117. *On a General Form of the Weyl Criterion in the Theory of Asymptotic Distribution. I*

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I. Introduction. There are a few examples of using probability theory methods in the theory of asymptotic distribution mod 1, and it seems quite natural to do so, since the theory in question deals with the problem of the distribution of the fractional parts of the elements of a sequence of real numbers and of the values of a realvalued function defined on $[0, \infty)$. In fact, the classical Weyl criterion concerning sequences of real numbers ([1]) has been viewed from the point of probability theory ([2]-[4]) and in [5] the method used in [4] is applied to the case of distributing the values mod 1 of functions defined on $[0, \infty)$. In the present paper we apply (Theorem 5) a (classic) result on the convergence of a class of distribution functions in order to find a very general form of the Weyl criterion, covering even the case of the Niven-Uchiyama criterion ([6], [7]) in the theory of the distribution of sequences of integers modulo m, and also the case of some summability-procedure distribution of sequences of real numbers, such as the Borel summability distribution of a sequence of real numbers (Application 6).

II. Definitions. Let L denote either the interval $[0, \infty)$ or the sequence $1, 2, \dots$.

The function $F(x)$ defined on the extended real line is said to be a *distribution function* (abbreviated d.f.) if $F(x)$ is bounded, non-decreasing and continuous on the left. We have then $F(-\infty)=\lim_{x\to-\infty} F(x)$ and $F(\infty)=\lim_{x\to\infty} F(x)$.

Let for each $t \in L$ a d.f. $F_t(x)$ be defined.

Now $F_t(x)$ is said to *converge weakly* to a d.f. $F(x)$ as $t\to\infty$, or $F_t(x)\xrightarrow{w}F(x)$, if

$$\lim_{t\to\infty} \{F_t(b)-F_t(a)\}=F(b)-F(a) \tag{1}$$

for every pair of continuity points a and b of $F(x)$. (See [8], p. 76, where the notion of vague convergence is defined, slightly different from the above notion of weak convergence. Also slightly different is Loève's description of weak convergence ([3], p. 178), where

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$F_n(x) \xrightarrow{w} F(x)$ implies that $F_n(x) \to F(x)$ on the continuity set of F.)

$F_t(x)$ is said to *converge completely* to a d.f. $F(x)$ as $t \to \infty$, or $F_t(x) \xrightarrow{c} F(x)$, if $F_t(x)$ converges weakly to $F(x)$ and if in addition

$$\lim_{t\to\infty} \{F_t(\infty) - F_t(-\infty)\} = F(\infty) - F(-\infty). \tag{2}$$

Example. Let $F_0(x)=0$ $(x \leq 0)$ and $=1$ $(x>0)$. Then define $F_n(x) = F_0(x+n)$. We have then $\lim_{n\to\infty} \{F_n(b) - F_n(a)\} = 0$ for every pair of real numbers a and b. So $F_n(x)$ converges weakly to every $F(x)$ that is identically equal to a constant. This implies that (2) is not satisfied. Hence $F_n(x)$ does not converge completely. See [3], p. 178.

Let $F_t(x)(t \in L)$ be a class of d.f. with the properties:

$$\begin{aligned} &F_t(\xi)=0 \text{ if } \xi \leq 0, \\ &F_t(\xi)=V(t) \text{ if } \xi>1, \\ &V(t) \leq M < \infty \text{ for all } t \in L. \end{aligned} \tag{3}$$

Suppose $F_t(x)$ is weakly convergent as $t \to \infty$, and let $F(x)$ be the weak limit. Then we have obviously $F(\xi)=V$ if $\xi>1$, where V is a constant. We suppose that $F(\xi)=0$ if $\xi \leq 0$, without loss of generality. We may show that $F_t(x)$ is also completely convergent. In fact this is an obvious conclusion from $F_t(b) - F_t(a) \to F(b) - F(a)$ as $t \to \infty$ $(a<0$ and $b>1)$, and $F_t(a) = F(a) = 0$. Moreover we have the following two theorems which are mentioned here without proof.

