 - \kappa)|\Gamma(4 - \delta)|[|\Gamma(4 - \vartheta)| + 3|\Gamma(3 + \vartheta)|]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \varpi - \delta + 1)} \|w_1\| \\
 & \quad + \frac{\alpha[|(3 - \delta)\Gamma(4 - \vartheta)| + 3|\Gamma(4 + \vartheta)|]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + 1)} (L\|w_1\| + O)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)| + 3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)} \|\sigma\| \\
 & + \frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)| + |\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta+1)} (L\|w_1\| + O) \\
 & + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)| + |\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)} \|\sigma\| \\
 & = (\mathcal{K}_0^* + L\mathcal{K}_1^*)\|w_1\| + \mathcal{K}_2^*\|\sigma\| + \mathcal{K}_1^*O \\
 & \leq (\mathcal{K}_0^* + L\mathcal{K}_1^*)r + \mathcal{K}_2^*\|\sigma\| + \mathcal{K}_1^*O \leq r
 \end{aligned}$$

for all $w_1, w_2 \in \mathcal{V}_r$. Hence, $\|\Upsilon_1 w_1 + \Upsilon_2 w_2\| \leq r$ and so $\Upsilon_1 w_1 + \Upsilon_2 w_2 \in \mathcal{V}_r$ for all $w_1, w_2 \in \mathcal{V}_r$. Now let $\{w_n\}_{n \geq 1}$ be a sequence in \mathcal{V}_r with $w_n \rightarrow w$ and $t \in [1, e]$. Then, we have

$$\begin{aligned}
 & |(\Upsilon_2 w_n)(t) - (\Upsilon_2 w)(t)| \\
 & \leq \frac{\beta}{\kappa\Gamma(\varrho+\mu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho+\mu-1} |\varphi(s, w_n(s)) - \varphi(s, w(s))| \frac{ds}{s} \\
 & \quad + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\vartheta+\mu-1} |\varphi(s, w_n(s)) - \varphi(s, w(s))| \frac{ds}{s} \\
 & \quad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2 + |\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\mu-\delta-1} |\varphi(s, w_n(s)) - \varphi(s, w(s))| \frac{ds}{s} \\
 & \leq \frac{\beta}{\kappa\Gamma(\varrho+\mu+1)} |\varphi(s, w_n(s)) - \varphi(s, w(s))| \\
 & \quad + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)| + 3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)} |\varphi(s, w_n(s)) - \varphi(s, w(s))| \\
 & \quad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)| + |\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)} |\varphi(s, w_n(s)) - \varphi(s, w(s))|.
 \end{aligned}$$

Since φ is continuous, $\|\Upsilon_2 w_n - \Upsilon_2 w\| \rightarrow 0$, and so the operator Υ_2 is continuous on the open ball \mathcal{V}_r . Now, we show that Υ_2 is uniformly bounded. Let $w \in \mathcal{V}_r$ and $t \in [1, e]$. Then, we get

$$\begin{aligned}
 |(\Upsilon_2 w)(t)| & \leq \frac{\beta}{\kappa\Gamma(\varrho+\mu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho+\mu-1} |\varphi(s, w(s))| \frac{ds}{s} \\
 & \quad + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\vartheta+\mu-1} |\varphi(s, w(s))| \frac{ds}{s} \\
 & \quad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2 + |\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
 & \quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\mu-\delta-1} |\varphi(s, w(s))| \frac{ds}{s}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\beta}{\kappa \Gamma(\varrho + \mu + 1)} \|\sigma\| + \frac{\beta[|(3 - \delta)\Gamma(4 - \vartheta)| + 3|\Gamma(4 + \vartheta)|]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu + 1)} \|\sigma\| \\
 &\quad + \frac{\beta|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)| + |\Gamma(4 - \vartheta)|]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta + 1)} \|\sigma\| \\
 &\leq \|\sigma\| \left[\frac{\beta}{\kappa \Gamma(\varrho + \mu + 1)} + \frac{\beta[|(3 - \delta)\Gamma(4 - \vartheta)| + 3|\Gamma(4 + \vartheta)|]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu + 1)} \right. \\
 &\quad \left. + \frac{\beta|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)| + |\Gamma(4 - \vartheta)|]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta + 1)} \right] \\
 &= \mathcal{K}_2^* \|\sigma\|
 \end{aligned}$$

