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On a Generalization of Blaschke's Rolling Theorem and the Smoothing of Surfaces

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A generalization of Blaschke's Rolling Theorem for not necessarily convex sets is proved that exhibits an intimate connection between a generalized notion of convexity, various concepts in mathematical morphology and image processing, and a certain smoothness condition. As a consequence a geometric characterization of Serra's regular model is obtained and various problems in image processing arisng from the smoothing of surfaces with Sternberg's rolling ball algorithm are addressed. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: Blaschke's Rolling Theorem; generalized convexity; smoothing of images; rolling ball algorithm; mathematical morphology; Serra's regular model

1. Introduction

Blaschke's Rolling Theorem gives necessary and sufficient conditions for a ball to roll freely inside a convex body S in $\mathbb{R}^2$ or $\mathbb{R}^3$; see [2]. Extensions of this theorem to higher dimensions and non-compact convex sets were treated by Koutroufiotis [7] and Delgado [4]. Firey [5], Goodey [6] and Weil [14] considered the case when a convex body L rolls freely in S and obtained results without differentiability assumptions on the boundaries of L and S. Brooks and Strantzen (1989) give a very general result of this kind. Their monograph is dedicated to a very thorough treatment of Blaschke's Rolling Theorem in $\mathbb{R}^n$.

In section 2 Blaschke's Rolling Theorem is generalized in a different way as a stepping stone towards solving various problems in image analysis in section 3: We consider the case where a ball rolls freely inside and outside a compact set S, which we will not require any more to be convex. It will be shown that this condition characterizes the smoothness of ∂S via a Lipschitz condition on the normal vectors of

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∂S and is closely connected to a generalized notion of convexity and various concepts of mathematical morphology and image processing. In particular, the generalized rolling theorem gives an exact geometric characterization of Serra's regular model from mathematical morphology.

In section 3, the theorem is then used to analyse Sternberg's [12] rolling ball algorithm. This algorithm was introduced as a morphological image processing tool to filter and smooth a grey-level function. Various conditions are given for this algorithm to successfully smooth surfaces. In particular, it is shown how modifications of the algorithm can be applied to an arbitrary bounded set S to generate a spectrum (not necessarily convex) sets with smooth boundaries that represent the shape of S to varying degrees of detail. The convex hull of S is obtained as a limiting case of this smoothing procedure.

All proofs are deferred to section 4.

2. Notation and the generalized rolling theorem

The setting throughout in $\mathbb{R}^d$ be equipped with the standard inner product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$. For $x \in \mathbb{R}^d \setminus \{0\}$ write $e(x) := x/|x|$. $B_r(x)$ denotes the closed ball with radius r centered at x, and $B := B_1(0)$. If $A \subset \mathbb{R}^d$ then A^c, $\bar{A}$, int A, ∂A and conv A denote the complement, closure, interior, boundary and convex hull of A, respectively. Further we define $\operatorname{diam} A := \sup_{x,y\in A} |x-y|$, the diameter of A, and $r(A) := \sup\{r\colon B_r(a) \subset A \text{ for some } a\}$, the inradius of A. For A, $C \subset \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ write $\lambda A := \{\lambda a : a \in A\}$ and denote by

$$A \oplus C = \{a + c\colon a \in A,\ c \in C\}$$

the *Minkowski addition* of A and C, and by

$$A \ominus C = \{x\colon x + C \subset A\}$$

the *Minkowski difference*, where we write $x + C$ for $\{x\} + C$. One then checks that

$$A \ominus C = (A^c \oplus (-1)C)^c. \tag{1}$$

Further, (1–5–9) of Matheron [9] gives

$$\operatorname{conv}(A \oplus C) = \operatorname{conv} A \oplus \operatorname{conv} C. \tag{CON}$$

For $\varepsilon \in \mathbb{R}$ write

$$A_\varepsilon = \begin{cases} A \oplus \varepsilon B & \text{if } \varepsilon \geqslant 0, \\ A \ominus |\varepsilon| B & \text{if } \varepsilon < 0, \end{cases}$$

and denote by $d_H(A, C) := \inf\{\varepsilon > 0\colon A \subset C_\varepsilon \text{ and } C \subset A_\varepsilon\}$ the Hausdorff distance between A and C. Minkowski addition and subtraction have become common tools in mathematical morphology and image processing. In particular, for some morphological problems it is helpful to restrict attention to compact sets A that satisfy $A = (A \oplus \lambda B) \ominus \lambda B = (A \ominus \lambda B) \oplus \lambda B$ for some $\lambda > 0$. The class consisting of these sets is called *Serra's regular model*, see [13, p. 144]. The use of this model is attractive

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in morphological problems because it avoids difficulties with the digitalization of sets and images, and it is particularly helpful when studying connectivity questions, see [13]. A more easily interpretable characterization of this model will be given in Theorem 1.

In a different context, Matheron [9, p. 24] defines for any $A \subset \mathbb{R}^d$ and $\lambda \geqslant 0$

$$\Psi_\lambda(A) := (A \ominus \lambda B) \oplus \lambda B = \bigcup_{B_\lambda(x) \subset A} B_\lambda(x) \tag{2}$$

and calls the mapping $\lambda \to \Psi_\lambda(A)$ the *granulometry* of the set A with respect to the *structuring element* B. The granulometry represents the 'size distribution' of the set A in the sense that the mapping $\lambda \to \mathrm{Leb}(\Psi_\lambda(A))$, where Leb denotes Lebesgue measure on $\mathbb{R}^d$, gives the volume occupied by the translates of λB that are included in A. It is helpful in this paper to extend the definition of the granulometry to the whole line by setting

$$\Psi_{-\lambda}(A) := (A \oplus \lambda B) \ominus \lambda B \quad (\lambda > 0). \tag{3}$$

Then (1) shows that for all $\lambda \in \mathbb{R}$

$$\Psi_{-\lambda}(A) = (\Psi_\lambda(A^c))^c = \Psi^*_\lambda(A), \tag{4}$$

where Ψ^*_λ is the so-called *dual mapping* of Ψ_λ, see p. 187 of [9].

The following properties of granulometries follow from Matheron [9, p. 24] together with (4):

$$\lambda \cdot \mu \geqslant 0 \text{ and } |\lambda| \geqslant |\mu| \text{ implies } \Psi_\lambda(\Psi_\mu(A)) = \Psi_\mu(\Psi_\lambda(A)) = \Psi_\lambda(A). \tag{ID}$$

$$\lambda, \mu \in \mathbb{R} \text{ and } \mu \leqslant \lambda \text{ implies } \Psi_\mu(A) \supset \Psi_\lambda(A), \text{ where } \Psi_0(A) = A. \tag{MON}$$

One also sees readily that

$$\lambda \in \mathbb{R} \text{ and } A \subset C \text{ implies } A_\lambda \subset C_\lambda, \text{ hence also } \Psi_\lambda(A) \subset \Psi_\lambda(C). \tag{MON II}$$

We will use the following notion of generalized convexity, for which Mani-Levitska [8] cites Perkal [9] as a reference: The set $A \subset \mathbb{R}^d$ is called *r-convex* ($r > 0$) if $A = C_r(A)$, where $C_r(A) = \bigcap_{\mathrm{int}\, B_r(x) \cap A = \emptyset} (\mathrm{int}\, B_r(x))^c$ is called the *r-convex hull* of A. To see why this is a generalized notion of convexity, note that $C_r(A) \subset C_s(A)$ for $r \leqslant s$, $C_r(A) \downarrow \bar{A}$ as $r \to 0$, and, under certain conditions of A (e.g. $\mathrm{int}(\mathrm{conv}\, A) \neq \emptyset$ is sufficient), $C_r(A) \uparrow C$ as $r \to \infty$ with $\bar{C} = \overline{\mathrm{conv}\, A}$. (The corresponding statement in Mani-Levitska [8] is erroneous; on the other hand, the assumptions on A made by Perkal [10] are clearly too strong).

Finally, if A and C are convex and compact, then C is said to *roll freely* in A if for each boundary point $a \in \partial A$ and each rotation $\rho \in \mathrm{SO}(d)$ there exists $x \in \mathbb{R}^d$ such that $a \in x + \rho(C) \subset A$, see Schneider [11]. We will be interested only in the case where C is a ball of radius r, so that the rolling condition then becomes $a \in B_r(x) \subset A$ for some x. If A is only closed, then for rB to roll freely in A we require in addition that $A \ominus rB$ be path-connected in order to preserve the physical meaning of rolling freely.