Theorem 1. *If a class of d.f.* $F_t(x)$, $t \in L$, *satisfies* (3), *then every sequence* $F_{t_n}(x)$ $(n=1, 2, \cdots)$ *contains a completely convergent subsequence* $F_{t_{n_k}}(x)$ $(k=1, 2, \cdots)$ (*Helly's theorem*); in other words: Every sequence of d.f. is weakly compact (see [3], p. 179).

Second we have

Theorem 2. *Let the class of d.f.* $F_t(x)$, $t \in L$, *satisfy* (3). *Let* $\mathcal{K}$ *be the class of all d.f.* $G(x)$ *such that to any* $G(x) \in \mathcal{K}$ *there exists a sequence* t_n $(n=1, 2, \cdots;\ t_n \in L)$ *such that* $F_{t_n}(x) \xrightarrow{c} G(x)$ *as* $n \to \infty$. *Then we have* $F_t(x) \xrightarrow{c} F(x)$ *as* $t \to \infty$ *if and only if* $\mathcal{K}$ *consists of one element* F *only*. We refer to [3], p. 180. Notice that Loève's definition of complete convergence differs from ours and that his statements about the variations such as $F(\infty) - F(-\infty)$ are set in terms of theorems.

III. Fourier-Stieltjes coefficients. Let F and G be d.f. with the properties

$$F(x)=0 \text{ if } x \leq 0,\ F(x)=V \text{ if } x>1, \tag{4}$$

$$G(x)=0 \text{ if } x \leq 0,\ G(x)=W \text{ if } x>1, \tag{5}$$

Define for all integers n:

$$a_n = \int_R \exp 2\pi i n x dF(x), \tag{6}$$

$$b_n=\int_R \exp 2\pi inx dG(x). \tag{7}$$

a_n and b_n are called the Fourier-Stieltjes coefficients of $F(x)$ and $G(x)$, resp. and $R=(-\infty, \infty)$.

Let $\Delta F(\xi)$ denote the increase of the d.f. $F(x)$ at $x=\xi$, or $\Delta F(\xi) =F(\xi+)-F(\xi)$.

Theorem 3. *Let F and G have the properties* (4) *and* (5), *resp. Assume that*

$$a_n=b_n \ (n=0, 1, \cdots), \tag{8}$$

then there is a constant c such that

$$G(x)=F(x)+c \ (0<x\leq 1), \tag{9}$$

$$\Delta G(0)+\Delta G(1)=\Delta F(0)+\Delta F(1). \tag{10}$$

Moreover, if $F(x)$ is continuous at $x=0$ and $x=1$, then $F(x)=G(x)$ $(-\infty<x<\infty)$.

Proof. Since $a_{-n}=\bar{a}_n$ and $b_{-n}=\bar{b}_n$, the assumption (8) holds also for $n=-1, -2, \cdots$. Evidently $a_0=b_0$ implies.

$$V=W. \tag{11}$$

Starting from (6) and (7) we find by integrating by parts:

$$a_n=V-2\pi in\int_R F(y) \exp 2\pi iny dy, \tag{12}$$

$$b_n=W-2\pi in\int_R G(y) \exp 2\pi iny dy \tag{13}$$

for all integers n. Now (8), (11), (12), and (13) imply that $F(y)$ and $G(y)$ have the same Fourier coefficients for all $n=\pm 1, \pm 2, \cdots$. This implies that there is a constant c such that

$$G(y)=F(y)+c, \quad \text{a.e. on } [0, 1].$$

Since $F(x)$ and $G(x)$ are continuous on the left we have therefore (9). Now we turn to the proof of (10). We have

$$\begin{aligned}\Delta G(0)+\Delta G(1)&=G(0+)-G(0)+G(1+)-G(1)\\&=W-\{G(1)-G(0+)\}\\&=V-\{F(1)-F(0+)\} \text{ (because of (9))}\\&=\Delta F(0)+\Delta F(1).\end{aligned}$$

Finally, if $\Delta F(0)=\Delta F(1)=0$ (or: if $F(x)$ is continuous at $x=0$ and at $x=1$), then by (10) we have $\Delta G(0)=\Delta G(1)=0$ and so $F(1)+c=G(1) =G(1+)=W$ and $F(1)=F(1+)=V$ which implies $c=0$ according to (11).