which implies that $\|\Upsilon_2 w\| \leq \mathcal{K}_2^* \|\sigma\|$. This shows that Υ_2 is uniformly bounded. Here, we prove that Υ_2 is equicontinuous. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$. We show that Υ_2 maps bounded sets into equicontinuous sets. For each $w \in \mathcal{V}_r$, we have

$$\begin{aligned}
 &|(\Upsilon_2 w)(t_2) - (\Upsilon_2 w)(t_1)| \\
 &\leq \frac{\beta}{\kappa \Gamma(\varrho + \mu)} \int_1^{t_1} \left[\left(\ln \frac{t_2}{s}\right)^{\varrho + \mu - 1} - \left(\ln \frac{t_1}{s}\right)^{\varrho + \mu - 1} \right] |\varphi(s, w(s))| \frac{ds}{s} \\
 &\quad + \frac{\beta}{\kappa \Gamma(\varrho + \mu)} \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s}\right)^{\varrho + \mu - 1} |\varphi(s, w(s))| \frac{ds}{s} \\
 &\quad + \frac{\beta[|(3 - \delta)\Gamma(4 - \vartheta)|[(\ln t_2)^3 - (\ln t_1)^3] + 3|\Gamma(4 + \vartheta)|[(\ln t_2)^2 - (\ln t_1)^2]]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu)} \\
 &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta + \mu - 1} |\varphi(s, w(s))| \frac{ds}{s} \\
 &\quad + \frac{\beta|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)|[(\ln t_2)^2 - (\ln t_1)^2] + |\Gamma(4 - \vartheta)|[(\ln t_2)^3 - (\ln t_1)^3]]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta)} \\
 &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \mu - \delta - 1} |\varphi(s, w(s))| \frac{ds}{s} \\
 &\leq \|\sigma\| \left(\frac{2\beta}{\kappa \Gamma(\varrho + \mu + 1)} \left(\ln \frac{t_2}{t_1}\right)^{\varrho + \mu} + \frac{\beta}{\kappa \Gamma(\varrho + \mu + 1)} |(\ln t_2)^{\varrho + \mu} - (\ln t_1)^{\varrho + \mu}| \right. \\
 &\quad + \frac{\beta[|(3 - \delta)\Gamma(4 - \vartheta)|[(\ln t_2)^3 - (\ln t_1)^3] + 3|\Gamma(4 + \vartheta)|[(\ln t_2)^2 - (\ln t_1)^2]]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu + 1)} \\
 &\quad \left. + \frac{\beta|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)|[(\ln t_2)^2 - (\ln t_1)^2] + |\Gamma(4 - \vartheta)|[(\ln t_2)^3 - (\ln t_1)^3]]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta + 1)} \right).
 \end{aligned}$$

Note that the right-hand side is independent of $w \in \mathcal{V}_r$ and converges to zero as $t_1 \rightarrow t_2$. This means that Υ_2 is equicontinuous. Consequently, the operator Υ_2 is relatively compact on \mathcal{V}_r and, by using the Arzela–Ascoli theorem, we conclude that Υ_2 is completely continuous. Hence, Υ_2 is compact on the open ball \mathcal{V}_r . Now, we show that Υ_1 is a contraction. Let $w_1, w_2 \in \mathcal{V}_r$ and $t \in [1, e]$. Then, we have