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All these notions are linked together in the following generalization of Blaschke's Rolling Theorem:

Theorem 1. *Let $S \neq \emptyset$ be a compact and path-connected subset of $\mathbb{R}^d$ and $r_0 > 0$. Then the following are equivalent:*

(i) $\Psi_\lambda(S) = S$ *for* $\lambda \in (-r_0, r_0]$.
(ii) *S and $\overline{S^c}$ are r_0-convex and* $\operatorname{int} S \neq \emptyset$.
(iii) *A ball of radius r rolls freely inside S and inside $\overline{S^c}$ for all* $0 \leqslant r \leqslant r_0$.
(iv) *For every $r_1 \in [0, r_0]$, $r_2 \in [0, r_0)$ there exist A, $D \subset \mathbb{R}^d$ with* $S = A \oplus r_1 B = D \ominus r_2 B$.
(v) *∂S is a $(d-1)$-dimensional C^1 submanifold in $\mathbb{R}^d$ with the outward pointing unit normal vector $n(s)$ at $s \in \partial S$ satisfying the Lipschitz condition*

$$|n(s) - n(t)| \leqslant \frac{1}{r_0}\,|s - t| \quad \textit{for all } s, t \in \partial S.$$

Moreover, for some $r_0 > 0$ above is equivalent to

(vi) *S belongs to Serra's regular model.*

Remark.

1. The theorem shows that the smoothness of ∂S is linked to the behaviour at the origin of the granulometry $\Psi_\lambda(S)$.
2. (iv) generalizes a well-known characterization of 'rolling freely' in the case where S is convex: Then $r_1 B$ rolls freely in S iff $r_1 B$ is a summand of S, i.e. there exists a convex, compact set A with $S = A \oplus r_1 B$; see Theorem 3.2.2 in [11].
3. One readily checks that if S is not assumed to be path-connected, then the theorem remains true if one requires in (ii) $\operatorname{int}(S_i) \neq \emptyset$ for each path-connected component $S_i \subset S$ and in (iii) one restricts 'rolling freely' to each path-connected component of S and $\overline{S^c}$.
4. Using notation from mathematical morphology gives a formal way of writing down how r-convexity generalizes convexity: The set $A \subset \mathbb{R}^d$ is said to be *closed with respect* to $\operatorname{int}(rB)$ if $A = (A \oplus \operatorname{int}(rB)) \ominus \operatorname{int}(rB)$. Then one has for $r > 0$:

$$C_r(A) = \bigcap_{\substack{C \supset A \\ C \text{ closed w.r.t. } \operatorname{int}(rB)}} C,$$

and, provided $\operatorname{int}(\operatorname{conv} A) \neq \emptyset$,

$$\overline{\operatorname{conv} A} = \bigcap_{\substack{C \supset A \\ C \text{ closed and convex}}} C = \bigcap_{\substack{C \supset A \\ C \text{ closed w.r.t. } \operatorname{int}(rB) \text{ for all } r > 0}} C.$$

This follows upon proving that C is closed w.r.t. $\operatorname{int}(rB)$ iff C is r-convex, and that if $\operatorname{int}(\operatorname{conv} C) \neq \emptyset$, then C is closed and convex iff C is closed w.r.t. $\operatorname{int}(rB)$ for all $r > 0$. The first statement is readily verified, the second can be proved e.g. with the help of Lemma D below.

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3. Applications to image analysis and smoothing

Sternberg [12] popularized the idea of filtering and smoothing a grey-scale image by applying Ψ_λ and $\Psi_{-\lambda}$, possibly iteratively, to the epigraph of the corresponding grey-level function. It follows from (2) that employing Ψ_λ can be visualized by rolling a ball with radius λ on the grey-level surface. As the ball cannot enter narrow pits, the smoothed image will be free of the corresponding small, dark details. Likewise, $\Psi_{-\lambda}$ can be visualized by rolling the ball along the underside of the grey-level surface. This depiction explains the name 'rolling ball algorithm' for this procedure. Figures 4 and 5 in Sternberg [12] show the algorithm in action, and an example of a grey-scale image before and after smoothing with the rolling ball algorithm.

This section analyses the rolling ball algorithm in terms of its properties as a smoothing operator, i.e. as a transformation that constructs a smooth surface which approximates the original (not necessarily smooth) surface given by the boundary of a set.

An important first smoothing property of this algorithm follows from the generalized Rolling Theorem established in section 2: The boundary of a set S is smooth in the sense described there if and only if the rolling ball algorithms Ψ_λ and $\Psi_{-\lambda}$ leave S invariant for some $\lambda > 0$. On the other hand, even repeated applications of these rolling ball algorithms may produce surfaces that are not smooth. As an example, consider the set $S \subset \mathbb{R}^2$ given by $S := B^1 \cup B^2 \cup B^3$, where $B^1 = B_1((0,0))$, $B^2 = B_1((1,0))$ and $B^3 = B_1((4.5,0))$. Then ∂S has cusps where ∂B^1 and ∂B^2 meet. Rolling B inside S leaves S invariant, rolling B along the outside of S smoothes these cusps but also introduces new cusps by enlarging B^2 and B^3. Rolling B again inside this set brings back S. This counterexample is easily generalized to connected sets and arbitrary dimension.

A naturally arising question is then under what conditions the rolling ball algorithm will smooth successfully and produce a smooth surface in the sense of Theorem 1. Furthermore, in view of the application to the filtering of images described above, it is of particular interest to obtain quantitative results concerning the smoothness in terms of the Lipschitz constant in Theorem 1(v). Such results will be given in Theorems 2 and 3 below.

Theorem 2. *Let* S, $T \subset \mathbb{R}^d$ *with* $T_{-\varepsilon} \subset S \subset T_\varepsilon$ *and* $\Psi_{R_i}(T) = T = \Psi_{-R_o}(T)$ *for some* R_i, $R_o > \varepsilon > 0$. *Consider the sets*

$$S_{io} := \Psi_{-r_o}(\Psi_{r_i}(S)) \quad \textit{and} \quad S_{oi} := \Psi_{r_i}(\Psi_{-r_o}(S))$$

with $0 < r_i < R_i - \varepsilon$ *and* $0 < r_o < R_o - \varepsilon$.

Then

$$T_{-\varepsilon} \subset S_{io} \subset T_\varepsilon, \quad T_{-\varepsilon} \subset S_{oi} \subset T_\varepsilon \tag{5}$$

and, independently of the dimension d, there exists some universal function

$$f(r_i, r_o, R_o, \varepsilon) \geqslant (r_i + r_o)\left(1 - \frac{8R_o\varepsilon}{(R_o - r_o + \varepsilon)r_o}\right)$$

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such that

$$S_{io} = \Psi_r(S_{io}) \quad \textit{for } r \in [-r_o, (f_{io} - r_o)^+), \tag{6}$$

where $f_{io} = f(r_i, r_o, R_o, \varepsilon)$. *Also,*

$$S_{oi} = \Psi_r(S_{oi}) \quad \textit{for } r \in (-(f_{oi} - r_i)^+, r_i], \tag{7}$$

where $f_{oi} = f(r_o, r_i, R_i, \varepsilon)$.

Equation (7) *remains true if instead of* $T_{-\varepsilon} \subset S \subset T_\varepsilon$ *one requires* $d_H(S, T) < \varepsilon < r_o$.

Note that (5) and assertion 1(c) of Lemma B below imply that $d_H(S_{io}, T) \leqslant \varepsilon$ and $d_H(S_{oi}, T) \leqslant \varepsilon$. Further, (ID) shows that the assertion of the theorem concerning S_{io} remains true if one has only $T_{-\varepsilon} \subset \Psi_{r_i}(S) \subset T_\varepsilon$ instead of $T_{-\varepsilon} \subset S \subset T_\varepsilon$; analogously for S_{oi} and $\Psi_{-r_o}(S)$. Finally, observe that as $\varepsilon \downarrow 0$ the endpoints of the intervals in (6) and (7) converge to $-r_o$ and r_i as expected.

By employing a proper choice of the smoothing parameters r_i and r_o one can ensure that some versions of these smoothing procedures work for any bounded set in $\mathbb{R}^d$. One way to achieve this is to exploit the well-known 'convexifying effect' of Minkowski addition. A quantitative statement of this effect is given by the theorem of Shapley, Folkman and Starr, see e.g. [11, p. 130]. However, in the cases under consideration here stronger statements are needed. Specifically, Lemma D below will derive a rate of convergence for the statement of the Corollary to Proposition 1–5–7 in Matheron [9]. Note that the latter proposition is false, as the counterexample $F = \{a, b\}$ with $a \neq b$ and $K = B$ shows.