Theorem 4. *Let the class of d.f. $F_t(x)$, $t\in L$, satisfy* (3). *Let $F(x)$ be a d.f. with* (4) *and such that $\Delta F(0)=\Delta F(1)=0$. Then we have*

$$F_t(x)\xrightarrow{c}F(x), \text{ as } t\to\infty,$$

if and only if for $n=0, 1, 2, \cdots$

$$\lim_{t\to\infty}\int_R \exp 2\pi inx dF_t(x)=\int_R \exp 2\pi inx dF(x). \tag{14}$$

Proof. According to the Helly-Bray theorem (see [3], p. 182) the complete convergence of $F_t(x)$ to $F(x)$, as $t\to\infty$, implies (14).

Now suppose that (14) is satisfied. Then apply Theorem 1. Let $F_{t_n}(x)$ be a sequence of d.f. contained in the class $F_t(x)$ which converges completely to a d.f. $G(x)$, say. Again apply the Helly-Bray theorem:

$$\lim_{n\to\infty}\int_R \exp 2\pi inx dF_{t_n}(x)=\int_R \exp 2\pi inx dG(x)$$

and combining this result with (14), we find

$$\int_R \exp 2\pi inx dG(x)=\int_R \exp 2\pi inx dF(x).$$

Hence according to Theorem 3 and the assumption on the continuity of $F(x)$ at $x=0$ and $x=1$ we find that $F(x)=G(x)$. This implies that the class $\mathcal{K}$ of Theorem 2 consists of only one function F, and that therefore $F_t(x)\xrightarrow{c}F(x)$ as $t\to\infty$.

Remark 1. If in Theorem 4 we drop the condition on the continuity of $F(x)$ at $x=0$ and $x=1$, then (14) implies that the class $\mathcal{K}$ of Theorem 2 consists of d.f. satisfying (9) and (10), according to the first part of Theorem 3. We can show however that in this case

$$\lim_{t\to\infty}\{F_t(b)-F_t(a)\}=F(b)-F(a) \tag{15}$$

for every pair a, b of continuity points of F with

$$0<a\leq b<1. \tag{16}$$

For, let t_n be a sequence from L with $t_n\to\infty$ such that

$$\lim_{n\to\infty}\{F_{t_n}(b)-F_{t_n}(a)\}=A.$$

Now t_n contains a subsequence τ_n such $F_{\tau_k}(x)\to G(x)$, say, as $k\to\infty$. Then $G(x)$ satisfies (9) according to Theorem 3. Hence

$$\lim_{k\to\infty}\{F_{\tau_k}(b)-F_{\tau_k}(a)\}=G(b)-G(a), \tag{17}$$

for every pair of continuity points a and b of G (definition of complete convergence), and thus this difference is equal to $F(b)-F(a)$. So $A=F(b)-F(a)$. As the passage to the limit in (15) holds for some subsequence of any sequence from L, according to the preceding argument, we have shown (15). We shall see that this statement has an important application.