$$\begin{aligned}
 &|(\Upsilon_1 w_1)(t) - (\Upsilon_1 w_2)(t)| \\
 &\leq \frac{|\kappa - 1|}{\kappa \Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi - 1} |w_1(s) - w_2(s)| \frac{ds}{s}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\alpha}{\kappa \Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho-1} |\psi(s, w_1(s)) - \psi(s, w_2(s))| \frac{ds}{s} \\
 & + \frac{(1-\kappa)[|3\Gamma(4+\vartheta)|(\ln t)^2 + |(\delta-3)\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1} |w_1(s) - w_2(s)| \frac{ds}{s} \\
 & + \frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(3+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\varpi-\delta-1} |w_1(s) - w_2(s)| \frac{ds}{s} \\
 & + \frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\vartheta-1} |\psi(s, w_1(s)) - \psi(s, w_2(s))| \frac{ds}{s} \\
 & + \frac{\alpha|\Gamma(4-\delta)|[|3\Gamma(3+\vartheta)|(\ln t)^2 + |\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)} \\
 & \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\delta-1} |\psi(s, w_1(s)) - \psi(s, w_2(s))| \frac{ds}{s} \\
 \leq & \left[\frac{|\kappa-1|}{\kappa\Gamma(\varrho-\varpi+1)} + \frac{(1-\kappa)[|3\Gamma(4+\vartheta)| + |(\delta-3)\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta+1)} \right. \\
 & + \frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)| + 3|\Gamma(3+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta+1)} + \frac{L\alpha}{\kappa\Gamma(\varrho+1)} \\
 & + \frac{L\alpha[|(3-\delta)\Gamma(4-\vartheta)| + 3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+1)} \\
 & \left. + \frac{L\alpha|\Gamma(4-\delta)|[|3\Gamma(3+\vartheta)| + |\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta+1)} \right] \|w_1 - w_2\| \\
 = & (\mathcal{K}_0^* + L\mathcal{K}_1^*) \|w_1 - w_2\|.
 \end{aligned}$$

Since $\mathcal{K}_0^* + L\mathcal{K}_1^* < 1$, Υ_1 is a contraction. Note that $\Upsilon = \Upsilon_1 + \Upsilon_2$. Now, by using Lemma 4, the operator Υ has a fixed point which is a solution for the Caputo–Hadamard boundary value problem (1)–(2). □

Here, we are going to investigate the existence of solutions for the Caputo–Hadamard problem (1)–(2) by considering different conditions.

Theorem 9 *Suppose that $\psi, \varphi : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions such that*

- (\mathcal{N}_3) *there are continuous nondecreasing functions $\xi_1, \xi_2 : [0, \infty) \rightarrow (0, \infty)$ and two maps $\theta_1, \theta_2 \in C([0, 1], \mathbb{R}^+)$ such that $|\psi(t, w)| \leq \theta_1(t)\xi_1(|w|)$ and $|\varphi(t, w)| \leq \theta_2(t)\xi_2(|w|)$ for all $(t, w) \in [1, e] \times \mathbb{R}$,*

- (\mathcal{N}_4) *$\mathcal{K}_0^* < 1$ and there is a constant $\Xi > 0$ such that $\frac{(1-\mathcal{K}_0^*)\Xi}{\mathcal{K}_1^*\|\theta_1\|\xi_1(\Xi) + \mathcal{K}_2^*\|\theta_2\|\xi_2(\Xi)} > 1$, where $\mathcal{K}_0^*, \mathcal{K}_1^*, \mathcal{K}_2^*$ are defined by (7).*

Then the Caputo–Hadamard problem (1)–(2) has at least one solution.

Proof We first show that the operator Υ maps bounded sets of \mathcal{E} into bounded sets. Let $\epsilon > 0$, $\mathcal{B}_\epsilon = \{w \in \mathcal{E} : \|w\| \leq \epsilon\}$ and $t \in [1, e]$. Then, we have

