ON A GENERALIZATION OF ANDO'S DILATION THEOREM

NIRUPAMA MALLICK AND K. SUMESH

Abstract. We introduce the notion of Q-commuting operators which includes commuting operators. We prove a generalized version of the commutant lifting theorem and Ando's dilation theorem in the context of Q-commuting operators.

1. Introduction

Throughout, $\mathcal{H}$ and $\mathcal{K}$ denote complex Hilbert spaces and $\mathcal{B}(\cdot)$ denotes the space of all bounded linear maps. Suppose $\mathcal{H}_i$ and $\mathcal{K}_i$ are Hilbert spaces such that $\mathcal{H}_i \subseteq \mathcal{K}_i,\ i = 1, 2$. Given $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ a bounded linear operator $S \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ is said to be an

(i) *extension* of T if $S(h) = T(h)$ for all $h \in \mathcal{H}_1$. (In such cases we write $S|_{\mathcal{H}_1} = T$.)

(ii) *lifting* of T if $S(\mathcal{H}_1^\perp) \subseteq \mathcal{H}_2^\perp$ and $T = P_{\mathcal{H}_2} S|_{\mathcal{H}_1}$ (equivalently $S^*|_{\mathcal{H}_2} = T^*$).

With respect to the decompositions $\mathcal{K}_1 = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$ and $\mathcal{K}_2 = \mathcal{H}_2 \oplus \mathcal{H}_2^\perp$, in (i) and (ii) the operator S has the matrix form $S = \begin{bmatrix} T & * \\ 0 & * \end{bmatrix}$ and $S = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix}$, respectively. Note that S is a lifting of T if and only if S^* is an extension of T^*.

An operator $S \in \mathcal{B}(\mathcal{K})$ is said to be a *dilation* of $T \in \mathcal{B}(\mathcal{H})$ if $\mathcal{H} \subseteq \mathcal{K}$ and $T^n = P_{\mathcal{H}} S^n|_{\mathcal{H}}$ for all $n \geq 0$, where $P_{\mathcal{H}} \in \mathcal{B}(\mathcal{K})$ is the orthogonal projection onto $\mathcal{H}$. In such case, with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, the operator S^n has the matrix form $S^n = \begin{bmatrix} T^n & * \\ * & * \end{bmatrix}$ for all $n \geq 0$. Clearly extension and lifting of $T \in \mathcal{B}(\mathcal{H})$ are dilations. It is well known that (see [6, 7]) given any contraction $T \in \mathcal{B}(\mathcal{H})$ there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an isometry $V \in \mathcal{B}(\mathcal{K})$ such that V is a dilation of T. Such a pair $(V, \mathcal{K})$ is called an *isometric dilation* of T. An isometric dilation $(V, \mathcal{K})$ is said to be *minimal* if

$$\mathcal{K} = \overline{\mathsf{span}}\{V^n(\mathcal{H}) : n \geq 0\}. \tag{1.1}$$

Minimal isometric dilations are unique upto unitary equivalence in the sense that if $(V, \mathcal{K})$ and $(V', \mathcal{K}')$ are two minimal isometric dilations, then there exists a unitary $U : \mathcal{K} \to \mathcal{K}'$ such that $U|_{\mathcal{H}} = I_{\mathcal{H}}$ and $UV = V'U$. Sz-Nagy proved ([5, 7]) that given a contraction $T \in \mathcal{B}(\mathcal{H})$ there exists a Hilbert space $\mathcal{K}$ and a unitary $U \in \mathcal{B}(\mathcal{K})$ such that U is a dilation of T. Such a pair $(U, \mathcal{K})$ is called a *unitary dilation* of T. Moreover, such a dilation is unique (up to unitary equivalence) if it is *minimal*, in the sense that

$$\mathcal{K} = \overline{\mathsf{span}}\{U^n(\mathcal{H}) : n \in \mathbb{Z}\}. \tag{1.2}$$

Date: October 31, 2019.

2010 *Mathematics Subject Classification.* 47A20.

Key words and phrases. Isometric dilation, unitary dilation, co-isometric extension, commutant lifting, Ando's dilation theorem.

Nirupama Mallick, Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam, Siruseri, Tamilnadu-603103, India. Email: `niru.mallick@gmail.com`.

K. Sumesh, Indian Institute of Technology Madras, Sardar Patel Road, Opposite to C. L.R.I, Adyar, Chennai-600036, India. Email: `sumeshkpl@gmail.com, sumeshkpl@iitm.ac.in`.

In [9] Schaffer gave an elementary proof of the existence of minimal unitary dilation of a contraction.

Given a contraction $T \in \mathcal{B}(\mathcal{H})$ there always exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a co-isometry $W \in \mathcal{B}(\mathcal{K})$ which extends T, and we call $(W, \mathcal{K})$ a *co-isometric extension* of T. In fact, by considering the lower right-hand corner of the matrix form of the Schaffer's construction (see [9, 7]), one can get a co-isometric extension $(W, \mathcal{K})$ of T which is *minimal* in the sense that

$$\mathcal{K} = \overline{\mathsf{span}}\{W^{*n}(\mathcal{H}) : n \geq 0\}. \tag{1.3}$$

Note that if $(V^*, \mathcal{K})$ is a minimal co-isometric extension of T^*, then $(V, \mathcal{K})$ is an isometric lifting and hence a dilation of T which is *minimal* in the sense that (1.1) holds. Now from the uniqueness property, it follows that $\mathcal{H}^\perp$ is invariant for every minimal isometric dilation of $T \in \mathcal{B}(\mathcal{H})$. Thus, a pair $(V, \mathcal{K})$ is a minimal isometric dilation of T if and only if $(V, \mathcal{K})$ is a minimal isometric lifting of T if and only if $(V^*, \mathcal{K})$ is a minimal co-isometric extension of T^*.

Ando ([1]) proved that given any two contractions $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ which are commuting (i.e., $T_1T_2 = T_2T_1$) there exists a Hilbert space $\mathcal{K}_0 \supseteq \mathcal{H}$ and commuting isometries $V_1, V_2 \in \mathcal{B}(\mathcal{K}_0)$ such that

$$T_1^n T_2^m = P_{\mathcal{H}} V_1^n V_2^m|_{\mathcal{H}}$$

for all $n, m \geq 0$. In fact, V_i can be chosen to be a lifting of $T_i, i = 1, 2$. Further, using Ito's theorem [3] he concluded that there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and commuting unitary operators $U_1, U_2 \in \mathcal{B}(\mathcal{K})$ such that

$$T_1^n T_2^m = P_{\mathcal{H}} U_1^n U_2^m|_{\mathcal{H}}$$

for all $n, m \geq 0$. This is known as the *Ando's dilation* theorem.

Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ be a contraction with isometric lifting $(V_i, \mathcal{K}_i)$ (respectively co-isometric extension $(W_i, \mathcal{K}_i)$), $i = 1, 2$. Suppose $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ intertwines T_1 and T_2, i.e., $XT_1 = T_2X$. Then, due to Sz-Nagy and Foias ([6],[7]) there exists a norm-preserving lifting (respectively extension) $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X which intertwine V_1 and V_2 (respectively W_1 and W_2). This result is called *intertwining lifting* (respectively *intertwining co-extension*) theorem. The case when $T_1 = T_2$ and $V_1 = V_2$ (respectively $W_1 = W_2$) is known as *commutant lifting* (respectively *commutant co-extension*) theorem.