Theorem 3. *Let* $S \subset \mathbb{R}^d$ *be bounded and for* $r_i, r_o > 0$ *consider the set* S_{io} *as defined in Theorem* 2 *and* $S_{oio} := \Psi_{-r_o}(\Psi_{r_i}(\Psi_{-r_o}(S)))$. *If* $r_o > \operatorname{diam} S$ *then, independently of the dimension* d, *there exists some universal function*

$$f = f(r_i, r_o, \operatorname{diam} S) \geqslant (r_i + r_o)\left(1 - \frac{4(r_o - \sqrt{r_o^2 - (\operatorname{diam} S)^2})}{r_o}\right)$$

such that

$$S_{io} = \Psi_r(S_{io}) \quad \textit{and} \quad S_{oio} = \Psi_r(S_{oio}) \quad \textit{for } r \in [-r_o, (f - r_o)^+) \tag{8}$$

and

$$(\operatorname{conv} \Psi_{r_i}(S))_{\sqrt{r_o^2 - (\operatorname{diam} S)^2} - r_o} \subset S_{io} \subset \operatorname{conv} \Psi_{r_i}(S), \tag{9}$$

$$(\operatorname{conv} S)_{\sqrt{r_o^2 - (\operatorname{diam} S)^2} - r_o - r_i} \subset S_{oio} \subset \operatorname{conv} S. \tag{10}$$

Moreover,

$$d_H(S_{oio}, \operatorname{conv} S) \leqslant (r_o - \sqrt{r_o^2 - (\operatorname{diam} S)^2} + r_i)\, \frac{\operatorname{diam} S - r(\operatorname{conv} S)}{r(\operatorname{conv} S)} \tag{11}$$

if r_i *and* r_o *are chosen so that* $r_o - \sqrt{r_o^2 - (\operatorname{diam} S)^2} + r_i < r(\operatorname{conv} S)$.

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Remark.

1. Note that $r_o - \sqrt{r_o^2 - (\operatorname{diam} S)^2} = O(r_o^{-1})$ as r_o increases. Hence proper choice of r_i and r_o ensures that S_{io} and S_{oio} are smooth in the sense of Theorem 1 if S is closed. In particular, S_{oio} gives a smooth approximation to S that represents the shape of S in decreasing detail as $r_i \to 0$ and $r_o \to \infty$ and that converges to $\operatorname{conv} S$ from inside at the rate $O(\max(r_i, r_o^{-1}))$ as follows from (11).
2. Equation (10) and (11) can be refined further: In a similar way as in Lemma F below one can prove that if $A \subset \mathbb{R}^d$ is bounded and convex and $r_i \in [0, r(A))$, then $A_{-\varepsilon} \subset \Psi_{r_i}(A)$, provided $\varepsilon > g(\operatorname{diam} A, r(A), r_i)$, where

$$g(p, r, r_i) := \frac{p - r}{2} - \sqrt{\left(\frac{p - r}{2}\right)^2 - (p - 2r) r_i}.$$

This bound is the best possible in that if $p \geqslant 2r > 0$ and $r_i \in [0, r)$, then by setting $\varepsilon := g(p, r, r_i)$ and $A := \operatorname{conv}(\operatorname{int} B_r(0) \cup B_\varepsilon((p - r - \varepsilon)e))$ for some unit vector e one obtains $r(A) = r$, $\operatorname{diam} A = p$ and $A_{-\varepsilon} \not\subset \Psi_{r_i}(A)$. Employing this and $\Psi_{r_i}((\operatorname{conv} S)_{\sqrt{r_o^2 - (\operatorname{diam} S)^2} - r_o}) \subset S_{oio}$ in the proof instead of (45) yields better but more unwieldy estimates for (10) and hence also (11).

4. Proofs

The results of sections 2 and 3 will be proved with the help of several lemmata. The first one is a simple geometric fact whose proof is omitted:

Lemma A. *Let $S \subset \mathbb{R}^d$ and $r_1, r_2 > 0$. Suppose there exist $s, x \in \mathbb{R}^d$ and sequences $\{s_n\}, \{x_n\} \subset \mathbb{R}^d$ with $s_n \to s$ such that $s_n \in B_{r_1}(x_n) \subset S$ for all n and $s \in B_{r_2}(x) \subset \overline{S^c}$. Then $x_n \to s + (r_1/r_2)(s - x)$. This continues to hold if the roles of S and $\overline{S^c}$ are switched.*

Lemma B.

1. *Let $S \subset \mathbb{R}^d$ satisfy $S = \Psi_{r_i}(S) = \Psi_{-r_o}(S)$ for some $r_i, r_o > 0$.*
   (a) *If $\varepsilon \geqslant -r_i$ then $S_\varepsilon = \Psi_r(S_\varepsilon)$ for $r \in [0, r_i + \varepsilon]$.*
   (b) *If $\varepsilon \leqslant r_o$ then $S_\varepsilon = \Psi_r(S_\varepsilon)$ for $r \in [\varepsilon - r_o, 0]$.*
   (c) *If $-r_i \leqslant r_1 \leqslant r_o$ and $r_2 \in \mathbb{R}$, then $(S_{r_1})_{r_2} = S_{r_1 + r_2}$.*
2. If $S \subset \mathbb{R}^d$ *satisfies the conditions of Theorem 1 for some $r_0 > 0$, then for $|\varepsilon| < r_0$, S_ε satisfies the conditions of Theorem 1 with $r_\varepsilon := r_0 - |\varepsilon|$.*

Proof of Lemma B. Part 1(a). The assertion is true if $S = \emptyset$, so assume $S \neq \emptyset$. We will show

$$S_\varepsilon \subset \Psi_{r_i + \varepsilon}(S_\varepsilon). \tag{12}$$

Then (MON) gives for $r \in [0, r_i + \varepsilon]$: $S_\varepsilon \subset \Psi_{r_i + \varepsilon}(S_\varepsilon) \subset \Psi_r(S_\varepsilon) \subset S_\varepsilon$, proving part (a). In the case $\varepsilon \geqslant 0$, (12) follows by using (2) to see that $S_\varepsilon = \bigcup (B_{r_i}(x) \colon B_{r_i}(x) \subset S) \oplus \varepsilon B \subset \bigcup (B_{r_i + \varepsilon}(x) \colon B_{r_i + \varepsilon}(x) \subset S_\varepsilon)$.

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In the case $-r_i \leqslant \varepsilon < 0$ let $s \in S_\varepsilon = S_{-|\varepsilon|}$. We need only look at the case where $S \neq \mathbb{R}^d$, so there exists $t \in \partial S$ such that $|s-t| = \min_{y\in\partial S}|s-y|$.

If $|s-t| > r_i$ then $S \supset B_{r_i}(s) = B_{r_i+\varepsilon}(s) \oplus |\varepsilon| B$, whence $s \in \Psi_{r_i+\varepsilon}(S_\varepsilon)$ by (2).

If $|s-t| \leqslant r_i$ we have

$$\operatorname{int} B_{|s-t|}(s) \subset S. \tag{13}$$

$S = \Psi_{-r_o}(S)$, (2) and (1) show $S^c = \bigcup(B_{r_o}(x)\colon B_{r_o}(x) \subset S^c)$. By considering a sequence $\{t_n\} \subset S^c$ converging to t one hence obtains

$$t \in B_{r_o}(x_2) \subset \overline{S^c} \quad \text{for some } x_2. \tag{14}$$

First consider the case $t \in S^c$. Then $|t-s| > |\varepsilon|$, because otherwise $s \notin S_{-|\varepsilon|}$. Let $\{t_n\} \subset S$ be a sequence converging to t. Then $S = \Psi_{r_i}(S)$ and (2) show that

$$t_n \in B_{r_i}(y_n) \subset S \quad \text{for some } \{y_n\}. \tag{15}$$

Lemma A, (13)–(15) yield $y_n \to x_1$ for some x_1 and $(x_1-t)/r_i = (t-x_2)/r_o = (s-t)/|s-t|$, which together with $|x_1-t| = r_i \geqslant |s-t|$ implies $|s-x_1| = r_i - |s-t| < r_i - |\varepsilon|$. Together with $y_n \to x_1$ we obtain for n large enough $s \in B_{r_i-|\varepsilon|}(y_n)$. Further, $B_{r_i-|\varepsilon|}(y_n) \oplus |\varepsilon| B = B_{r_i}(y_n) \subset S$, whence $s \in B_{r_i-|\varepsilon|}(y_n) \subset S_{-|\varepsilon|}$, so (12) follows by (2).

The case $t \in S$ can be treated quite similarity.

(b) Using the duality relation (4) and $(S_\varepsilon)^c = (S^c)_{-\varepsilon}$, which is a consequence of (1), one sees that under the assumptions made in 1, (b) is equivalent to (a).

(c) It is an immediate consequence of the definition of '$\oplus$' and '$\ominus$' that in the case $r_1,\ r_2 \geqslant 0$ for any set $\tilde{S}$, $(\tilde{S}_{r_1})_{r_2} = \tilde{S}_{r_1+r_2}$. Using this, one obtains in the case $(-r_1)(r_1+r_2) \geqslant 0$: $(S_{r_1})_{r_2} = (S_{r_1})_{-r_1+(r_1+r_2)} = ((S_{r_1})_{-r_1})_{r_1+r_2} = (\Psi_{-r_1}(S))_{r_1+r_2} = S_{r_1+r_2}$ by parts (a) and (b) with $\varepsilon = 0$.