$$\begin{aligned}
 |(\Upsilon w)(t)| &\leq \frac{|\kappa - 1|}{\kappa \Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi - 1} \|w\| \frac{ds}{s} \\
 &\quad + \frac{\alpha}{\kappa \Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - 1} \|\theta_1\| \xi_1(\|w\|) \frac{ds}{s} \\
 &\quad + \frac{\beta}{\kappa \Gamma(\varrho + \mu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho + \mu - 1} \|\theta_2\| \xi_2(\|w\|) \frac{ds}{s} \\
 &\quad + \frac{(1 - \kappa)[|3\Gamma(4 + \vartheta)|(\ln t)^2 + |(\delta - 3)\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(\varrho - \varpi + \vartheta)} \\
 &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} \|w\| \frac{ds}{s} \\
 &\quad + \frac{(1 - \kappa)|\Gamma(4 - \delta)|[|\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(3 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \varpi - \delta)} \\
 &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} \|w\| \frac{ds}{s} \\
 &\quad + \frac{\alpha[|(3 - \delta)\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(4 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta)} \\
 &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} \|\theta_1\| \xi_1(\|w\|) \frac{ds}{s} \\
 &\quad + \frac{\beta[|(3 - \delta)\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(4 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu)} \\
 &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta + \mu - 1} \|\theta_2\| \xi_2(\|w\|) \frac{ds}{s} \\
 &\quad + \frac{\alpha|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)|(\ln t)^2 + |\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \delta)} \\
 &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} \|\theta_1\| \xi_1(\|w\|) \frac{ds}{s} \\
 &\quad + \frac{\beta|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)|(\ln t)^2 + |\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta)} \\
 &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \mu - \delta - 1} \|\theta_2\| \xi_2(\|w\|) \frac{ds}{s} \\
 &\leq \mathcal{K}_0^* \|w\| + \mathcal{K}_1^* \|\theta_1\| \xi_1(\|w\|) + \mathcal{K}_2^* \|\theta_2\| \xi_2(\|w\|).
 \end{aligned}$$

Hence, $\|\Upsilon w\| \leq \mathcal{K}_0^* \epsilon + \mathcal{K}_1^* \|\theta_1\| \xi_1(\epsilon) + \mathcal{K}_2^* \|\theta_2\| \xi_2(\epsilon)$. Now, we prove that Υ maps bounded sets into equicontinuous sets of \mathcal{E} . Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$ and $w \in \mathcal{B}_\epsilon$. Then, we get

$$\begin{aligned}
 &|(\Upsilon w)(t_2) - (\Upsilon w)(t_1)| \\
 &\leq \frac{|\kappa - 1|\epsilon}{\kappa \Gamma(\varrho - \varpi)} \left[\int_1^{t_1} \left[\left(\ln \frac{t_2}{s}\right)^{\varrho + \varpi - 1} - \left(\ln \frac{t_1}{s}\right)^{\varrho + \varpi - 1} \right] \frac{ds}{s} + \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s}\right)^{\varrho + \varpi - 1} \frac{ds}{s} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\alpha \|\theta_1\| \xi_1(\epsilon)}{\kappa \Gamma(\varrho)} \left[\int_1^{t_1} \left[\left(\ln \frac{t_2}{s} \right)^{\varrho-1} - \left(\ln \frac{t_1}{s} \right)^{\varrho-1} \right] \frac{ds}{s} + \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s} \right)^{\varrho-1} \frac{ds}{s} \right] \\
 & + \frac{\beta \|\theta_2\| \xi_2(\epsilon)}{\kappa \Gamma(\varrho + \mu)} \left[\int_1^{t_1} \left[\left(\ln \frac{t_2}{s} \right)^{\varrho+\mu-1} - \left(\ln \frac{t_1}{s} \right)^{\varrho+\mu-1} \right] \frac{ds}{s} + \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s} \right)^{\varrho+\mu-1} \frac{ds}{s} \right] \\
 & + \frac{(1-\kappa)[|3\Gamma(4+\vartheta)|[(\ln t_2)^2 - (\ln t_1)^2] + |(\delta-3)\Gamma(4-\vartheta)|[(\ln t_2)^3 - (\ln t_1)^3]]\epsilon}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta+1)} \\
 & + ((1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|[(\ln t_2)^3 - (\ln t_1)^3] \\
 & + 3|\Gamma(3+\vartheta)|[(\ln t_2)^2 - (\ln t_1)^2]]\epsilon) \\
 & / (6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta+1)) \\
 & + (\alpha[(3-\delta)\Gamma(4-\vartheta)|[(\ln t_2)^3 - (\ln t_1)^3] \\
 & + 3|\Gamma(4+\vartheta)|[(\ln t_2)^2 - (\ln t_1)^2]]\|\theta_1\|\xi_1(\epsilon)) \\
 & / (6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+1)) \\
 & + (\beta[(3-\delta)\Gamma(4-\vartheta)|[(\ln t_2)^3 - (\ln t_1)^3] \\
 & + 3|\Gamma(4+\vartheta)|[(\ln t_2)^2 - (\ln t_1)^2]]\|\theta_2\|\xi_2(\epsilon)) \\
 & / (6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)) \\
 & + (\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|[(\ln t_2)^2 - (\ln t_1)^2] \\
 & + |\Gamma(4-\vartheta)|[(\ln t_2)^3 - (\ln t_1)^3]]\|\theta_1\|\xi_1(\epsilon)) \\
 & / (6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta+1)) \\
 & + (\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|[(\ln t_2)^2 - (\ln t_1)^2] \\
 & + |\Gamma(4-\vartheta)|[(\ln t_2)^3 - (\ln t_1)^3]]\|\theta_2\|\xi_2(\epsilon)) \\
 & / (6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)).
 \end{aligned}$$