One may ask how these dilation theorems of commuting pair of contractions can be generalized to the setting of noncommuting pair of contractions T_1, T_2? In [8] Sebestyen proved analogues of commutant lifting theorem and Ando's dilation theorem for *anticommuting* pair (i.e., $T_2T_1 = -T_1T_2$) of contractions. In [4] Keshari and Mallick considered *q-commuting* operators (i.e., $T_2T_1 = qT_1T_2$) where $q \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. They proved a generalized version of the commutant lifting theorem, intertwining lifting theorem and Ando's dilation theorem in the context of q-commuting contractions, and called them as *q-commutant lifting* theorem, *q-intertwining lifting* theorem and *q-commutant dilation* theorem respectively. In this article we consider operators T_1 and T_2 which are *Q-commuting* i.e., T_2T_1 equals either QT_1T_2, T_1QT_2 or T_1T_2Q for some bounded operator Q. Our main aim is to prove an analogue of the commutant lifting theorem and Ando's dilation theorem to the setting of Q-commuting operators. As a first step we characterize (Theorem 2.6) Q-commutants of a contraction $T \in \mathcal{B}(\mathcal{H})$ in terms of $\overline{Q}$-commutants of its minimal isometric dilation $(V, \mathcal{K})$, where $\overline{Q} = Q \oplus Q' \in \mathcal{B}(\mathcal{K})$ with $Q' \in \mathcal{B}(\mathcal{H}^\perp)$. This is a generalized version of (q-)commutant lifting theorem ([4, 6]). The proof uses Schaffer construction. Further using ideas from [2] we prove generalized versions (see Theorem 2.8, 2.15) of (q-)intertwining lifting theorem. In Theorem 2.14 we characterize Q-commutants of a contraction in terms of $\overline{Q}$-commutants of its minimal unitary dilation. Finally, we prove our main theorems that Q-commuting contractions can be dilated into $\overline{Q}$-commuting isometries (Theorem 2.16) and further into $\overline{Q}$-commuting unitaries (Theorem 2.20). These results generalize Ando's dilation theorem and q-commutant dilation theorem. The proofs consist of standard dilation theoretic arguments.

2. Main results

Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces such that $\mathcal{H} \subseteq \mathcal{K}$. Given any $Q \in \mathcal{B}(\mathcal{H})$ we let $\overline{Q}_{\mathcal{K}}$ (or simply $\overline{Q}$) denotes any bounded operator on $\mathcal{K}$ such that $\mathcal{H}$ is a reducing subspace for $\overline{Q}$ and $\overline{Q}|_{\mathcal{H}} = Q$. Note that $(Q \oplus qI_{\mathcal{H}^{\perp}}) \in \mathcal{B}(\mathcal{K})$ is an example for such an operator $\overline{Q}$ for every $q \in \mathbb{C}$. If Q is a contraction or (co-)isometry or unitary, then we require $\overline{Q}$ also to be a contraction or (co-)isometry or unitary, respectively.

Definition 2.1. Given $Q \in \mathcal{B}(\mathcal{H})$ two operators $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are said to be Q-*commuting* if one of the following happens:

$$T_2T_1 = QT_1T_2 \text{ or } T_2T_1 = T_1QT_2 \text{ or } T_2T_1 = T_1T_2Q. \tag{2.1}$$

If $Q = I_{\mathcal{H}}$ (respectively $Q = -I_{\mathcal{H}}$), then Q-commuting means commuting (respectively anti-commuting).

Example 2.2. Let $T_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$ in $M_2(\mathbb{C})$. Note that T_1, T_2 are not commuting. In fact, there does not exists any $q \in \mathbb{C}$ such that $T_2T_1 = qT_1T_2$. But $Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_2(\mathbb{C})$ are such that $T_2T_1 = T_1T_2Q = T_1Q'T_2$. Note that there is no $Q \in M_2(\mathbb{C})$ such that $T_2T_1 = QT_1T_2$.

Example 2.3. Suppose $L, R, Q, Q' \in \mathcal{B}(\ell^2)$ are the linear operators given by

$$\begin{aligned} L(x_1, x_2, x_3, \cdots) &:= (x_2, x_3, x_4, \cdots) \\ R(x_1, x_2, x_3, \cdots) &:= (0, x_1, x_2, x_3, \cdots) \\ Q(x_1, x_2, x_3, \cdots) &:= (0, x_2, x_3, x_4, \cdots) \\ Q'(x_1, x_2, x_3, \cdots) &:= (0, 0, x_3, x_4, \cdots) \end{aligned}$$

Clearly $qLR \neq RL$ for all $q \in \mathbb{C}$. Note that $RL = QLR = LQ'R = LRQ$.

Example 2.4. Suppose $T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ in $M_2(\mathbb{C})$. Note that there does not exists any $Q \in M_2(\mathbb{C})$ such that T_1 and T_2 are Q-commuting.

The above example shows that given two operators $S, T \in \mathcal{B}(\mathcal{H})$, there may not exist always an operator $Q \in \mathcal{B}(\mathcal{H})$ such that S and T are Q-commuting. However, the next Lemma says that given two operators $T, Q \in \mathcal{B}(\mathcal{H})$, under some suitable conditions, there always exists an operator $S \in \mathcal{B}(\mathcal{H})$ such that S and T are Q-commuting. This is a generalization of [4, Lemma 3.5]. Recall that $T \in \mathcal{B}(\mathcal{H})$ is called a pure co-isometry if $TT^* = I$ and $T^{*n} \to 0$ in the strong operator topology.

Lemma 2.5. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is a pure co-isometry and $Q \in \mathcal{B}(\mathcal{H})$ is an isometry. If $T^*(\mathcal{H})$ is invariant for Q, then there exists a co-isometry $S \in \mathcal{B}(\mathcal{H})$ such that $TS = STQ$.*

Proof. Since T^* is an isometry $\mathcal{W} = (T^*\mathcal{H})^\perp$ is a wandering subspace for T^*, i.e., $T^{*m}(\mathcal{W}) \perp T^{*n}(\mathcal{W})$ for all $m \neq n \in \mathbb{N}$. Moreover, since T is pure co-isometry $\mathcal{H} = \bigoplus_{n=0}^\infty T^{*n}(\mathcal{W})$. As $T^*(\mathcal{H}) \subseteq \mathcal{W}^\perp$ and $Q^*(\mathcal{W}) \subseteq \mathcal{W}$, for $m < n$ and $w, w' \in \mathcal{W}$ we have

$$\langle (QT^*)^n w, (QT^*)^m w' \rangle = \langle (QT^*)^{n-m} w, w' \rangle = \langle T^*(QT^*)^{n-m-1} w, Q^* w' \rangle = 0.$$

Thus $(QT^*)^m(\mathcal{W}) \perp (QT^*)^n(\mathcal{W})$ for all $m \neq n \in \mathbb{N} \cup \{0\}$. Define $S_0 : \mathcal{H} \to \mathcal{H}$ by

$$S_0\Big(\sum_{n=0}^\infty T^{*n} w_n\Big) = \sum_{n=0}^\infty (QT^*)^{n+1} w_n.$$

