84. On a Generalization of Bochner's Tube Theorem for Generic CR-Submanifolds

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The classical Bochner's tube theorem states that every holomorphic function on a connected tube domain $\boldsymbol{R}^n+i\Omega\subset\boldsymbol{C}^n$ can be extended holomorphically to the convex hull of the tube $\boldsymbol{R}^n+ich(\Omega)$.

H. Komatsu [4] has obtained a simple proof of the local version of this theorem by using Cauchy's integral formula. By making use of the theory of Fourier-Bros-Iagolnitzer transform, M. S. Baouendi and F. Treves [1] have generalized the result above. In particular they have obtained the microlocal version of Bochner's tube theorem for generic CR-manifolds.

In this paper we shall give a simple proof of this result. In the section 1, we formulate Bochner's tube theorem for generic CR-submanifolds by employing the notion of specialization of sheaf of holomorphic functions (cf. [5]). In the section 2 we give the new proof of the theorem by reducing the problem to the totally real case.

1. Statement of the result. Let N be a real analytic submanifold of a complex manifold X. For $p\in N$, we denote by $H_p(N)$ the complex tangent space to N at p. The submanifold N is said to be generic, if for all $p\in N$, $\dim_{\boldsymbol{C}} H_p(N)=\dim_{\boldsymbol{C}} X-\operatorname{codim}_{\boldsymbol{R}} N$. Let us assume hereafter that N is generic.

Let S_NX be the spherical normal bundle $T_NX-\{0\}/\boldsymbol{R}^+$ with the projection $\tau: S_NX\to N$. The disjoint union ${}^{\tilde{N}}X=(X-N)\coprod S_NX$ has the structure of real analytic manifold with boundary S_NX. It is called the real monoidal transform of X with center N.

Let i (resp. $\tilde{i}$) be the embedding $i: N\to X$ (resp. $\tilde{i}: S_NX\to{}^{\tilde{N}}X$) and j (resp. $\tilde{j}$) be the natural inclusion map $j: X-N\to X$ (resp. $\tilde{j}: X-N\to{}^{\tilde{N}}X$). We set $\tilde{\mathcal{A}}_{N|X}=\tilde{i}^{-1}(\tilde{j}_*(j^{-1}\mathcal{O}_X))$.

Recall that a subset V of S_NX is said to be convex if each fiber of τ is convex. For every subset U in S_NX, we denote by $ch(U)$ the smallest convex set containing U.

Theorem (cf. [1]). *Let N be a real analytic submanifold of X. We assume that N is a generic* CR-*submanifold. Let U be an open connected subset of S_NX. Then the following assertions hold.*

(a) $\Gamma(U;\tilde{\mathcal{A}}_{N|X})=\Gamma(ch(U);\tilde{\mathcal{A}}_{N|X})$.

(b) *If $ch(U)=\tau^{-1}(\tau(U))$, then $\Gamma(U;\tilde{\mathcal{A}}_{N|X})=\Gamma(\tau(U);\mathcal{O}_{X|N})$.*

We note here that the last statement imply the classical Kneser's theorem [3].

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2. Proofs. Let Y be a complexification of N, i.e., Y is a complex manifold in which N can be embedded as a generic totally real submanifold. We denote by $\tilde{\mathcal{A}}_N$ the sheaf $\tilde{\mathcal{A}}_{N|Y}$ which can be defined in the same way as in the section 1. Let $i_C: Y \to X$ be the natural complexification of the embedding $i: N \to X$. Since N is generic, i_C is a submersion in a neighborhood of N. We denote by ϕ the natural mapping $\phi: S_N Y - S_N(i_C^{-1}(N)) \to S_N X$.

Proof of (a). We have the followings:

$$\Gamma(U;\tilde{\mathcal{A}}_{N|X})=\Gamma(\phi^{-1}(U);\phi^{-1}\tilde{\mathcal{A}}_{N|X}),$$
$$\Gamma(\phi^{-1}(U);\phi^{-1}\tilde{\mathcal{A}}_{N|X})\subset\Gamma(\phi^{-1}(U);\tilde{\mathcal{A}}_N).$$

By using the microlocal version of Bochner's tube theorem ([4]), we have

$$\Gamma(\phi^{-1}(U);\tilde{\mathcal{A}}_N)=\Gamma(ch(\phi^{-1}(U));\tilde{\mathcal{A}}_N).$$

Hence, every section of $\phi^{-1}\tilde{\mathcal{A}}_{N|X}$ on $\phi^{-1}(U)$ can be continued to $ch(\phi^{-1}(U))$ as a section of $\tilde{\mathcal{A}}_N$. By definition, $\phi^{-1}(\tilde{\mathcal{A}}_{N|X})$ is the sheaf of boundary value of holomorphic functions which satisfy the partial de Rham system defined by the submersion i_C. This yields

$$\begin{aligned}\Gamma(\phi^{-1}(U);\phi^{-1}(\tilde{\mathcal{A}}_{N|X}))&=\Gamma(ch(\phi^{-1}(U));\phi^{-1}(\tilde{\mathcal{A}}_{N|X}))\\&=\Gamma(\phi^{-1}(ch(U));\phi^{-1}(\tilde{\mathcal{A}}_{N|X})).\end{aligned}$$

Therefore,

$$\Gamma(U;\tilde{\mathcal{A}}_{N|X})=\Gamma(ch(U);\tilde{\mathcal{A}}_{N|X})$$

which completes the proof of (a).

Proof of (b). From the above arguments we have

$$\Gamma(U;\tilde{\mathcal{A}}_{N|X})=\Gamma(\tau^{-1}(\tau(U));\tilde{\mathcal{A}}_{N|X})=\Gamma(\tau(U);\tau_*\tilde{\mathcal{A}}_{N|X}).$$

Since N is generic the sheaf $\tau_*\tilde{\mathcal{A}}_{N|X}$ is isomorphic to the sheaf $i^{-1}\mathcal{O}_X$ ([6]). We have

$$\Gamma(U;\tilde{\mathcal{A}}_{N|X})=\Gamma(\tau(U);\mathcal{O}_{X|N}).$$

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