Note that the right-hand side tends to zero independently of $w \in \mathcal{B}_\epsilon$ as $t_2 \rightarrow t_1$. By using the Arzela–Ascoli theorem, we deduce that $\Upsilon : \mathcal{E} \rightarrow \mathcal{E}$ is completely continuous. Here, we prove that the set of all solutions of the equation $w = \lambda(\Upsilon w)$ is bounded for each $\lambda \in [0, 1]$. Let $\lambda \in [0, 1]$, w be such that $w = \lambda(\Upsilon w)$ and $t \in [1, e]$. Then by using computations used in the first step, we obtain $\|w\| \leq \mathcal{K}_0^* \|w\| + \mathcal{K}_1^* \|\theta_1\| \xi_1(\|w\|) + \mathcal{K}_2^* \|\theta_2\| \xi_2(\|w\|)$. Thus, we conclude that $\frac{(1-\mathcal{K}_0^*)\|w\|}{\mathcal{K}_1^* \|\theta_1\| \xi_1(\|w\|) + \mathcal{K}_2^* \|\theta_2\| \xi_2(\|w\|)} \leq 1$. By using the assumption (\mathcal{N}_4) , we can choose a number $\mathcal{E} > 0$ such that $\|w\| \neq \mathcal{E}$ and $\frac{(1-\mathcal{K}_0^*)\mathcal{E}}{\mathcal{K}_1^* \|\theta_1\| \xi_1(\mathcal{E}) + \mathcal{K}_2^* \|\theta_2\| \xi_2(\mathcal{E})} > 1$. Consider the set $\mathcal{U} = \{w \in \mathcal{E} : \|w\| < \mathcal{E}\}$. Note that the operator $\Upsilon : \overline{\mathcal{U}} \rightarrow \mathcal{E}$ is continuous and completely continuous and also we can not find $w \in \partial\mathcal{U}$ such that $w = \lambda(\Upsilon w)$ holds for some $\lambda \in (0, 1)$. Now, by using Lemma 5, the operator Υ has a fixed point in $\overline{\mathcal{U}}$ which is a solution for the Caputo–Hadamard fractional integro-differential boundary value problem (1)–(2). \square

Now by using the Banach contraction principle, we review the Caputo–Hadamard problem (1)–(2) under some different conditions.

Theorem 10 *Suppose that $\psi : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying assumption (\mathcal{N}_1) . Assume that the function $\varphi : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following condition:*

(\mathcal{N}_5) there is a positive constant \tilde{L} such that for each $|\varphi(t, w_1) - \varphi(t, w_2)| \leq \tilde{L}|w_1 - w_2|$ for all $w_1, w_2 \in \mathbb{R}$ and $t \in [1, e]$.

If $\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^* < 1$, then the Caputo–Hadamard problem (1)–(2) has a unique solution, where $\mathcal{K}_0^*, \mathcal{K}_1^*$, and \mathcal{K}_2^* are given by (7).