IV. The Weyl criterion and its generalizations. Let $f(t)$ be a realvalued Borel measurable function defined on $[0, \infty)$. Let $[f(t)]$, $(f(t))$ denote the integral and the fractional part of $f(t)$ respectively. Let $B(t)$ be a nondecreasing function defined on $[0, \infty)$ and continuous on the left. Let $\chi_{[0,\xi)}(x)$ be the characteristic function of the interval $[0, \xi)$. Define the following class of d.f.:

$$F_T(\xi)=\frac{1}{B(T)}\int_0^T \chi_{[0,\xi)}((f(t)))dB(t) \quad (0\leq\xi<1) \tag{18}$$

for all T with $B(T)>0$, and

$$F_T(\xi)=0 \ (\xi \leq 0),$$
$$F_T(\xi)=1 \ (\xi > 1). \tag{19}$$

It is easily seen that $F_T(\xi)$ is continuous on the left. For,

$$F_T(\xi)-F_T(\xi-\delta)=\frac{1}{B(T)}\int_0^T \chi_{[\xi-\delta,\xi)}((f(t)))dB(t),$$

and the integral on the right of this equality tends to zero as $\delta \to 0+$ by the Lebesgue bounded convergence theorem.

Let $F(\xi)$ be a d.f. with

$$F(\xi)=0 \ (\xi \leq 0) \text{ and } F(\xi)=1 \ (\xi > 1). \tag{20}$$

Theorem 5 (Weyl criterion). *Let the class of d.f. $F_T(\xi)$ and the d.f. $F(\xi)$ be defined as in the first lines of* IV. *It is assumed that $\Delta F(0)=\Delta F(1)=0$. Then we have*

$$F_T(\xi) \xrightarrow{c} F(\xi) \text{ as } T \to \infty, \tag{21}$$

if and only if for $k=1, 2, \cdots$

$$\lim_{T\to\infty} \frac{1}{B(T)}\int_0^T \exp 2\pi i k f(t) dB(t) = \int_0^1 \exp 2\pi i k x dF(x) \tag{22}$$

Proof. According to Theorem 4 we have (21) if and only if for $k=1, 2, \cdots$

$$\lim_{T\to\infty} \int_R \exp 2\pi i k x dF_T(x) = \int_R \exp 2\pi i k x dF(x). \tag{23}$$

We show that the first member of (23) is equal to the first member of (22). Let T be fixed. Consider the measure space $(\Omega, \mathscr{B}, \mu)$ where $\Omega=[0, T)$, $\mathscr{B}$ the class of all Borel sets on $[0, T)$, and let for every set $A \in \mathscr{B}$

$$\mu(A)=\frac{1}{B(T)}\int_A dB(t).$$

Now $(f(t))$ is Borel measurable on $(\Omega, \mathscr{B}, \mu)$. Hence (18) implies $F_T(\xi)=\mu(E_\xi)$, where

$$E_\xi=\{t : t \in \Omega, 0 \leq (f(t)) < \xi\}.$$

By applying a well-known transformation theorem (see [9], sec. 39, Theorem C) one finds

$$\int_R \exp 2\pi i k x dF_T(x) = \int_\Omega \exp 2\pi i k x f(t) d\mu(t) = \frac{1}{B(T)}\int_0^T \exp 2\pi i k f(t) dB(t).$$

(Direct computation gives the same result:

$$\begin{aligned}\int_R \exp 2\pi i k x dF_T(x) &= \int_0^1 \exp 2\pi i k x d \frac{1}{B(T)}\int_0^T \chi_{[0,x)}((f(t)))dB(t)\\ &= \frac{1}{B(T)}\int_0^T \left\{\int_0^1 \exp 2\pi i k x d\, \chi_{[0,x)}((f(t)))\right\} dB(t)\\ &= \frac{1}{B(T)}\int_0^T \exp 2\pi i k f(t) dB(t).\Big)\end{aligned}$$

Now letting $T \to \infty$ one finds the conclusion of the theorem.

Remark 2. If the condition $\Delta F(0)=\Delta F(1)=0$ is removed from Theorem 5, in which case we have $\Delta F(0)+\Delta F(1)>0$, then (22) is necessary and sufficient in order to have

$$\lim_{T\to\infty}\{F_T(b)-F_T(a)\}=F(b)-F(a) \tag{24}$$

for every two points of continuity a, b of $F(x)$ with $0<a\leq b<1$, according to Remark 1.

(*To be continued. References will be found at the end of the second note.*)