Then,

$$\begin{aligned}
\Big\|S_0\Big(\sum_{n\geq 0} T^{*n} w_n\Big)\Big\|^2 &= \sum_{n\geq 0}\sum_{m\geq 0} \langle (QT^*)^{n+1} w_n, (QT^*)^{m+1} w_m \rangle \\
&= \sum_{n\geq 0} \langle (QT^*)^{n+1} w_n, (QT^*)^{n+1} w_n \rangle \\
&= \sum_{n\geq 0} \langle w_n, w_n \rangle \\
&= \sum_{n\geq 0} \langle T^{*n} w_n, T^{*n} w_n \rangle \\
&= \sum_{n\geq 0}\sum_{m\geq 0} \langle T^{*n} w_n, T^{*m} w_m \rangle \\
&= \Big\|\sum_{n\geq 0} T^{*n} w_n\Big\|^2.
\end{aligned}$$

Thus S_0 is a well-defined isometry. Moreover, for $w_n \in \mathcal{W}, n \geq 1$ we have

$$S_0 T^*\Big(\sum_{n\geq 0} T^{*n} w_n\Big) = \sum_{n\geq 0} (QT^*)^{n+2} w_n = QT^* S_0\Big(\sum_{n\geq 0} T^{*n} w_n\Big),$$

hence $S_0 T^* = QT^* S_0$. Take adjoint on both sides to get $TS = STQ$, where $S = S_0^*$. ☐

2.1. **Lifting theorems.** We recall some basic facts which we will be using frequently. Suppose $T \in \mathcal{B}(\mathcal{H})$ is a contraction with dilation $(V, \mathcal{K})$ and let $Y \in \mathcal{B}(\mathcal{K})$ be an extension of $X \in \mathcal{B}(\mathcal{H})$. Then w.r.t to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ we have $Y = \begin{bmatrix} X & * \\ 0 & * \end{bmatrix}$ and $V^n = \begin{bmatrix} T^n & * \\ * & * \end{bmatrix}$ for all $n \geq 0$. Note that $V^n Y^m = \begin{bmatrix} T^n X^m & * \\ * & * \end{bmatrix}$, so that $T^n X^m = P_\mathcal{H} V^n Y^m|_\mathcal{H}$ for all $n, m \geq 0$. Similarly if Y is any lifting of X, then $X^n T^m = P_\mathcal{H} Y^n V^m|_\mathcal{H}$ for all $n, m \geq 0$.

Now we prove an analogue of the (q-)commutant lifting theorem for Q-commuting operators.

Theorem 2.6 (Q-commutant lifting). *Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction with isometric lifting $(V, \mathcal{K})$, and let $X \in \mathcal{B}(\mathcal{H})$. Suppose $Q \in \mathcal{B}(\mathcal{H})$ and $\overline{Q} \in \mathcal{B}(\mathcal{K})$ are contractions.*

(i) *If* $XT = QTX$, *then there exists a lifting* $Y \in \mathcal{B}(\mathcal{K})$ *of* X *such that* $YV = \overline{Q}VY$.

(ii) *If* $XT = TQX$, *then there exists a lifting* $Y \in \mathcal{B}(\mathcal{K})$ *of* X *such that* $YV = V\overline{Q}Y$.

Further assume that Q *and* $\overline{Q}$ *are unitary.*

(iii) *If* $XT = TXQ$, *then there exists a lifting* $Y \in \mathcal{B}(\mathcal{K})$ *of* X *such that* $YV = VY\overline{Q}$.

In all cases $T^nX^m = P_{\mathcal{H}}V^nY^m|_{\mathcal{H}}$ *and* $X^nT^m = P_{\mathcal{H}}Y^nV^m|_{\mathcal{H}}$ *for all* $n, m \geq 0$. *Moreover,* Y *can be chosen such that* $\|Y\| = \|X\|$.

Proof. (i) Set $\widehat{T} = \begin{bmatrix} QT & 0 \\ 0 & T \end{bmatrix}$ and $\widehat{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. Let $D = (I - \overline{Q}^*\overline{Q})^{\frac{1}{2}} \in \mathcal{B}(\mathcal{K})$ and $\mathcal{K}_0 = \overline{\mathsf{ran}}(DV) \subseteq \mathcal{K}$. Let $\widetilde{\mathcal{K}} = \mathcal{K} \bigoplus (\bigoplus_1^\infty \mathcal{K}_0)$. We consider $\mathcal{H} \subseteq \mathcal{K} \subseteq \widetilde{\mathcal{K}}$ through the canonical identification. Now define $\widetilde{V} \in \mathcal{B}(\widetilde{\mathcal{K}})$ by

$$\widetilde{V} = \begin{bmatrix} \overline{Q}V & 0 & 0 & \dots & \cdots \\ DV & 0 & 0 & \dots & \cdots \\ 0 & I_{\mathcal{K}_0} & 0 & \dots & \cdots \\ 0 & 0 & I_{\mathcal{K}_0} & \dots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

Note that $\widetilde{V}$ is an isometry. Also since $\overline{Q}^*|_{\mathcal{H}} = Q^*$ and $V^*|_{\mathcal{H}} = T^*$ we have

$$\widetilde{V}^*h = (\overline{Q}V)^*h = V^*(\overline{Q}^*h) = T^*Q^*h = (QT)^*h$$

for all $h \in \mathcal{H}$, i.e., $\widetilde{V}^*|_{\mathcal{H}} = (QT)^*$. Thus $\widetilde{V}$ is an isometric lifting of QT. Set $\widehat{V} = \begin{bmatrix} \widetilde{V} & 0 \\ 0 & V \end{bmatrix} \in \mathcal{B}(\widetilde{\mathcal{K}} \oplus \mathcal{K})$. Clearly $\widehat{V}$ is an isometric lifting of the contraction $\widehat{T}$. Since $\widehat{T}\widehat{X} = \widehat{X}\widehat{T}$, by commutant lifting theorem there exists $\widehat{Y} \in \mathcal{B}(\widetilde{\mathcal{K}} \oplus \mathcal{K})$ such that $\widehat{V}\widehat{Y} = \widehat{Y}\widehat{V}$, $\widehat{Y}^*|_{\mathcal{H}\oplus\mathcal{H}} = \widehat{X}^*$ and $\|\widehat{Y}\| = \|\widehat{X}\|$. Let $\widehat{Y} = \begin{bmatrix} * & B \\ * & * \end{bmatrix} \in \mathcal{B}(\widetilde{\mathcal{K}} \oplus \mathcal{K})$ where $B = \begin{bmatrix} Y & Y_1 & Y_2 & Y_3 \dots \end{bmatrix}^{tr} \in \mathcal{B}(\mathcal{K}, \widetilde{\mathcal{K}})$ with respect to the decomposition $\widetilde{\mathcal{K}} = \mathcal{K} \bigoplus (\bigoplus_1^\infty \mathcal{K}_0)$. Then

$$\widehat{V}\widehat{Y} = \widehat{Y}\widehat{V} \implies \widetilde{V}B = BV \implies \overline{Q}VY = YV,$$

where $Y \in \mathcal{B}(\mathcal{K})$. Also

$$\widehat{Y}^*|_{\mathcal{H}\oplus\mathcal{H}} = \widehat{X}^* \implies B^*|_{\mathcal{H}} = X^* \implies Y^*|_{\mathcal{H}} = X^*,$$

so that Y is a lifting of X. Hence $\|X\| \leq \|Y\| \leq \|B\| \leq \|\widehat{Y}\| = \|\widehat{X}\| = \|X\|$.