Proof Put $K^* = \sup_{t \in [1, e]} |\psi(t, 0)| < \infty$ and $N^* = \sup_{t \in [1, e]} |\varphi(t, 0)| < \infty$. Choose $r > 0$ such that $r \geq \frac{N^*\mathcal{K}_2^* + K^*\mathcal{K}_1^*}{1 - (\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^*)}$. Let $\mathcal{B}_r = \{w \in \mathcal{E} : \|w\| \leq r\}$. We show that $\Upsilon\mathcal{B}_r \subset \mathcal{B}_r$. Let $w \in \mathcal{B}_r$. By using assumptions (\mathcal{N}_1) and (\mathcal{N}_5), we have

$$\begin{aligned} \|\Upsilon w\| &\leq \frac{|\kappa - 1|}{\kappa\Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi - 1} \|w\| \frac{ds}{s} \\ &\quad + \frac{\alpha}{\kappa\Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - 1} (L\|w\| + K^*) \frac{ds}{s} \\ &\quad + \frac{\beta}{\kappa\Gamma(\varrho + \mu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho + \mu - 1} (\tilde{L}\|w\| + N^*) \frac{ds}{s} \\ &\quad + \frac{(1 - \kappa)[|3\Gamma(4 + \vartheta)|(\ln t)^2 + |(\delta - 3)\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(\varrho - \varpi + \vartheta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} \|w\| \frac{ds}{s} \\ &\quad + \frac{(1 - \kappa)|\Gamma(4 - \delta)|[|\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(3 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \varpi - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} \|w\| \frac{ds}{s} \\ &\quad + \frac{\alpha[|(3 - \delta)\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(4 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} (L\|w\| + K^*) \frac{ds}{s} \\ &\quad + \frac{\beta[|(3 - \delta)\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(4 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta + \mu - 1} (\tilde{L}\|w\| + N^*) \frac{ds}{s} \\ &\quad + \frac{\alpha|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)|(\ln t)^2 + |\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} (L\|w\| + K^*) \frac{ds}{s} \\ &\quad + \frac{\beta|\Gamma(4 - \delta)|[3|\Gamma(3 + \vartheta)|(\ln t)^2 + |\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \mu - \delta - 1} (\tilde{L}\|w\| + N^*) \frac{ds}{s} \\ &\leq (\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^*)r + \mathcal{K}_2^*N^* + \mathcal{K}_1^*K^* < r. \end{aligned}$$

Hence, $\mathcal{Y}\mathcal{B}_r \subset \mathcal{B}_r$. Let $t \in [1, e]$ and $w_1, w_2 \in \mathbb{R}$. Then, we have

$$\begin{aligned} \|(\mathcal{Y}w_1)(t) - (\mathcal{Y}w_2)(t)\| &\leq \frac{|\kappa - 1|}{\kappa \Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi - 1} |w_1(s) - w_2(s)| \frac{ds}{s} \\ &\quad + \frac{\alpha}{\kappa \Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - 1} |\psi(s, w_1(s)) - \psi(s, w_2(s))| \frac{ds}{s} \\ &\quad + \frac{\beta}{\kappa \Gamma(\varrho + \mu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho + \mu - 1} |\varphi(s, w_1(s)) - \varphi(s, w_2(s))| \frac{ds}{s} \\ &\quad + \frac{(1 - \kappa)[|3\Gamma(4 + \vartheta)|(\ln t)^2 + |(\delta - 3)\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(\varrho - \varpi + \vartheta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} |w_1(s) - w_2(s)| \frac{ds}{s} \\ &\quad + \frac{(1 - \kappa)|\Gamma(4 - \delta)|[|\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(3 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \varpi - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} |w_1(s) - w_2(s)| \frac{ds}{s} \\ &\quad + \frac{\alpha[|(3 - \delta)\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(4 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} |\psi(s, w_1(s)) - \psi(s, w_2(s))| \frac{ds}{s} \\ &\quad + \frac{\beta[|(3 - \delta)\Gamma(4 - \vartheta)|(\ln t)^3 + 3|\Gamma(4 + \vartheta)|(\ln t)^2]}{6\kappa(\delta + \vartheta)\Gamma(\varrho + \vartheta + \mu)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta + \mu - 1} |\varphi(s, w_1(s)) - \varphi(s, w_2(s))| \frac{ds}{s} \\ &\quad + \frac{\alpha|\Gamma(4 - \delta)|[|3\Gamma(3 + \vartheta)|(\ln t)^2 + |\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} |\psi(s, w_1(s)) - \psi(s, w_2(s))| \frac{ds}{s} \\ &\quad + \frac{\beta|\Gamma(4 - \delta)|[|3\Gamma(3 + \vartheta)|(\ln t)^2 + |\Gamma(4 - \vartheta)|(\ln t)^3]}{6\kappa(\delta + \vartheta)\Gamma(3 + \vartheta)\Gamma(\varrho + \mu - \delta)} \\ &\quad \times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \mu - \delta - 1} |\varphi(s, w_1(s)) - \varphi(s, w_2(s))| \frac{ds}{s} \\ &\leq (\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^*) \|w_1 - w_2\|. \end{aligned}$$