In the case $(-r_2)(r_1+r_2) \geqslant 0$, $(S_{r_1})_{r_2} = (S_{(r_1+r_2)-r_2})_{r_2} = ((S_{r_1+r_2})_{-r_2})_{r_2} = \Psi_{r_2}(S_{r_1+r_2}) = S_{r_1+r_2}$ by parts (a) and (b), as $r_2 \in [(r_1+r_2)-r_o, r_i+(r_1+r_2)]$ and $-r_i \leqslant r_1+r_2 \leqslant r_o$ follow from $-r_i \leqslant r_1 \leqslant r_o$ and $\min(0,r_1) \leqslant r_1+r_2 \leqslant \max(0,r_1)$ which is a consequence of $(-r_2)(r_1+r_2) \geqslant 0$.

Part 2: As S is compact and path-connected, it is easily checked that S_ε is compact for $\varepsilon \in \mathbb{R}$ and path-connected if $\varepsilon \geqslant 0$. For $-r_o < \varepsilon < 0$ the path-connectedness of $S_\varepsilon = S \ominus |\varepsilon| B$ follows directly form Theorem 1(iii) and $S_\varepsilon \neq \emptyset$ by Theorem 1(i). The assertion follows then form (a), (b) of part 1. □

Lemma C. *Let $A, S \subset \mathbb{R}^d$ be non-empty. Assume*:

(i) $\Psi_{r_1+r_1}(A) = A$ *for some* $r_1, r_2 > 0$,
(ii) $\Psi_{-R}(S) = S$ *for some* $R > 0$,
(iii) $S_{-h} \subset A \subset S$ *for some* $h > 0$.

Then there exists a universal function

$$\tilde{f} = \tilde{f}(r, r_1, R, h) \geqslant (r+r_1)\,\frac{Rr - 4Rh - 3rh - 2h^2}{(R+h)r}$$

such that $b \in \partial A \cap B_r(c)$ *for some* c *with* $\operatorname{int} B_r(c) \subset A$ *implies* $\operatorname{int} B_{\tilde{f}}(b + \tilde{f}e(c-b)) \subset A$.

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Proof of Lemma C. A standard computation shows that if $p, x, y \in \mathbb{R}^d$, then

$$(\operatorname{int} B_R(x)) \cap (\operatorname{int} B_r(y)) = \emptyset,\ p \in \partial B_r(y) \text{ and } |p - x| \leqslant R + h \text{ imply}$$

$$\langle e(p - x), e(y - p) \rangle \geqslant \frac{2Rr - 2Rh - h^2}{2(R + h)r}. \tag{16}$$

Now let $b \in \partial A \cap B_r(c)$ for some c with $\operatorname{int} B_r(c) \subset A$. (2), (4) and (ii) show $S^c = \bigcup (B_R(x)\colon B_R(x) \subset S^c)$. $S_{-h} \subset A$ implies the existence of some $\{s_n\} \subset S^c$ with $|b - s_n| \leqslant h + 1/n$. Hence there exists some $\{x_n\}$ with $s_n \in B_R(x_n) \subset S^c$, and so $|b - x_n| \leqslant R + h + 1/n$. As $A \cap S^c = \emptyset$ implies $\operatorname{int} B_r(c) \cap B_R(x_n) = \emptyset$, and necessarily $b \in \partial B_r(c)$, we can apply (16) with $p := b$ and $y := c$ to find

$$\langle e(b - x_n), e(c - b) \rangle \geqslant \frac{2Rr - 2R(h + 1/n) - (h + 1/n)^2}{2(R + h + 1/n)r} \quad \text{for all } n. \tag{17}$$

Write $\mathscr{Y}_b := \{ y : \operatorname{int} B_{r + r_1}(y) \subset A, b \in B_{r + r_1}(y) \}$. Equation (20) below will show that $\mathscr{Y}_b$ is not empty. Using $B_{r+r_1}(y)$ with $y \in \mathscr{Y}_b$ instead of $B_r(c)$ in the argument leading to (17) we obtain

$$\langle e(b - x_n), e(y - b) \rangle \geqslant \frac{2R(r + r_1) - 2(h + 1/n) - (h + 1/n)^2}{2(R + h + 1/n)(r + r_1)} \tag{18}$$

for any $y \in \mathscr{Y}_b$ and all n. Employing the fact that $\langle u_1, u_3 \rangle \geqslant 4 \min (\langle u_1, u_2 \rangle, \langle u_2, u_3 \rangle) - 3$ for any unit vectors u_1, u_2 and u_3, one deduces from (17) and (18)

$$\langle e(y - b), e(c - b) \rangle \geqslant \frac{Rr - 4Rh - 3rh - 2h^2}{(R + h)r} =: \frac{\tilde{f}}{r + r_1} \tag{19}$$

for all $y \in \mathscr{Y}_b$. Next, we show

$$\text{If } n \text{ satisfies } \langle n, c - b \rangle = 0 \text{ then there exists } \tilde{y} \in \mathscr{Y}_b \text{ with } (\tilde{y} - b, n) \geqslant 0. \tag{20}$$

It is enough to prove (20) for the case where n is a unit vector. For $k \geqslant 1$ define $p_k := \frac{1}{2}(b + c) + \sqrt{(r^2/4 - 1/k^2)}\, e(b - c) + (1/k)n \in \operatorname{int} B_r(c) \subset A$. Using (i) and (2) one obtains $p_k \in B_{r+r_1}(c_k) \subset A$ for $k \geqslant 1$ and some $\{c_k\}$. Hence for all $k \geqslant 1$: $(r + r_1)^2 \geqslant |p_k - c_k|^2 \geqslant 1/k^2 + (r + r_1)^2 + 2\langle p_k - b, b - c_k \rangle$, as $b \in \partial A$ implies $|b - c_k| \geqslant r + r_1$. It follows that $\limsup_{k \to \infty} \langle c_k - b, e(b - p_k) \rangle \leqslant 0$. Further, $\lim_{k \to \infty} |n + e(b - p_k)| = 0$ as can be verified by computation, more quickly, by drawing a picture in the $(n, e(b - c))$-plane and observing that $p_k \in \partial B_{r/2}(\frac{1}{2}(b + c))$, $p_k \to b$ and that n is a tangent vector to $\partial B_{r/2}(\frac{1}{2}(b + c))$ at b. Hence

$$\begin{aligned} \liminf_{k \to \infty} \langle c_k - b, n \rangle &= -\limsup (\langle c_k - b, e(b - p_k) \rangle - \langle c_k - b, n + e(b - p_k) \rangle) \\ &\geqslant 0 - \limsup |c_k - b| \cdot |n + e(b - p_k)| \\ &\geqslant 0 \text{ as } |c_k - b| \text{ is bounded.} \end{aligned} \tag{21}$$

Boundedness of $|c_k - b|$ also implies the existence of a subsequence $\{k'\}$ such that $\lim_{k' \to \infty} c_{k'} = \tilde{y}$ for some $\tilde{y}$. Then $\operatorname{int} B_{r+r_1}(\tilde{y}) \subset A$, and $b \in B_{r+r_1}(\tilde{y})$ as $p_k \to b$ and $|p_k - c_k| \leqslant r + r_1$. So $\tilde{y} \in \mathscr{Y}_b$ and $\langle \tilde{y} - b, n \rangle \geqslant 0$ follows from (21). This proves (20).

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Now let $x \in \operatorname{int} B_{\tilde f}(b + \tilde f e(c-b))$ and set $n := x - b - \langle x - b, e(c-b)\rangle e(c-b)$. Then $\langle n, e(c-b)\rangle = 0$, so (20) gives a $\tilde y$ with

$$\langle \tilde y - b, n\rangle \geqslant 0 \text{ and } \tilde y \in \mathscr{Y}_b, \text{ hence } |\tilde y - b| = r + r_1. \tag{22}$$

We will show $|x - \tilde y| < |b - \tilde y|$, i.e. $x \in \operatorname{int} B_{r+r_1}(\tilde y) \subset A$, proving the lemma. From the definition of n, $\langle x - b, b - \tilde y\rangle = \langle x - b, e(c-b)\rangle\langle e(c-b), b - \tilde y\rangle + \langle n, b - \tilde y\rangle \leqslant \langle x - b, e(c-b)\rangle(-\tilde f)$ by (19) and (22) and because one easily verifies $\langle x - b, e(c-b)\rangle \geqslant 0$. Hence,

$$\begin{aligned}|x - \tilde y|^2 &= |x - b|^2 + |b - \tilde y|^2 + 2\langle x - b, b - \tilde y\rangle\\ &\leqslant |b - \tilde y|^2 + |x - b|^2 - 2\tilde f\langle x - b, e(c-b)\rangle\\ &< |b - \tilde y|^2 \text{ as } x \in \operatorname{int} B_{\tilde f}(b + \tilde f e(c-b)).\end{aligned}$$

□

Lemma D. *Let* $A \subset \mathbb{R}^d$ *and* $r > 0$. *Then*

$$d_H(\operatorname{conv}(A \oplus rB), A \oplus rB) \leqslant (\operatorname{diam} A)^2/2r.$$

Moreover, if $r > \operatorname{diam} A$ *then* $\operatorname{conv}(A \oplus \sqrt{r^2 - (\operatorname{diam} A)^2}B) \subset A \oplus rB$.