(ii) Set $\widehat{T} = \begin{bmatrix} TQ & 0 \\ 0 & T \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. Let $\widehat{X}, D, \widetilde{\mathcal{K}}$ be as in case (i) with $\mathcal{K}_0 = \overline{\mathsf{ran}}(VD) \subseteq \mathcal{K}$. Define $\widetilde{V} \in \mathcal{B}(\widetilde{\mathcal{K}})$ by

$$\widetilde{V} = \begin{bmatrix} V\overline{Q} & 0 & 0 & \dots & \cdots \\ VD & 0 & 0 & \dots & \cdots \\ 0 & I_{\mathcal{K}_0} & 0 & \dots & \cdots \\ 0 & 0 & I_{\mathcal{K}_0} & \dots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

Note that $\widehat{V} = \begin{bmatrix} \widetilde{V} & 0 \\ 0 & V \end{bmatrix} \in \mathcal{B}(\widetilde{\mathcal{K}} \oplus \mathcal{K})$ is an isometric lifting of the contraction $\widehat{T}$, and since $\widehat{T}\widehat{X} = \widehat{X}\widehat{T}$, by proceeding as in case (i) we can get $Y \in \mathcal{B}(\mathcal{K})$ such that $V\overline{Q}Y = YV, Y^*|_{\mathcal{H}} = X^*$ and $\|Y\| = \|X\|$.

(iii) Suppose Q is a unitary. Set $\widehat{T} = \begin{bmatrix} TQ^* & 0 \\ 0 & T \end{bmatrix}$ and $\widehat{X} = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$, and $\widehat{V} = \begin{bmatrix} V\overline{Q}^* & 0 \\ 0 & V \end{bmatrix}$ on $\mathcal{K} \oplus \mathcal{K}$. Note that $(\widehat{V}, \mathcal{K} \oplus \mathcal{K})$ is an isometric lifting of $\widehat{T}$. Since $\widehat{T}\widehat{X} = \widehat{X}\widehat{T}$, by commutant lifting theorem there exists a lifting $\widehat{Y} = \begin{bmatrix} * & * \\ Y & * \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K})$ of $\widehat{X}$ such that $\widehat{Y}\widehat{V} = \widehat{V}\widehat{Y}$ and $\|\widehat{Y}\| = \|\widehat{X}\|$. Observe that Y is the required lifting of X. This completes the proof. □

Remark 2.7. In Theorem 2.6(iii) suppose Q is only a co-isometry, so that $TX = XTQ^*$. The above proof shows that, in such case also we can get a lifting Y of X satisfying all properties except the equality $VY\overline{Q} = YV$, but we get $VY = YV\overline{Q}^*$.

Theorem 2.8 (Q-intertwining lifting). *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ be a contraction with isometric lifting $(V_i, \mathcal{K}_i)$, $i = 1, 2$, and let $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Suppose $Q \in \mathcal{B}(\mathcal{H}_2)$ and $\overline{Q} \in \mathcal{B}(\mathcal{K}_2)$ are contractions.*

(i) *If $XT_1 = QT_2X$, then there exists a lifting $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X such that $YV_1 = \overline{Q}V_2Y$.*

(ii) *If $XT_1 = T_2QX$, then there exists a lifting $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X such that $YV_1 = V_2\overline{Q}Y$.*

Suppose $Q \in \mathcal{B}(\mathcal{H}_1)$ and $\overline{Q} \in \mathcal{B}(\mathcal{K}_1)$ are unitary.

(iii) *If $XT_1 = T_2XQ$, then there exists a lifting $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X such that $YV_1 = V_2Y\overline{Q}$. In all cases $T_2^nX = P_{\mathcal{H}_2}V_2^nY|_{\mathcal{H}_1}$ and $XT_1^n = P_{\mathcal{H}_2}YV_1^n|_{\mathcal{H}_1}$ for all $n \geq 0$. Moreover, Y can be chosen such that $\|Y\| = \|X\|$.*

Proof. First assume that $XT_1 = QT_2X$. Set

$$\widehat{T} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \ \widehat{X} = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}, \ \mathcal{Q} = \begin{bmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & Q \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad \text{and}$$

$$\overline{\mathcal{Q}} = \begin{bmatrix} I_{\mathcal{K}_1} & 0 \\ 0 & \overline{Q} \end{bmatrix}, \ \widehat{V} = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{K}_2).$$

Note that $\mathcal{H}_1 \oplus \mathcal{H}_2$ is reducing for the contraction $\overline{\mathcal{Q}}$, and $\overline{\mathcal{Q}}|_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \mathcal{Q}$. Since $\widehat{X}\widehat{T} = \mathcal{Q}\widehat{T}\widehat{X}$ and $\widehat{V}$ is an isometric lifting of $\widehat{T}$, by Theorem 2.6 there exists a lifting $\widehat{Y} = \begin{bmatrix} * & * \\ Y & * \end{bmatrix} \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ of $\widehat{X}$ such that $\widehat{Y}\widehat{V} = \overline{\mathcal{Q}}\widehat{V}\widehat{Y}$ and $\|\widehat{Y}\| = \|\widehat{X}\|$. As $\widehat{Y}\widehat{V} = \overline{\mathcal{Q}}\widehat{V}\widehat{Y}$ we get $YV_1 = \overline{Q}V_2Y$. Also since $\widehat{Y}^*|_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \widehat{X}^*$ we have $Y^*|_{\mathcal{H}_2} = X^*$, i.e., Y is a lifting of X. Hence $\|X\| \leq \|Y\| \leq \|\widehat{Y}\| = \|\widehat{X}\| = \|X\|$. The case when $T_2QX = XT_1$ can be proved similarly since $\widehat{T}\mathcal{Q}\widehat{X} = \widehat{X}\widehat{T}$. For the case when $XT_1 = T_2XQ$ repeat the above process by taking $\mathcal{Q} = \begin{bmatrix} Q & 0 \\ 0 & I_{\mathcal{H}_2} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\overline{\mathcal{Q}} = \begin{bmatrix} \overline{Q} & 0 \\ 0 & I_{\mathcal{K}_2} \end{bmatrix} \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$. □

