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ON A GENERALIZATION OF CLOSE-TO-CONVEXITY

K. INAYAT NOOR

Mathematics Department
Science College of Education for Girls
Sitteen Road, Riyadh, Saudi Arabia

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ABSTRACT. A class T_k of analytic functions in the unit disc is defined in which the concept of close-to-convexity is generalized. A necessary condition for a function f to belong to T_k, raduis of convexity problem and a coefficient result are solved in this paper.

KEY WORDS AND PHRASES. Close-to-convex functions, functions of bounded boundary rotation, necessary condition, radius of convexity, generalized Koebe function.

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1. INTRODUCTION.

This paper is directed to mathematical specialists or non-specialists familiar with multivalent functions [1], and to close-to-convex functions [2].

Let V_k be the class of functions of bounded boundary rotation and K be the class of close-to-convex functions. We generalize the concept of close-to-convexity in the following direction.

<u>Definition</u>. Let f with $f(z) = cz + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E = \{z:|z|<1\}$, $|c|=1$ and $f'(z) \neq 0$. Then $f\varepsilon T_k$, $k\geq 2$, if there exist a function $g\varepsilon V_k$ such that, for $z\varepsilon E$

$$\operatorname{Re} \frac{f'(z)}{g'(z)} > 0. \tag{1.1}$$

It is clear that $T_2 \equiv K$.

Using a method by Kaplan [2], we have

<u>THEOREM 1</u>. Let $f\varepsilon T_k$. Then with $z = re^{i\theta}$ and $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\frac{k}{2}\pi \tag{1.2}$$

REMARK 1. From theorem 1, we can interpret some geometric meaning for the class T_k. For simplicity, let us suppose that the image domain is bounded by an analytic curve C. At a point on C, the outward drawn normal has an angle $\arg[e^{i\theta}f'(e^{i\theta})]$. Then from (1.2), it follows that the angle of the outward drawn normal turns back at most $\frac{k}{2}\pi$. This is a necessary condition for a function f to belong to T_k. It will be interesting to see if this condition is also sufficient.

REMARK 2. Goodman [3] defines the class $K(\beta)$ of functions as follows. Let f with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and $f'(z) \neq 0$. Then for $\beta \geq 0$, $f \varepsilon K(\beta)$, if for $z=re^{i\theta}$ and $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[\frac{(zf'(z))'}{f'(z)}\right] d\theta > -\beta\pi$$

We note that $T_k \subset K\left(\frac{k}{2}\right)$.

2. MAIN RESULTS

From remark 2 and results given in [3] for the class $K(\beta)$, we have at once

THEOREM 2. Let $f \varepsilon T_k$.

(i) Denote by $L(r,f)$ the length fo the image of the circle $|z| = r$ under f and by $A(r,f)$ the area of $f(|z|=r)$. Then for $0 \leq r < 1$,

(a) $L(r,f) \leq L(r,F_k)$,

(b) $A(r,f) \leq A(r,F_k)$,

where F_k is defined by, for $z \varepsilon E$,

$$F_k(z) = \frac{1}{(k+2)}\left[\left(\frac{1+k}{1-z}\right)^{\frac{1}{2}k+1} - 1\right]$$
$$= z + \sum_{n=2}^{\infty} A_n(k) z^n \tag{2.1}$$

and clearly $F_k \varepsilon T_k$.

(ii) $|a_n| \leq A_n(k)$, $n = 2,3, \ldots\ldots\ldots\ldots$, $k \geq 2$

where $A_n(k)$ is defined by (2.1). This result is sharp for each $n \geq 2$.

(iii) For $z = re^{i\theta}$, $0 \leq r < 1$,

$$\frac{(1-r)^{\frac{1}{2}k}}{(1+r)^{\frac{1}{2}k+2}} \leq |f'(z)| \leq \frac{(1+r)^{\frac{1}{2}k}}{(1-r)^{\frac{1}{2}k+2}}$$

These bounds are sharp, equality being attained for the function F_k defined by (2.1).

We also need the following result.