Proof of Lemma D. Let $x \in \operatorname{conv}(A \oplus rB)$. (CON) shows $x = y + \bar r e$ where $y \in \operatorname{conv} A$, $\bar r \in [0, r]$ and $e \in \partial B$. Consider the closed half-space $H_1 := \{z : \langle z - y, e\rangle \geqslant 0\}$. Then there exists $a_1 \in A \cap H_1$ as otherwise $\operatorname{conv} A \subset H_1^c$, contradicting $y \in \operatorname{conv} A \cap H_1$.

Now $|y - a_1| \leqslant \operatorname{diam} A$. To see this in the case $y \neq a_1$ consider the closed half-space $H_2 := \{x : \langle z - y, y - a_1\rangle \geqslant 0\}$ and observe as above that $y \in \operatorname{conv} A \cap H_2$ implies the existence of some $a_2 \in A \cap H_2$. But then $|y - a_1|^2 \leqslant |a_1 - a_2|^2 \leqslant (\operatorname{diam} A)^2$. Now, $|x - a_1|^2 = |x - y|^2 + |y - a_1|^2 + 2\langle x - y, y - a_1\rangle \leqslant r^2 + (\operatorname{diam} A)^2$ as $x - y = \bar r e$ and $a_1 \in A_1$. Together with $\sqrt{[r^2 + (\operatorname{diam} A)^2]} - r \leqslant (\operatorname{diam} A)^2/2r$ one obtains $x \in (a_1 \oplus rB) \oplus [(\operatorname{diam} A)^2/2r]B$, proving the first assertion. If instead $x \in \operatorname{conv}(A \oplus \sqrt{[r^2 - (\operatorname{diam} A)^2]}B)$ and $r > \operatorname{diam} A$, then replacing r by $\sqrt{[r^2 - (\operatorname{diam} A)^2]}$ in above proof shows that there exists $a_1 \in A$ with $|x - a_1|^2 \leqslant \sqrt{[r^2 - (\operatorname{diam} A)^2]}^2 + (\operatorname{diam} A)^2 = r^2$, so $x \in a_1 \oplus rB$, proving the second assertion. □

Lemma E. *Let* $A \subset \mathbb{R}^d$ *and* $R \geqslant 0$.

(a) $0 \leqslant r \leqslant R$ *implies* $(\operatorname{conv} A \oplus RB)_{-r} = \operatorname{conv}(A \oplus (R - r)B)$.
(b) $A = \Psi_R(A)$ *implies* $\operatorname{conv} A = \Psi_r(\operatorname{conv} A)$ *for* $r \in (-\infty, R]$.

Proof of Lemma E. (a) The proof of Proposition 1–5–3 in [9] can readily be modified to show that

$$\operatorname{conv} A = \Psi_{-r}(\operatorname{conv} A) \tag{23}$$

for arbitrary, not necessarily closed $A \subset \mathbb{R}^d$. If $x \notin \operatorname{conv} A$ then Theorem 1.3.4 in Schneider (1993) yields a closed half-space $H \supset \operatorname{conv} A$ with $x \in H^c \cup \partial H$, so it is still possible to find a y with $x \in B_r(y) \subset (\operatorname{conv} A)^c$. (23) then follows from (2) and (4).

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Now (CON) shows that $(\operatorname{conv} A \oplus RB))_{-r} = (\operatorname{conv}(A \oplus (R-r)B) \oplus rB)_{-r} = \Psi_{-r}$ $(\operatorname{conv}(A \oplus (R-r)B)) = \operatorname{conv}(A \oplus (\mathrm{R}-r)B)$.

(b) Part (a) shows the assertion for $r \leqslant 0$. For the case $0 < r \leqslant R$ let $x \in \operatorname{conv} A$. Then $x = \sum_{i=1}^{n} \alpha_i a_i$, where $\sum_i \alpha_i = 1$, $\alpha_i \geqslant 0$ and $a_i \in A$ for all i. $\Psi_R(A) = A$ together with (2) shows that for all i, $a_i \in B_R(x_i) \subset A$ for some $x_i \in A$. Then $x' := \sum_i \alpha_i x_i \in \operatorname{conv} A$ and $|x - x'| \leqslant \sum_i \alpha_i |a_i - x_i| \leqslant R$, so $x \in B_R(x')$. Further $B_R(x') \subset \operatorname{conv} A$: $y \in B_R(x')$ implies $y = x' + \bar{r}e$ with $\bar{r} \in [0, R]$ and $e \in \partial B$. $B_R(x_i) \subset A$ shows $x_i + \bar{r}e \in A$ for all i, hence $y = x' + \bar{r}e = \sum_i \alpha_i (x_i + \bar{r}e) \subset \operatorname{conv} A$.

Now the assertion in the case $0 < r \leqslant R$ follows from $x \in B_R(x') \subset \operatorname{conv} A$ together with (2) and (MON). □

Lemma F. *Let $A \subset \mathbb{R}^d$ be bounded and convex. Then*

$$d_H(A, A_{-\varepsilon}) \leqslant \varepsilon \, \frac{\operatorname{diam} A - r(A)}{r(A)} \; \textit{for } \varepsilon \in [0, r(A)).$$

The inequality is tight as can be seen by considering the set $A := \operatorname{conv}(\{a\} \cup B_r(0))$ for various $|a|$ and r.

Proof of Lemma F. We may assume $r(A) > 0$. Let $0 < r < r(A)$. Then there exists b with $B_r(b) \subset A$. Now let $a \in A$. Then $C_a := \operatorname{conv}(\{a\} \cup B_r(b)) \subset A$ as A is convex. One readily checks that if $\lambda \in [0,1]$ and $a_\lambda := a + \lambda(b - a)$, then $B_{\lambda r}(a_\lambda) \subset C_a$. So if $\varepsilon \in [0, r]$ then by setting $\lambda := \varepsilon/r$ one obtains $\inf_{x \in A_{-\varepsilon}} |a - x| \leqslant |a - a_{\varepsilon/r}| = (\varepsilon/r)|b - a| \leqslant (\varepsilon/r)(\operatorname{diam} A - r)$ as $B_r(b) \subset A$. Hence the assertion of the lemma is true with any $r \in (0, r(A))$ in place of $r(A)$, and thus also for $r(A)$. □

Proof of Theorem 1. (i) ⇒ (iv): Set $A := S \ominus r_1 B$ and $D := S \oplus r_2 B$.
(iv) ⇒ (v): We will successively prove (24)–(29):

For every $s \in \partial S$ there exists a unique unit vector $n(s)$ such that $B_{r_o}(s - r_o n(s)) \subset S$ and $\operatorname{int} B_{r_o}(s + r_o n(s)) \subset S^{c}$. (24)

$$\text{For every } s, u \in \partial S, \ |n(s) - n(u)| \leqslant \frac{1}{r_o} |s - u|. \tag{25}$$

Now fix $s \in \partial S$ and define the hyperplane $H_s := s + (n(s))^{\perp}$, where $^{\perp}$ denotes the orthogonal complement, the $(d-1)$-dimensional neighbourhood $U'_s := \{x \in H_s$: $|x - s| < r_o/4\}$, and the neighbourhood U_s of s by $U_s := \{x$: $x = u' + \alpha n(s)$, $u' \in U'_s$, $\alpha \in \mathbb{R}$, $|\alpha| < r_o/4\}$.

$$\text{For every } u \in U_s \cap \partial S, \ \langle n(s), n(u) \rangle \geqslant \tfrac{15}{16}. \tag{26}$$

Let $u \in \partial S \cap U_s$. Then the line $g_u(\alpha) := u + \alpha n(s)$, $\alpha \in \mathbb{R}$, meets $\partial S \cap U_s$ in u only. (27)

$$\text{If } x \in U'_s, \text{ then } g_x(\cdot) \cap \partial S \cap U_s \neq \emptyset. \tag{28}$$