Remark 2.9. Theorem 2.8 can also be deduced from the classical intertwining theorem as follows: To prove (i) of Theorem 2.8 let $(\widetilde{V}, \widetilde{\mathcal{K}})$ be an isometric lifting of the contraction $\overline{Q}V_2 = \begin{bmatrix} QT_2 & 0 \\ * & * \end{bmatrix} \in \mathcal{B}(\mathcal{K}_2)$. Then $(\widetilde{V}, \widetilde{\mathcal{K}})$ is also an isometric lifting of QT_2. Since $XT_1 = (QT_2)X$, the classical intertwining lifting theorem yields a lifting $\widehat{Y} \in \mathcal{B}(\mathcal{K}_1, \widetilde{\mathcal{K}})$ of X such that $\widehat{Y}V_1 = \widetilde{V}\widehat{Y}$ and $\|\widehat{Y}\| = \|X\|$. With respect to the decomposition $\widetilde{\mathcal{K}} = \mathcal{K}_2 \oplus \mathcal{K}_2^{\perp}$ let $\widetilde{V} = \begin{bmatrix} \overline{Q}V_2 & 0 \\ * & * \end{bmatrix}$ and $\widehat{Y} = \begin{bmatrix} Y \\ Y_0 \end{bmatrix}$ with $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. Then $\widehat{Y}V_1 = \widetilde{V}\widehat{Y}$ implies that $YV_1 = \overline{Q}V_2Y$. Also, since $\widehat{Y}$ is a lifting of X we have Y is a lifting of X, so that $\|X\| \leq \|Y\| \leq \|\widehat{Y}\| = \|X\|$. Part (ii) is proved similarly by replacing $\overline{Q}V_2$ with $V_2\overline{Q}$. For (iii), rewrite the assumption as $X(T_1Q^*) = T_2X$, and note that $V_1\overline{Q}^*$ is already an isometric lifting of T_1Q^*, so the result follows even more directly from classical intertwining lifting theorem. Specializing Theorem 2.8 to $T_1 = T_2$ and $V_1 = V_2$, we obtain Theorem 2.6.

Suppose $T \in \mathcal{B}(\mathcal{H})$. Recall that $(V, \mathcal{K})$ is an isometric lifting of a T if and only if $(V^*, \mathcal{K})$ is an co-isometric extension of T^*. So we can restate the Theorems 2.6, 2.8 as follows. We will be using this versions later.

Theorem 2.10 (Q-commutant extension). *Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction with co-isometric extension $(W, \mathcal{K})$, and let $X \in \mathcal{B}(\mathcal{H})$. Suppose $Q \in \mathcal{B}(\mathcal{H})$ and $\overline{Q} \in \mathcal{B}(\mathcal{K})$ are contractions.*

(i) *If $XTQ = TX$, then there exists an extension $Y \in \mathcal{B}(\mathcal{K})$ of X such that $YW\overline{Q} = WY$.*

(ii) *If $XQT = TX$, then there exists an extension $Y \in \mathcal{B}(\mathcal{K})$ of X such that $Y\overline{Q}W = WY$.*

Further assume that Q and $\overline{Q}$ are unitary.

(iii) *If $QXT = TX$, then there exists an extension $Y \in \mathcal{B}(\mathcal{K})$ of X such that $\overline{Q}YW = WY$.*

In all cases $T^nX^m = P_{\mathcal{H}}W^nY^m|_{\mathcal{H}}$ and $X^nT^m = P_{\mathcal{H}}Y^nW^m|_{\mathcal{H}}$ for all $n, m \geq 0$. Moreover, Y can be chosen such that $\|Y\| = \|X\|$.

Theorem 2.11 (Q-intertwining extension). *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ be a contraction with co-isometric extension $(W_i, \mathcal{K}_i)$, $i = 1, 2$ and let $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Suppose $Q \in \mathcal{B}(\mathcal{H}_1)$ and $\overline{Q} \in \mathcal{B}(\mathcal{K}_1)$ are contractions.*

(i) *If $XT_1Q = T_2X$, then there exists an extension $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X such that $YW_1\overline{Q} = W_2Y$.*

(ii) *If $XQT_1 = T_2X$, then there exists an extension $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X such that $Y\overline{Q}W_1 = W_2Y$.*

Suppose $Q \in \mathcal{B}(\mathcal{H}_2)$ and $\overline{Q} \in \mathcal{B}(\mathcal{K}_2)$ are unitary.

(iii) *If $QXT_1 = T_2X$, then there exists an extension $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X such that $\overline{Q}YW_1 = W_2Y$.*

In all cases $T_2^nX = P_{\mathcal{H}_2}V_2^nY|_{\mathcal{H}_1}$ and $XT_1^n = P_{\mathcal{H}_2}YV_1^n|_{\mathcal{H}_1}$ for all $n \geq 0$. Moreover, Y can be chosen such that $\|Y\| = \|X\|$.

Remark 2.12. Note that case (i) is a stronger version of [10, Theorem 3]. In [10] Sebestyen considered $\overline{Q} \in \mathcal{B}(\mathcal{K}_1)$ with the additional assumption that $\overline{\mathsf{span}}\{W_1^{*k}h : h \in \mathcal{H}_1, 0 \leq k \leq n\}$ reduces $\overline{Q}$ for every $n \geq 0$.

Recall that the minimal isometric dilation is an isometric lifting. Thus, Theorem 2.6 characterizes the operators X which are Q-commutant to T in terms of the operators Y which are $\overline{Q}$-commutant to the minimal isometric dilation V of T. Next we characterizes the operators X which are Q-commutant to T in terms of the operators Y which are $\overline{Q}$-commutant to the minimal unitary dilation of T, provided Q is a unitary. To prove our result we use the following lemma.

Lemma 2.13 ([2]). *Suppose $T \in \mathcal{B}(\mathcal{H})$ is a contraction with the unique minimal co-isometric extension $(W, \mathcal{K}_0)$. Let $(U^*, \mathcal{K})$ be the unique minimal co-isometric extension of W^*. Then U^* is a unitary, and $(U, \mathcal{K})$ is the unique minimal unitary dilation of T.*

Theorem 2.14. *Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction with the minimal unitary dilation $(U, \mathcal{K})$. Suppose $Q \in \mathcal{B}(\mathcal{H})$ is a unitary and let $X \in \mathcal{B}(\mathcal{H})$.*

(i) *If $XT = QTX$, then there exist $\overline{Q} \in \mathcal{B}(\mathcal{K})$ unitary and a dilation $Y \in \mathcal{B}(\mathcal{K})$ of X such that $YU = \overline{Q}UY$.*

(ii) *If $XT = TXQ$, then there exist $\overline{Q} \in \mathcal{B}(\mathcal{K})$ unitary and a dilation $Y \in \mathcal{B}(\mathcal{K})$ of X such that $YU = UY\overline{Q}$.*

In fact, given $q \in \mathbb{T}$ we can choose $\overline{Q} = Q \oplus qI_{\mathcal{H}^\perp}$. In all cases $X^nT^m = P_{\mathcal{H}}Y^nU^m|_{\mathcal{H}}$ and $T^nX^m = P_{\mathcal{H}}U^nY^m|_{\mathcal{H}}$ for all $n, m \geq 0$. Moreover, Y can be chosen such that $\|Y\| = \|X\|$.