Since we have $\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^* < 1$, \mathcal{Y} is a contraction. By using the Banach contraction principle, \mathcal{Y} has a unique fixed point which is the unique solution of the Caputo–Hadamard problem (1)–(2). This completes the proof. \square

4 Examples

In this section, we provide three numerical examples to examine the validity of our theoretical findings. To do this, we consider constants $\kappa = 0.78$, $\alpha = 0.69$, $\beta = 0.73$, $\varrho = 3.95$, $\varpi = 3.87$, $\mu = 1.3$, $\delta = 1.92$, and $\vartheta = 0.001$ with $\delta + \vartheta = 1.921 \neq 0$ for our examples. The next example illustrates Theorem 8.

Example 1 Consider the Caputo–Hadamard fractional integro-differential equation

$$[0.78^{\text{CH}}\mathcal{D}_{1^+}^{3.95} + 0.22^{\text{CH}}\mathcal{D}_{1^+}^{3.87}]w(t) = 0.69 \frac{0.01t|w(t)|}{7 + |w(t)|} + 0.73^H\mathcal{I}_{1^+}^{1.3} \ln t(\sin w(t)) \tag{8}$$

with boundary value conditions

$$\begin{cases} w(1) = 0, & {}^{\text{CH}}\mathcal{D}_{1^+}^{1.92}w(e) = 0, \\ {}^{\text{CH}}\mathcal{D}_{1^+}w(1) = 0, & \frac{1}{\Gamma(0.001)} \int_1^e (\ln \frac{e}{s})^{0.001-1} w(s) \frac{ds}{s} = 0, \end{cases} \tag{9}$$

where $t \in [1, e]$. Define the continuous functions $\psi, \varphi : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ by $\psi(t, w) = \frac{0.99t|w|}{7+|w|}$ and $\varphi(t, w) = \ln t(\sin w)$. Note that $|\psi(t, w_1) - \psi(t, w_2)| \leq L|w_1 - w_2|$ holds for all $w_1, w_2 \in \mathbb{R}$, where $L = 0.01e > 0$. Also, the continuous function $\sigma(t) = \ln t$ on $[1, e]$ is such that $|\varphi(t, w)| \leq \sigma(t) = \ln t$ for all $w \in \mathbb{R}$. In this case, we have $\|\sigma\| = \sup_{t \in [1, e]} \sigma(t) = 1$. Some calculations show that $\mathcal{K}_0^* = 0.8968$ and $\mathcal{K}_1^* = 0.3559$. Hence, $\mathcal{K}_0^* + L\mathcal{K}_1^* = 0.9064 < 1$. By using Theorem 8, the Caputo–Hadamard problem (8)–(9) has a solution.

Next example illustrates Theorem 9.

Example 2 Consider the Caputo–Hadamard fractional integro-differential equation

$$[0.78^{\text{CH}}\mathcal{D}_{1^+}^{3.95} + 0.22^{\text{CH}}\mathcal{D}_{1^+}^{3.87}]w(t) = 0.69 \frac{1}{16+t} \left(\frac{3}{4} + \frac{|w(t)|}{2+|w(t)|} \right) + 0.73^H\mathcal{I}_{1^+}^{1.3} \frac{1}{3+\sin \frac{\pi t}{2}} \left(\frac{4}{5} + \frac{|w(t)|}{3+|w(t)|} \right) \tag{10}$$

with boundary value conditions

$$\begin{cases} w(1) = 0, & {}^{\text{CH}}\mathcal{D}_{1^+}^{1.92}w(e) = 0, \\ {}^{\text{CH}}\mathcal{D}_{1^+}w(1) = 0, & \frac{1}{\Gamma(0.001)} \int_1^e (\ln \frac{e}{s})^{0.001-1} w(s) \frac{ds}{s} = 0, \end{cases} \tag{11}$$