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Equations (27) and (28) show that in U_s, ∂S is the graph of some real-valued function f_s defined on $U'_s \subset H_s$ and graphed with the ordinate axis in direction $n(s)$: f_s is given at $x \in U'_s$ as the unique $\alpha \in (-r_o/4, r_o/4)$ that satisfies $g_x(\alpha) \in \partial S$.

$$\text{Let } u' \in U'_s, \text{ so } u := u' + f_s(u')n(s) \in U_s \cap \partial S. \text{ Then for any } x \in U'_s$$
$$f_s(x) = f_s(u') - \tfrac{1}{\langle n(s), n(u)\rangle}\langle n(u), x - u'\rangle + O(|x - u'|^2). \tag{29}$$

Thus f_s is a C^1 function on U'_s provided the map $u' \mapsto (1/\langle n(s), n(u)\rangle)n(u)$ is continuous. But this a consequence of (25) and (26), once continuity of the map $u' \mapsto u$ is established. That in turn is a consequence of the continuity of f_s which follows form (29). Using theorem 2.1.2 in Berger and Gostiaux [1] one then concludes that ∂S is a $(d-1)$-dimensional C^1 submanifold in $\mathbb{R}^d$. Hence at each $s \in \partial S$ there exists a unique outward pointing unit normal vector, which is given by $n(s)$ as can be computed from (29) or directly deduced from (24). The asserted Lipschitz condition for $n(s)$ is given by (25).

It remains to prove (24)–(29). Equation (24) follows from (iv) with a standard argument involving Lemma A.

To see (25) observe that the inclusions in (24) imply for $s, u \in \partial S$: $|(s - r_o n(s)) - (u + r_o n(u))| \geqslant 2r_o$ and $|(u - r_o n(u)) - (s + r_o n(s))| \geqslant 2r_o$. The first inequality is equivalent to $|s-u|^2 + r_o^2|n(s) + n(u)|^2 - 2r_o\langle s - u, n(s) + n(u)\rangle \geqslant 4r_o^2$, the second to $|s-u|^2 + r_o^2|n(s)+n(u)|^2 + 2r_o\langle s-u, n\langle s\rangle + n(u)\rangle \geqslant 4r_o^2$. Adding both inequalities and using $|n(s)+n(u)|^2 = 4 - |n(s)-n(u)|^2$ gives (25).

Equation (26) follows from (25), using $|s-u|^2 \leqslant r_o^2/8$ for $u \in U_s$.

To prove (27), let $u \in U_s \cap \partial S$ and denote by $u' \in U'_s$ the orthogonal projection of u onto H_s. For an arbitrary $x \in U'_s$ an elementary calculation together with (24) and (26) shows that

$$\begin{aligned} &g_x(\alpha) \in \operatorname{int} B_{r_o}(u + r_o n(u)) \subset S^c \text{ for } \bar{\alpha}(x) < \alpha < r_o/4, \\ &g_x(\alpha) \in \operatorname{int} B_{r_0}(u - r_o n(u)) \subset \operatorname{int} S \text{ for } -r_o/4 < \alpha < \underline{\alpha}(x), \end{aligned} \tag{30}$$

with $\bar{\alpha}(x) = \langle n(s), u - u'\rangle + r_o\langle n(s), n(u)\rangle - \sqrt{[r_o^2\langle n(s), n(u)\rangle^2 + 2r_o\langle n(u), x - u'\rangle - |x-u'|^2]}$ and $\underline{\alpha}(x) = \langle n(s), u - u'\rangle - r_o\langle n(s), n(u)\rangle + \sqrt{[r_o^2\langle n(s), n(u)\rangle^2 - 2r_o\langle n(u), x - u'\rangle - |x-u'|^2]}$. To prove (27) now, set $x = u'$. Then $\underline{\alpha}(x) = \bar{\alpha}(x) = \langle n(s), u - u'\rangle$, so $g_{u'}(\cdot)$ meets $\partial S \cap U_s$ only in $g_{u'}(\langle n(s), u - u'\rangle) = u$, and hence so does $g_u(\cdot)$.

To prove (28), set $u = s$. Then $u' = s$ and thus for $x \in U'_s$, $-r_o/4 < \underline{\alpha}(x)$ and $\bar{\alpha}(x) < r_o/4$ imply that $g_x(\alpha) \in U_s \cap \partial S$ for some $\alpha \in (-r_o/4, r_o/4)$.

As for (29), let u' and u as given there. Then u' is the projection of u onto H_s, $f_s(u') = \langle n(s), u - u'\rangle$ and for $x \in U'_s$, $\underline{\alpha}(x) \leqslant f_s(x) \leqslant \bar{\alpha}(x)$ by (30). But the Taylor series expansions $\sqrt{(1+x)} = 1 + \frac{1}{2}x + O(|x|^2)$ shows that

$$\underline{\alpha}(x) = \langle n(s), u - u'\rangle - \frac{1}{\langle n(s), n(u)\rangle}\langle n(u), x - u'\rangle + O(|x - u'|^2)$$

and the same expansion holds for $\bar{\alpha}(x)$, so (29) entails. □

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(v) $\Rightarrow$ (iii): Let $s \in \partial S$. We will show below:

$$\text{There exists some } 0 < \tilde{r} < \tilde{l} \leqslant r_o \text{ with } \operatorname{conv}(\{s\} \cup B_{\tilde{r}}(s - \tilde{l}n(s))) \subset S. \tag{31}$$

$$\text{If } 0 < r < l \leqslant r_o \text{ and } \operatorname{conv}(\{s\} \cup B_r(s - ln(s))) \subset S, \text{ then } B_l(s - ln(s)) \subset S. \tag{32}$$

Now take $\tilde{r}, \tilde{l}$ as given by (31). If $\tilde{l} < r_o$ then (32) shows that for any $\tilde{l}' \in (\tilde{l}, 2\tilde{l})$ there exists $0 < \tilde{r}' < \tilde{l}'$ so that (31) remains true for $\tilde{r}'$ and $\tilde{l}'$. Iterating this way if necessary one sees that (31) holds for $\tilde{l} = r_0$ and some $0 < \tilde{r} < \tilde{l}$. Then (32) shows that $B_r(s - rn(s)) \subset S$ for all $0 < r \leqslant r_0$. To prove that $S \ominus rB$ is connected for $0 < r \leqslant r_0$, let $a, b \in S \ominus rB$. Write r_a for the maximal $\tilde{r}$ such that $B_{\tilde{r}}(a) \subset S$, so $r_a \geqslant r$, and pick a point $s_a \in \partial S \cap B_{r_a}(a)$. The tangent hyperplane of ∂S at s_a must necessarily be a tangent hyperplane to $B_{r_a}(a)$ at s_a, so $a = s - r_a n(s_a)$. As $r_a \geqslant r$, the map $\Gamma_a : t \in [0, r_a - r] \mapsto s_a - (r_a - t)n(s_a)$ is a path in $S \ominus rB$ connecting a and $s_a - rn(s_a)$. Let Γ_b be an analogous path for b, and $\Gamma \subset \partial S$ a path connecting s_a to s_b. Then $\Gamma_{ab} := \{s - rn(s), s \in \Gamma\}$ is a subset of $S \ominus rB$ as $B_r(s - rn(s)) \subset S$. Further, $n(s)$ is a continuous function of s by the Lipschitz condition, so Γ_{ab} is a path connecting $s_a - rn(s_a)$ to $s_b - rn(s_b)$. Hence a ball of radius $r \leqslant r_0$ rolls freely in S. That such a ball rolls also freely in $\overline{S^c}$ can be shown in a similar way using $\partial\overline{S^c} \subset \partial S$. It remains to prove (31) and (32).

Theorem 2.1.2 in Berger and Gostiaux [1] and a standard differentiability argument show that for every choice of $0 < r < l$, there exists a neighborhood $N_{r,l}$ of s with $\operatorname{conv}(\{s\} \cup B_r(s - ln(s))) \cap N_{r,l} \subset S$. Equation (31) is a direct consequence of this. To prove (32) let r and l as given there and set $C_{r,l} := \operatorname{conv}(\{s\} \cup B_r(s - ln(s)))$. It follows from the compactness of S that there exists a maximal $\bar{r} \leqslant l$ with $C_{\bar{r},l} \subset S$. Suppose $\bar{r} < l$. Then $\partial C_{\bar{r},l}$ must meet ∂S in some point $t \neq s$ because otherwise $\bar{r}$ cannot be maximal as $C_{(\bar{r}+l)/2,l} \cap N_{(\bar{r}+l)/2,l} \subset S$ and $(\bar{r} + l)/2 > \bar{r}$.