Proof. We prove only the case (i). Case (ii) can be proved similarly. Suppose $q \in \mathbb{T}$ and $(W, \mathcal{K}_0)$ is the minimal co-isometric extension of T. From Lemma 2.13 and the uniqueness of minimal unitary dilation we can assume that $(U^*, \mathcal{K})$ is the minimal co-isometric extension of W^*. Note that $\mathcal{H} \subseteq \mathcal{K}_0 \subseteq \mathcal{K}$. Let $Q_0 = Q \oplus qI_{\mathcal{K}_0 \ominus \mathcal{H}} \in \mathcal{B}(\mathcal{K}_0)$. Since $Q^*XT = TX$, by Theorem 2.10 (iii) there exists an extension $Y_0 \in \mathcal{B}(\mathcal{K}_0)$ of X with $\|Y_0\| = \|X\|$ such that $Q_0^*Y_0W = WY_0$, $T^nX^m = P_{\mathcal{H}}W^nY_0^m|_{\mathcal{H}}$ and $X^nT^m = P_{\mathcal{H}}Y_0^nW^m|_{\mathcal{H}}$ for all $n, m \geq 0$. Again since $Y_0^*W^*Q_0^* = W^*Y_0^*$, by Theorem 2.10 (i) there exists an extension $Y^* \in \mathcal{B}(\mathcal{K})$ of Y_0^* with $\|Y^*\| = \|Y_0^*\|$ such that $Y^*U^*(Q \oplus qI_{\mathcal{K} \ominus \mathcal{H}})^* = U^*Y^*$, $W^{*n}Y_0^{*m} = P_{\mathcal{K}_0}U^{*n}Y^{*m}|_{\mathcal{K}_0}$ and $Y_0^{*n}W^{*m} = P_{\mathcal{K}_0}Y^{*n}U^{*m}|_{\mathcal{K}_0}$ for all $n, m \geq 0$. Note that Y has the required properties. □

Theorem 2.15. *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ be a contraction with the minimal unitary dilation $(U_i, \mathcal{K}_i)$, $i = 1, 2$ and $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.*

(i) *Suppose $Q \in \mathcal{B}(\mathcal{H}_2)$ is a unitary such that $XT_1 = QT_2X$. Then there exist $\overline{Q} \in \mathcal{B}(\mathcal{K}_2)$ unitary and $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that $YU_1 = \overline{Q}U_2Y$.*

(ii) *Suppose $Q \in \mathcal{B}(\mathcal{H}_1)$ is a unitary such that $XT_1 = T_2XQ$. Then there exist $\overline{Q} \in \mathcal{B}(\mathcal{K})$ unitary and $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that $YU_1 = U_2Y\overline{Q}$.*

In fact, given $q \in \mathbb{T}$ we can choose $\overline{Q} = Q \oplus qI$. In all cases $XT_1^n = P_{\mathcal{H}_2}YU_1^n|_{\mathcal{H}_1}$ and $T_2^nX = P_{\mathcal{H}_2}U_2^nY|_{\mathcal{H}_1}$ for all $n \geq 0$. Moreover, Y can be chosen such that $\|Y\| = \|X\|$.

Proof is similar to that of Theorem 2.8.

2.2. **Dilation theorems.** In this section we prove an analogue of Ando's dilation theorem for Q-commuting contractions.

Theorem 2.16 (Q-commuting isometric dilation). *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ be contractions and $Q \in \mathcal{B}(\mathcal{H})$ be a unitary such that $T_2T_1 = QT_1T_2$. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, isometries $V_1, V_2 \in \mathcal{B}(\mathcal{K})$ and $\overline{Q} \in \mathcal{B}(\mathcal{K})$ unitary such that*

(i) *$V_2V_1 = \overline{Q}V_1V_2$; and*

(ii) *V_i is a lifting (and hence a dilation) of T_i so that $T_1^nT_2^m = P_{\mathcal{H}}V_1^nV_2^m|_{\mathcal{H}}$ and $T_2^nT_1^m = P_{\mathcal{H}}V_2^nV_1^m|_{\mathcal{H}}$ for all $n, m \geq 0$.*

In fact, given $q \in \mathbb{T}$ we can choose $\overline{Q} = Q \oplus qI_{\mathcal{K}\ominus\mathcal{H}}$.

Proof. Fix $q \in \mathbb{T}$. Let $(\widehat{V}_1, \widehat{\mathcal{K}})$ be the minimal isometric dilation of T_1. Since $T_2T_1 = QT_1T_2$, by Theorem 2.6(i) there exists $\widehat{V}_2 \in \mathcal{B}(\widehat{\mathcal{K}})$ such that

$$\widehat{V}_2\widehat{V}_1 = (Q \oplus qI_{\widehat{\mathcal{K}}\ominus\mathcal{H}})\widehat{V}_1\widehat{V}_2, \quad \widehat{V}_2^*|_{\mathcal{H}} = T_2^* \quad \text{and } \|\widehat{V}_2\| = \|T_2\| \leq 1.$$

Suppose $(V_2, \mathcal{K})$ is the minimal isometric dilation of $\widehat{V}_2$. Note that $\mathcal{H} \subseteq \widehat{\mathcal{K}} \subseteq \mathcal{K}$. Since $\widehat{V}_1\widehat{V}_2 = (Q \oplus qI_{\widehat{\mathcal{K}}\ominus\mathcal{H}})^*\widehat{V}_2\widehat{V}_1$, from Theorem 2.6(i) we get $V_1 \in \mathcal{B}(\mathcal{K})$ such that

$$V_1V_2 = (Q \oplus qI_{\mathcal{K}\ominus\mathcal{H}})^*V_2V_1, \quad V_1^*|_{\widehat{\mathcal{K}}} = \widehat{V}_1^* \quad \text{and } \|V_1\| = \|\widehat{V}_1\| \leq 1.$$

Let $V_1 = \begin{bmatrix} \widehat{V}_1 & 0 \\ A & B \end{bmatrix}$ w.r.t the decomposition $\mathcal{K} = \widehat{\mathcal{K}} \oplus \widehat{\mathcal{K}}^{\perp}$. Since

$$0 \leq \widehat{V}_1^*\widehat{V}_1 + A^*A \leq \|\widehat{V}_1^*\widehat{V}_1 + A^*A\|\, I \leq \|V_1^*V_1\|\, I \leq I$$

and $\widehat{V}_1$ is an isometry we have $A = 0$, so that $V_1|_{\widehat{\mathcal{K}}} = \widehat{V}_1$. Since Q and V_2 are isometries we have

$$\|V_1V_2^nk\| = \|(Q \oplus qI_{\mathcal{H}^\perp})V_1V_2^nk\| = \left\|V_2V_1V_2^{n-1}k\right\| = \left\|V_1V_2^{n-1}k\right\| \quad \forall\, n \geq 1.$$