where $t \in [1, e]$. Define continuous maps $\psi, \varphi : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ by $\psi(t, w) = \frac{1}{16+t} (\frac{3}{4} + \frac{|w|}{2+|w|})$ and $\varphi(t, w) = \frac{1}{3+\sin \frac{\pi t}{2}} (\frac{4}{5} + \frac{|w|}{3+|w|})$. Note that $|\psi(t, w(t))| \leq \frac{1}{16+t} (1 + \|w\|)$ and $|\varphi(t, w(t))| \leq \frac{1}{3+\sin \frac{\pi t}{2}} (1 + \|w\|)$ for all $w \in \mathbb{R}$ and $t \in [1, e]$. Put $\theta_1(t) = \frac{1}{16+t}$, $\theta_2(t) = \frac{1}{3+\sin \frac{\pi t}{2}}$, and $\xi_1(\|w\|) = \xi_2(\|w\|) = 1 + \|w\|$. Then, we have $\psi(t, w) \leq \theta_1(t)\xi_1(\|w\|)$ and $\varphi(t, w) \leq \theta_2(t)\xi_2(\|w\|)$. Note that $\|\theta_1\| = \frac{1}{17} = 0.0588$, $\|\theta_2\| = \frac{1}{4} = 0.25$, and $\xi_1(\mathcal{E}) = \xi_2(\mathcal{E}) = 1 + \mathcal{E}$. Also, $\mathcal{K}_0^* = 0.8968 < 1$, $\mathcal{K}_1^* = 0.3559$, and $\mathcal{K}_2^* = 0.2995$. By considering assumption (\mathcal{N}_4) , choose $\mathcal{E} > 12.76$. Now by using Theorem 9, the Caputo–Hadamard problem (10)–(11) has a solution.

Next example illustrates Theorem 10.

Example 3 Consider the Caputo–Hadamard fractional integro-differential equation

$$[0.78^{\text{CH}}\mathcal{D}_{1^+}^{3.95} + 0.22^{\text{CH}}\mathcal{D}_{1^+}^{3.87}]w(t) = 0.69 \frac{\cos t|w(t)|}{1+|w(t)|} + 0.73^H\mathcal{I}_{1^+}^{1.3} \frac{2}{5+t} \left(\frac{|\arctan w(t)|}{|\arctan w(t)| + 1} \right) \tag{12}$$

with boundary value conditions

$$\begin{cases} w(1) = 0, & {}^{\text{CH}}\mathcal{D}_{1^+}^{1.92}w(e) = 0, \\ {}^{\text{CH}}\mathcal{D}_{1^+}w(1) = 0, & \frac{1}{\Gamma(0.001)} \int_1^e (\ln \frac{e}{s})^{0.001-1} w(s) \frac{ds}{s} = 0, \end{cases} \tag{13}$$

where $t \in [1, e]$. Define continuous maps $\psi, \varphi : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ by $\psi(t, w) = \frac{\cos t|w|}{1+|w|}$ and $\varphi(t, w) = \frac{2}{7+t} (\frac{|\arctan w(t)|}{|\arctan w(t)|+1})$. Note that $|\psi(t, w_1(t)) - \psi(t, w_2(t))| \leq \cos t(|w_1(t) - w_2(t)|)$ and $|\varphi(t, w_1(t)) - \varphi(t, w_2(t))| \leq \frac{2}{7+t}(|w_1(t) - w_2(t)|)$. Put $L = |\cos(e)| = 0.9117$ and $\tilde{L} = 0.25$. Some calculations show that $\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^* = 0.98127 < 1$. Now by using Theorem 10, the Caputo–Hadamard problem (12)–(13) has a unique solution.

5 Conclusions

It is known that we should increase our ability for studying of different types of fractional integro-differential equations. In this case, we could create modern software in the future by using advanced modelings of distinct phenomena. In this way, we should try to review different types of fractional integro-differential equations. In this work, we study the existence of solutions for a Caputo–Hadamard fractional integro-differential equation with boundary value conditions involving the Hadamard fractional operators via different orders. Also, we provide three examples to illustrate our main results.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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