Now $C_{\bar{r},l} \subset S$ implies that the tangent hyperplane of ∂S at t must coincide with some supporting hyperplane of the convex set $C_{\bar{r},l}$ at t. Thus $\langle n(t), t - s\rangle \geqslant 0$. By definition of $C_{\bar{r},l}$ and as $t \neq s$ we can write $t = \alpha s + (1 - \alpha)(s - ln(s) + re)$ for some $\alpha \in [0, 1)$ and some $e \in \partial B_1(0)$. Standard arguments of convex geometry readily show that one can take $e = n(t)$, hence $t - s = (1 - \alpha)(\bar{r}n(t) - ln(s))$ for some $\alpha \in [0, 1)$. This identity and the Lipschitz condition give $(1 - \bar{r}/l)\langle n(s), n(t)\rangle = \frac{1}{2}(2 - |n(s) - n(t)|^2) - \bar{r}/l\langle n(s), n(t)\rangle \geqslant \frac{1}{2}(2 - (1/r_0^2)|s - t|^2) - (\bar{r}/l)\langle n(s), n(t)\rangle \geqslant \frac{1}{2}(2 - (1/r_0^2)(\bar{r}^2 + l^2 - 2\bar{r}l\langle n(s), n(t)\rangle)) - \bar{r}/l\langle n(s), n(t)\rangle$. Now $r_0 \geqslant l$ shows that the last term is at least $\frac{1}{2}(1 - \bar{r}^2/l^2) > (1 - \bar{r}/l)\bar{r}/l \geqslant (1 - \bar{r}/l)(\bar{r}/l - [1/(1 - \alpha)l]\langle t - s, n(t)\rangle)$ as $\bar{r} < l$ and $\langle n(t), t - s\rangle \geqslant 0$. But the last term equals $(1 - \bar{r}/l)\langle n(s), n(t)\rangle$ by the identity for $t - s$ above. This contradiction shows that we must have in fact $\bar{r} \geqslant l$. □

(iii) $\Rightarrow$ (ii): One readily checks that (iii) and the fact that S^c is open imply

$$\partial\overline{S^c} = \partial S \quad \text{and } \operatorname{int}\overline{S^c} = S^c. \tag{33}$$

We will show first:

$$s \in \operatorname{int} S \quad \text{implies} \quad s \in \operatorname{int} B_{r_0}(x) \subset \operatorname{int} S \quad \text{for some } x \in \mathbb{R}^d. \tag{34}$$

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As ∂S is compact there exists $t \in \partial S$ such that $r_s := |s - t| = \inf_{y \in \partial S} |s - y| > 0$. If $r_s < r_0$, then it follows from (iii), (33) and Lemma A that there exists $x \in \mathbb{R}^d$ with $t \in B_{r_s}(s) \subset B_{r_0}(x) \subset S$, proving (34).

Proceeding in the same way with the closed set $\overline{S^c}$ and using (33) shows that (34) holds with $\operatorname{int} S$ replaced by S^c. These two versions of (34) yield $\operatorname{int} S = \bigcup_{\operatorname{int} B_{r_0}(x) \subset \operatorname{int} S} \operatorname{int} B_{r_0}(x)$ and $S^c = \bigcup_{\operatorname{int} B_{r_0}(x) \subset S^c} \operatorname{int} B_{r_0}(x)$. (iii) implies $\operatorname{int} S \neq \emptyset$. So (ii) follows from the general fact that $\operatorname{int} S = (\overline{S^c})^c$. □

(ii) $\Rightarrow$ (i). Is S is r_0-convex then one has for all $0 < \lambda < r_0$: $S^c = \bigcup_{\operatorname{int} B_{r_0}(x) \subset S^c} \operatorname{int} B_{r_0}(x) \subset \bigcup_{B_\lambda(x) \subset S^c} B_\lambda(x) \subset S^c$, and thus $\Psi_\lambda(S) = S$ for $\lambda \in (-r_0, 0)$ follows.

r_0-convexity of $\overline{S^c}$ together with $\operatorname{int} S = (\overline{S^c})^c$ and closedness of S gives

$$\operatorname{int} S = \bigcup_{\operatorname{int} B_{r_0}(x) \subset \operatorname{int} S} \operatorname{int} B_{r_0}(x) \subset \bigcup_{B_{r_0}(x) \subset S} B_{r_0}(x). \tag{35}$$

Let $s \in \partial S$. We will show in a moment that

$$\inf_{y \in \operatorname{int} S} |s - y| = 0. \tag{36}$$

Together with (35) one sees that there exists a sequence $\{s_n\} \subset \operatorname{int} S$ converging to s and a sequence $\{x_n\}$ such that $s_n \subset \operatorname{int} B_{r_0}(x_n) \subset \operatorname{int} S$. Let $\bar{x}$ be a cluster point of the bounded sequence $\{x_n\}$. One concludes $s \in B_{r_0}(\bar{x}) \subset S$ because S is closed, whence $\partial S \subset \bigcup_{B_{r_0}(x) \subset S} B_{r_0}(x)$. Together with (35) one obtains for $0 \leqslant \lambda \leqslant r_0$: $S \subset \bigcup_{B_{r_0}(x) \subset S} B_{r_0}(x) \subset \bigcup_{B_\lambda(x) \subset S} B_\lambda(x) \subset S$ and thus $\Psi_\lambda(S) = S$ for $\lambda \in [0, r_0]$ follows.

It remains to prove (36). Suppose it were not true. As $\operatorname{int} S \neq \emptyset$ and S is path-connected it is possible to find a $t \in \partial S$ such that $d := \inf_{y \in \operatorname{int} S} |t - y|$ satisfies $0 < d < r_0/10$. Together with (35) this yields the existence of a x with

$$0 < |x - t| \leqslant r_0 + d \quad \text{and} \quad \operatorname{int} B_{r_0}(x) \subset \operatorname{int} S. \tag{37}$$

Set $b := t + [d/2|x - t|](x - t)$ and observe $\inf_{y \in \operatorname{int} S} |b - y| > 0$, so there exists a sequence $\{b_n\} \subset S^c$ with $\lim_{n \to \infty} b_n = b$ and a sequence $\{c_n\}$ with

$$b_n \in \operatorname{int} B_{r_0}(c_n) \subset S^c. \tag{38}$$

Now $t \in S$ shows that for all n, $|c_n - t|^2 \geqslant r_0^2 > |c_n - b_n|^2 = |c_n - t|^2 + |t - b_n|^2 + 2\langle c_n - t, t - b_n \rangle$, which together with the definition of b and the boundedness of $\{c_n - t\}$ implies $\limsup_n \langle c_n - t, t - x \rangle = \limsup_n \langle c_n - t, t - b \rangle 2|x - t|/d = \limsup_n \langle c_n - \mathrm{t}, t - b_n \rangle 2|x - t|/d \leqslant -\limsup_n |t - b_n|^2/2 \cdot 2|x - t|/d = -|t - b|^2 |x - t|/d \leqslant 0$.

Together with (37) and (38) this shown $\limsup_n |c_n - x|^2 \leqslant \limsup_n (|c_n - b_n| + |b_n - b| + |b - t|)^2 + |t - x|^2 \leqslant (r_0 + d/2)^2 + (r_0 + d)^2 < 3r_0^2$ as $d < r_0/10$, contradicting $|c_n - x| \geqslant 2r_0$ for all n as implied by (37) and (38). This proves (36). □

Proof of Theorem 2. Lemma B 1(a), (b) give $T_{-\varepsilon} = \Psi_{r_i}(T_{-\varepsilon}) = \Psi_{-r_o}(T_{-\varepsilon})$ and $T_\varepsilon = \Psi_{r_i}(T_\varepsilon) = \Psi_{-r_o}(T_\varepsilon)$. So by (MON II)

$$T_{-\varepsilon} \subset \Psi_{r_i}(S) \subset T_\varepsilon \quad \text{and} \quad T_{-\varepsilon} \subset S_{io} \subset T_\varepsilon \quad \text{as well as } T_{-\varepsilon} \subset S_{oi} \subset T_\varepsilon. \tag{39}$$

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Now consider first S_{io}. (ID) shows $\Psi_{-r_o}(S_{io}) = S_{io}$ whence (MON) yields $S_{io} = \Psi_r(S_{io})$ for $r \in [-r_o, 0]$. (6) is proved once it is shown that for $f_{io} > r_o$ and $r \in (0, f_{io} - r_o)$

$$s \in S_{io} \quad \text{implies } s \in B_r(x) \subset S_{io} \quad \text{for some } x, \tag{40}$$

because (40) is equivalent to $S_{io} = \Psi_r(S_{io})$ by (2). We need only consider the case where $S_{io} \neq \mathbb{R}^d$. Then there exists $c \in \partial S_{io}$ such that $|s - c| = \min_{y \in \partial S_{io}} |s - y|$. If $|s - c| > r$, then $B_r(s) \subset S_{io}$ and (40) follows. If $|s - c| \leqslant r$ then set $A := \Psi_{r_i}(S) \oplus r_o B$. Then $S_{io} = A \ominus r_o B$, so $c \in \partial(A \ominus r_o B)$. It is readily checked that this implies

$$\operatorname{int} B_{r_o}(c) \subset A \quad \text{and} \quad \operatorname{int} B_{r_o}(b) \subset (S_{io})^{\mathrm{c}} \quad \text{for some } b \text{ with } |b - c| = r_o. \tag{41}$$