Repeating the above step recursively we get

$$\|V_1V_2^nk\| = \|V_1V_2k\| = \|(Q \oplus qI_{\mathcal{H}^\perp})V_1V_2k\| = \|V_2V_1k\| = \|V_1k\| = \|\widehat{V}_1k\| = \|k\| = \|V_2^nk\|$$

for every $k \in \widehat{\mathcal{K}}$ and $n \geq 1$. Clearly the above equality holds for $n = 0$. Hence

$$\|(I - V_1^*V_1)^{\frac{1}{2}}V_2^nk\|^2 = \langle V_2^nk, V_2^nk\rangle - \langle V_2^nk, V_1^*V_1V_2^nk\rangle = \|V_2^nk\|^2 - \|V_1V_2^nk\|^2 = 0$$

for all $k \in \widehat{\mathcal{K}}$ and $n \geq 0$. Since $\mathcal{K} = \overline{\text{span}}\{V_2^nk : k \in \widehat{\mathcal{K}}, n \geq 0\}$ from above equation we get $(I - V_1^*V_1) = 0$, i.e., V_1 is an isometry. Moreover, since minimal isometric dilations are liftings we have $V_i^*|_{\mathcal{H}} = (V_i^*|_{\widehat{\mathcal{K}}})|_{\mathcal{H}} = \widehat{V}_i^*|_{\mathcal{H}} = T_i^*$ for $i = 1, 2$. Thus V_1, V_2 are isometric lifting of T_1, T_2 respectively, so that (ii) follows. This completes the proof. □

Corollary 2.17 (Q-commuting co-isometric dilation). *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ be contractions and $Q \in \mathcal{B}(\mathcal{H})$ be a unitary such that $T_2T_1 = T_1T_2Q$. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, co-isometries $W_1, W_2 \in \mathcal{B}(\mathcal{K})$ and $\overline{Q} \in \mathcal{B}(\mathcal{K})$ unitary such that*

(i) *$W_2W_1 = W_1W_2\overline{Q}$; and*

(ii) *W_i is an extension (and hence a dilation) of T_i so that $T_1^nT_2^m = P_{\mathcal{H}}W_1^nW_2^m|_{\mathcal{H}}$ and $T_2^nT_1^m = P_{\mathcal{H}}W_2^nW_1^m|_{\mathcal{H}}$ for all $n, m \geq 0$.*

In fact, given $q \in \mathbb{T}$ we can choose $\overline{Q} = Q \oplus qI_{\mathcal{K}\ominus\mathcal{H}}$.

Proof. Fix $q \in \mathbb{T}$. Since $T_1^*T_2^* = Q^*T_2^*T_1^*$, by Theorem 2.16 there exists Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an isometric lifting $W_i^* \in \mathcal{B}(\mathcal{K})$ of T_i^* such that $W_1^*W_2^* = (Q \oplus qI_{\mathcal{H}^\perp})^*W_2^*W_1^*$. Note that $W_i \in \mathcal{B}(\mathcal{K}),\ i = 1, 2$ are the required co-isometric extensions. □

Theorem 2.18. *Let $V_1, V_2 \in \mathcal{B}(\mathcal{H})$ be isometries and $Q \in \mathcal{B}(\mathcal{H})$ be a unitary such that $V_2V_1 = QV_1V_2$. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and unitaries $\overline{Q}, U_1, U_2 \in \mathcal{B}(\mathcal{K})$ such that*

(i) *$U_2U_1 = \overline{Q}U_1U_2$; and*

(ii) *U_i is an extension (and hence a dilation) of V_i so that $V_1^nV_2^m = P_{\mathcal{H}}U_1^nU_2^m|_{\mathcal{H}}$ and $V_2^nV_1^m = P_{\mathcal{H}}U_2^nU_1^m|_{\mathcal{H}}$ for all $n,m \geq 0$.*

In fact, given $q \in \mathbb{T}$ we can choose $\overline{Q} = Q \oplus qI_{\mathcal{K}\ominus\mathcal{H}}$.

Proof. Fix $q \in \mathbb{T}$. Suppose $(\widehat{V}_2, \widehat{\mathcal{H}})$ is the minimal unitary dilation of V_2. Then $\widehat{\mathcal{H}} = \overline{\mathsf{span}}\{\widehat{V}_2^n\mathcal{H} : n \in \mathbb{Z}\}$ and $\widehat{V}_2$ is an extension of V_2. Define $\widehat{V}_1 : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ by

$$\widehat{V}_1(\widehat{V}_2^nh) = (\widehat{Q}^*\widehat{V}_2)^nV_1h \qquad \forall\, h \in \mathcal{H}, n \in \mathbb{Z},$$

where $\widehat{Q} = (Q \oplus qI_{\widehat{\mathcal{H}}\ominus\mathcal{H}}) \in \mathcal{B}(\widehat{\mathcal{H}})$. Then for all $h, h' \in \mathcal{H}$ and $n \geq m$,

$$\begin{aligned}
\langle \widehat{V}_1(\widehat{V}_2^nh), \widehat{V}_1(\widehat{V}_2^mh')\rangle &= \langle (\widehat{Q}^*\widehat{V}_2)^{n-m}V_1h, V_1h'\rangle \\
&= \langle (\widehat{Q}^*\widehat{V}_2)^{n-m-1}\widehat{Q}^*\widehat{V}_2V_1h, V_1h'\rangle \\
&= \langle (\widehat{Q}^*\widehat{V}_2)^{n-m-1}Q^*V_2V_1h, V_1h'\rangle \\
&= \langle (\widehat{Q}^*\widehat{V}_2)^{n-m-1}V_1V_2h, V_1h'\rangle \\
&= \langle V_1V_2^{n-m}h, V_1h'\rangle \qquad \text{(by repeating above steps)} \\
&= \langle \widehat{V}_2^{n-m}h, h'\rangle \qquad (\because \widehat{V}_2|_{\mathcal{H}} = V_2 \text{ and } n-m \geq 0) \\
&= \langle \widehat{V}_2^nh, \widehat{V}_2^mh'\rangle.
\end{aligned}$$

Thus $\widehat{V}_1$ is a well defined isometry. Clearly $\widehat{V}_1$ is an extension of V_1. Moreover, $\widehat{Q}\widehat{V}_1\widehat{V}_2 = \widehat{V}_2\widehat{V}_1$ on $\widehat{\mathcal{H}}$. Suppose $(U_1, \mathcal{K})$ is the minimal unitary dilation (and hence an extension) of $\widehat{V}_1$, so that $\mathcal{K} = \overline{\mathsf{span}}\{U_1^n(\widehat{\mathcal{H}}) : n \in \mathbb{Z}\} = \overline{\mathsf{span}}\{U_1^n(\widehat{\mathcal{H}}) : n \leq 0\}$ as U_1 leaves $\widehat{\mathcal{H}}$ invariant. Define $U_2 : \mathcal{K} \to \mathcal{K}$ by

$$U_2(U_1^n\widehat{h}) = (\overline{Q}U_1)^n\widehat{V}_2\widehat{h} \qquad \forall\, \widehat{h} \in \widehat{\mathcal{H}}, n \in \mathbb{Z},$$

where $\overline{Q} = (Q \oplus qI_{\mathcal{K}\ominus\mathcal{H}}) \in \mathcal{B}(\mathcal{K})$. As in the case of $\widehat{V}_1$, it can be verified that U_2 is also a well defined isometric extension of $\widehat{V}_2$. Clearly $U_2U_1 = \overline{Q}U_1U_2$. Now we shall prove that U_2 is onto, so that it is a unitary. For, if $n > 0$ let

$$\mathcal{K}_n = \overline{\mathsf{span}}\{U_1^{*j}\widehat{h} : 0 \leq j \leq n, \widehat{h} \in \widehat{\mathcal{H}}\} = \mathsf{span}\{U_1^{*j}\widehat{h} : 0 \leq j \leq n, \widehat{h} \in \widehat{\mathcal{H}}\}.$$