Next, we will show

$$\operatorname{int} B_{f_{io} - r_o}(c + (f_{io} - r_o) e(c - b)) \subset S_{io}. \tag{42}$$

Set $Z := T_{\varepsilon + r_o}$. By the definition of A we can write $A = (S_{-r_i}) \oplus (r_i + r_o) B$. We get

$$\begin{aligned} &\Psi_{r_i + r_o}(A) = A \quad \text{by (MON)},\\ &\Psi_{-(R_o - r_o - \varepsilon)}(Z) = Z \quad \text{by Lemma B 1(b) as } \varepsilon + r_o < R_o,\\ &Z_{-2\varepsilon} \subset A \subset Z \quad \text{by (39) and as Lemma B 1(c) yields } (T_{-\varepsilon})_{r_o} = Z_{-2\varepsilon}. \end{aligned} \tag{43}$$

$s \in S_{io}$ implies $A, Z \neq \emptyset$, so (41), (43) and Lemma C show $B_{\tilde f}(b + \tilde f e(c - b)) \subset A$, where

$$\tilde f = \tilde f(r_o, r_i, R_o - r_o - \varepsilon, 2\varepsilon) = (r_i + r_o)\left(1 - \frac{8 R_o \varepsilon}{(R_o - r_o + \varepsilon) r_o}\right) = f_{io}.$$

Hence $\operatorname{int} B_{f_{io} - r_o}(c + (f_{io} - r_o) e(c - b)) = \operatorname{int} B_{f_{io}}(b + f_{io} e(c - b)) \ominus r_o B \subset A \ominus r.B = S_{io}$, proving (42).

Now we can finish the proof of (40) for the case $|s - c| \leqslant r < f_{io} - r_o$. By the definition of c, if $|s - c| > 0$, then $\operatorname{int} B_{|s - c|}(s) \subset S_{io}$, which together with (41) implies $e(s - c) = -e(b - c)$. Hence $|s - (c + (f_{io} - r_o) e(c - b))| < f_{io} - r_o$, which together with (42) and $r < f_{io} - r_o$ yields (40). In the case $|s - c| = 0$ we have $c = s \in S_{io}$, whence (42) gives $s \in B_r(c + re(c - b)) \subset \operatorname{int} B_{f_{io} - r_o}(c + (f_{io} - r_o) e(c - b)) \cup \{c\} \subset S_{io}$, completing the proof of (40) and of (6).

As for (7), (1) and (4) show that $\tilde S := S^{\mathrm{c}}$ and $\tilde T := T^{c}$ satisfy $\tilde T_{-\varepsilon} \subset \tilde S \subset \tilde T_{\varepsilon}$ and $\Psi_{-R_i}(\tilde T) = \tilde T = \Psi_{R_o}(\tilde T)$. Then the already proven assertion (6) shows that the set $\tilde{\tilde S} := \Psi_{-r_i}(\Psi_{r_o}(\tilde S))$ satisfies $\tilde{\tilde S} = \Psi_r(\tilde{\tilde S})$ for $r \in [r_i, (f_{oi} + r_i)^+)$. But using (4) one obtains $\tilde{\tilde S} = \Psi_{-r_i}((\Psi_{-r_o}(S))^{\mathrm{c}}) = (\Psi_{r_i}(\Psi_{-r_o}(S)))^{\mathrm{c}} = (S_{oi})^{\mathrm{c}}$, so (4) gives $S_{oi} = \Psi_r(S_{oi})$ for $r \in (-(f_{oi} - r_i)^+, r_i]$.

Finally, observe that $d_H(S, T) < \varepsilon$ gives $T \subset S_{\varepsilon} \subset T_{2\varepsilon}$ (it is enough to require $d_H(S, T) \leqslant \varepsilon$ if S and T are both closed), so the inclusions in the following statement result from (MON II); the equalities follow from Lemma B 1(c) and $0 < r_o - \varepsilon < R_o$, $2\varepsilon < R_o$ and $\varepsilon + r_o < R_o$, respectively:

$$T_{-\varepsilon} = (T_{r_o - \varepsilon})_{-r_o} \subset ((S_{\varepsilon})_{r_o - \varepsilon})_{-r_o} \subset ((T_{2\varepsilon})_{r_o - \varepsilon})_{-r_o} \subset (T_{\varepsilon + r_o})_{-r_o} = T_{\varepsilon}.$$

Further, $r_o - \varepsilon > 0$ shows $(S_{\varepsilon})_{r_o - \varepsilon} = (S \oplus \varepsilon B) \oplus (r_o - \varepsilon) B = S_{r_o}$, whence $\tilde S := \Psi_{-r_o}(S) = ((S_{\varepsilon})_{r_o - \varepsilon})_{-r_o}$. We thus have $T_{-\varepsilon} \subset \tilde S \subset T_{\varepsilon}$, so Theorem 2 applies for $\tilde S$. But $(\tilde S)_{oi} = \Psi_{r_i}(\Psi_{-r_o}(\tilde S)) = \Psi_{r_i}(\Psi_{-r_o}(S)) = S_{oi}$ by (ID). □

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Proof of Theorem 3. First consider S_{io}. $\Psi_r(S_{io}) = S_{io}$ for $r \in [-r_o, 0]$ follows from (ID) and (MON). To prove the case $r \in (0, (f - r_o)^+)$ we will show as in the proof of (6) of Theorem 2 that $f > r_o$ and $r \in (0, (f - r_o)$ imply (40). The proof of (40) follows that for Theorem 2 except that we set here $Z := (\operatorname{conv} \Psi_{r_i}(S))) \oplus r_o B$. (CON) and Lemma E(a) show $\Psi_{-R_o}(Z) = Z$ for every $R_o > 0$, and Lemma D shows

$$\operatorname{conv}(\Psi_{r_i}(S) \oplus \sqrt{(r_o^2 - (\operatorname{diam} S)^2 B)}) \subset \Psi_{r_i}(S) \oplus r_o B =: A \tag{44}$$

(note $\operatorname{diam} S \geqslant \operatorname{diam} \Psi_{r_i}(S)$). Whence Lemma E(a) and (CON) give $Z_{-(r_o - \sqrt{[r_o^2 - (\operatorname{diam} S)^2]})} \subset A \subset Z$. As in the proof of (6) one hence finds that for every $R_o > 0$, (40) holds for $r \in (0, (\tilde{f}(r_o, r_i, R_o, r_o - \sqrt{[r_o^2 - (\operatorname{diam} S)^2]} - r_o)^+)$, and hence for $0 < r < \lim_{R_o \to \infty} (\tilde{f}(r_o, r_i, R_o, r_o - \sqrt{[r_o^2 - (\operatorname{diam} S)^2]} - r_o)^+ = (f - r_o)^+$.

Finally, the first inclusion in (9) follows by applying $\ominus r_o B$ in (44) and using (CON) and Lemma E(a); further (MON II) and Lemma E(b) show that for any set $C \subset \mathbb{R}^d$, $\Psi_{-r_o}(C) \subset \Psi_{-r_o}(\operatorname{conv} C) = \operatorname{conv} C$, so the second inclusion in (9) follows by setting $C := \Psi_{r_i}(S)$.

Concerning S_{oio}, we just saw that $S \subset \Psi_{-r_o}(S) \subset \operatorname{conv} S$, but then it follows that $\operatorname{diam} \Psi_{-r_o}(S) = \operatorname{diam} S$. So applying the already proved part of (8) concerning S_{io} to the set $\Psi_{-r_o}(S)$ instead of S shows the assertion concerning S_{oio} in (8). Further, Lemmas D and E(a) yield $(\operatorname{conv} S)_{-(r_o - \sqrt{[r_o^2 - (\operatorname{diam} S)^2]}} \subset \Psi_{-r_o}(S)$, hence

$$(\operatorname{conv} S)_{\sqrt{[r_o^2 - (\operatorname{diam} S)^2]} - r_o - r_i} \subset (\Psi_{-r_o}(S))_{-r_i} \subset \Psi_{r_i}(\Psi_{-r_o}(S)) \subset S_{oio} \subset \Psi_{-r_o}(S), \tag{45}$$

using (MON) and (ID) for the last two inclusions. We already saw $\Psi_{-r_o}(S) \subset \operatorname{conv} S$, so (10) follows. Equation (11) is a consequence of (10) and Lemma F. □

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