We prove by induction that U_2 maps $\mathcal{K}_n$ onto $\mathcal{K}_n$ for every $n > 0$. Suppose $n = 1$. Then for all $0 \leq j \leq 1$ and $\widehat{h} \in \widehat{\mathcal{H}}$ we have $U_1^{*j}\widehat{V}_2^*\widehat{Q}^j\widehat{h} \in \mathcal{K}_1$ and

$$U_2(U_1^{*j}\widehat{V}_2^*\widehat{Q}^j\widehat{h}) = (\overline{Q}U_1)^{*j}\widehat{V}_2\widehat{V}_2^*\widehat{Q}^j\widehat{h} = U_1^{*j}\overline{Q}^{*j}\widehat{Q}^j\widehat{h} = U_1^{*j}\widehat{h}.$$

Thus $U_2(\mathcal{K}_1) = \mathcal{K}_1$. Now assume that U_2 maps $\mathcal{K}_n$ onto $\mathcal{K}_n$. To prove that U_2 maps $\mathcal{K}_{n+1}$ onto $\mathcal{K}_{n+1}$ it is enough to prove that $U_1^{*(n+1)}\widehat{h}$ has a pre-image for every $\widehat{h} \in \widehat{\mathcal{H}}$. Since $U_1^{*n}\widehat{h} \in \mathcal{K}_n$ there exists $x \in \mathcal{K}_n$ such that $U_2(x) = U_1^{*n}\widehat{h}$. Note that $\mathcal{H} \subseteq \widehat{\mathcal{H}} \subseteq \mathcal{K}_n \subseteq \mathcal{K}$, hence $\mathcal{K}_n$ is reducing for $\overline{Q}$. Therefore there exists $z \in \mathcal{K}_n$ such that $U_2(z) = \overline{Q}U_2x$. Clearly $U_1^*(z) \in \mathcal{K}_{n+1}$, and

$$U_2U_1^*z = U_1^*\overline{Q}^*U_2(z) = U_1^*U_2x = U_1^{*n+1}\widehat{h}.$$

Thus U_2 maps $\mathcal{K}_{n+1}$ onto $\mathcal{K}_{n+1}$. By induction we conclude that $U_1^{*n}\widehat{h}$ has a pre-image under U_2 for all $n > 0, \widehat{h} \in \widehat{\mathcal{H}}$. Since $\mathcal{K} = \overline{\mathsf{span}}\{U_1^n(\widehat{\mathcal{H}}) : n \leq 0\}$ we conclude that U_2 is onto. Note that $(\overline{Q}, U_1, U_2, \mathcal{K})$ is the required quadruple. ☐

Corollary 2.19. *Let $W_1, W_2 \in \mathcal{B}(\mathcal{H})$ be co-isometries and $Q \in \mathcal{B}(\mathcal{H})$ be a unitary such that $W_2W_1 = W_1W_2Q$. Then there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and unitaries $\overline{Q}, U_1, U_2 \in \mathcal{B}(\mathcal{K})$ such that*

(i) $U_2U_1 = U_1U_2\overline{Q}$; *and*

(ii) U_i *is a lifting (and hence a dilation) of* W_i *so that* $W_1^nW_2^m = P_{\mathcal{H}}U_1^nU_2^m|_{\mathcal{H}}$ *and* $W_2^nW_1^m = P_{\mathcal{H}}U_2^nU_1^m|_{\mathcal{H}}$ *for all* $n, m \geq 0$.

In fact, given $q \in \mathbb{T}$ *we can choose* $\overline{Q} = Q \oplus qI_{\mathcal{K}\ominus\mathcal{H}}$.

Proof. Since W_i^*'s are isometries satisfying $W_2^*W_1^* = QW_1^*W_2^*$, from above theorem there exists Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and unitaries $U_1, U_2 \in \mathcal{B}(\mathcal{K})$ such that U_i^*'s are extensions of W_i^*'s with $U_2^*U_1^* = (Q \oplus qI_{\mathcal{H}^\perp})U_1^*U_2^*$. □

Combining Theorems 2.16, 2.18 and Corollaries 2.17, 2.19 we have the following analogue of Ando's theorem for Q-commuting contractions.

Theorem 2.20 (Q-commuting unitary dilation). *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ be contractions and $Q \in \mathcal{B}(\mathcal{H})$ be a unitary such that $T_2T_1 = QT_1T_2$ (respectively $T_2T_1 = T_1T_2Q$). Then there exist Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and unitaries $\overline{Q}, U_1, U_2 \in \mathcal{B}(\mathcal{K})$ such that*

(i) $U_2U_1 = \overline{Q}U_1U_2$ *(respectively* $U_2U_1 = U_1U_2\overline{Q}$*); and*

(ii) $T_1^nT_2^m = P_{\mathcal{H}}U_1^nU_2^m|_{\mathcal{H}}$ *and* $T_2^nT_1^m = P_{\mathcal{H}}U_2^nU_1^m|_{\mathcal{H}}$ *for all* $n, m \geq 0$.

In fact, given $q \in \mathbb{T}$ *we can choose* $\overline{Q} = Q \oplus qI_{\mathcal{K}\ominus\mathcal{H}}$.

Remark 2.21. If $Q = qI_{\mathcal{H}}$ for some $q \in \mathbb{T}$, then Theorem 2.20 reduces to q-commuting dilation theorem.

Acknowledgment. The first author thanks the Department of Atomic Energy (DAE), Government of India for financial support and IMSc Chennai for providing necessary facilities to carry out this work. We thank B. V. R. Bhat for helpful suggestions.

References

[1] T. Ando, *On a pair of commutative contractions*, Acta Sci. Math. (Szeged) **2**4 (1963), 88–90.

[2] R. G. Douglas, P. S. Muhly and C. Pearcy, *Lifting commuting operators*, Michigan Math. J. **1**5 (1968), 385–395.

[3] T. Ito, *On the commutative family of subnormal operators*, J. Fac. Sci. Hokkaido Univ. Ser. I **1**4 (1958), 1–15.

[4] D. K. Keshari and N. Mallick, *q-commuting dilation*, Proc. Amer. Math. Soc. 147 (2019), no. 2, 655–669.

[5] B. Sz.-Nagy, *Sur les contractions de l'espace de Hilbert*, Acta Sci. Math. Szeged **1**5 (1953), 87–92.

[6] B. Sz.-Nagy and C. Foias, *Dilatation des commutants d'opérateurs*, C. R. Acad. Sci. Paris Sér. A-B **2**66 (1968), A493–A495. MR0236755

[7] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, Translated from the French and revised, North-Holland Publishing Co., Amsterdam, 1970.

[8] Z. Sebestyen, *Anticommutant lifting and anticommuting dilation*, Proc. Amer. Math. Soc. **1**21 (1994), no. 1, 133–136.

[9] J. J. Schaffer, *On unitary dilations of contractions*, Proc. Amer. Math. Soc. **6** (1955), 322.

[10] Z. Sebestyen, *Lifting intertwining operators*, Period. Math. Hungar. **2**8 (1994), no. 3, 235–240.