Lemma 1 [4]. Let $g\varepsilon V_k$. Then there are two starlike functions s_1 and s_2 such that for $z\varepsilon E$

$$g'(z) = \frac{(s_1(z)/z)^{\frac{1}{4}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{4}k-\frac{1}{2}}}$$

THEOREM 3. $f\varepsilon T_k$ if and only if

$$f'(z) = \frac{(k_1'(z))^{\frac{1}{4}k+\frac{1}{2}}}{(k_2'(z))^{\frac{1}{4}k-\frac{1}{2}}}, \qquad k_1, k_2 \varepsilon k$$

PROOF: From definition 1, we have

$$f'(z) = g'(z)h(z), \qquad g\varepsilon V_k \text{ and } \operatorname{Re} h(z)>0.$$

Using lemma 1, we know that there are two starlike functions s_1 and s_2 such that $z\varepsilon E$,

$$g'(z) = \frac{(s_1(z)/z)^{\frac{1}{4}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{4}k-\frac{1}{2}}}$$

Thus

$$\begin{aligned} f'(z) &= \frac{(s_1(z)/z)^{\frac{1}{4}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{4}k-\frac{1}{2}}}\, h(z) = \frac{((s_1(z)h(z))/z)^{\frac{1}{4}k+\frac{1}{2}}}{((s_2(z)h(z))/z)^{\frac{1}{4}k-\frac{1}{2}}} \\ &= \frac{(k_1'(z))^{\frac{1}{4}k+\frac{1}{2}}}{(k_2'(z))^{\frac{1}{4}k-\frac{1}{2}}} \end{aligned}$$

where k_1 and k_2 are two suitable selected close-to-convex functions.

Lemma 2. Let H be analytic and be defined as

$$H(z)g'(z) = (zg'(z))', \quad g\varepsilon V_k \text{ and } H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z),$$

$$\operatorname{Re} h_i(z) > 0, \; i=1,2, \; h_i(0)= 1$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} |H(z)|^2 d\theta \le \frac{1 + (k^2-1)r^2}{1-r^2} \qquad (z = re^{i\theta})$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} |H'(z)|\, d\theta \le \frac{k}{1-r^2}$$

PROOF: By the representation formula due to Paatero [5], we can write

$$H(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{it}}{1-ze^{it}}\, d\mu(t),$$

where

$$\int_0^{2\pi} d\mu(t) = 2\pi, \text{ and } \qquad \int_0^{2\pi} |d\mu(t)| \leq k\pi$$

Let $H(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$

Then

$$c_n = \frac{1}{\pi}\int_0^{2\pi} e^{-int} d\mu(t), \text{ and so for } n\geq 1,$$

$$|c_n| \leq \frac{1}{\pi}\int_0^{2\pi} |d\mu(t)| \leq k$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} |H(z)|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq (1+k^2 \sum_{n=1}^{\infty} r^{2n}) = \frac{1+(k^2-1)r^2}{1-r^2}$$

Also

$$H'(z) = \frac{1}{\pi}\int_0^{2\pi} \frac{e^{it}}{(1-ze^{it})^2}\, d\mu(t)$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} |H'(z)| d\theta \leq \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-re^{i(\theta+t)}|^2}\, d\theta |d\mu(t)| \leq \frac{1}{1-r^2}\,\frac{1}{\pi}\int_0^{2\pi} |d\mu(t)| \leq \frac{k}{1-r^2}$$

<u>THEOREM 4</u>: Let $f\epsilon T_k$. Then for $n\geq 1$,

$$\Big|\,|a_{n+1}| - |a_n|\,\Big| \leq c(k)n^{\frac{k}{2}-1},$$

where $c(k)$ is a constant and depends only on k.

<u>PROOF</u>: Since $f\epsilon T_k$, we have for $z\epsilon E$,

$$f'(z) = g'(z)h(z), \qquad g\epsilon V_k \text{ and } \mathrm{Re}\, h(z)>0$$

Set

$$F(z) = z(zf'(z))' = zg'(z)[H(z)h(z) + zh'(z)], \tag{2.2}$$

where $\mathrm{Re}\, h(z) > 0$ and $H(z)g'(z) = (zg'(z))'$, with

$$H(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z), \qquad \mathrm{Re}\, h_i(z) > 0,\ i=1,2,\ h_i(0)=1$$

Thus, for $\xi \,\epsilon\, E$ and $n\geq 1$;

$$|(n+1)^2\xi a_{n+1}-n^2a_n| \le \frac{1}{2\pi r^{n+1}}\int_0^{2\pi}|z-\xi|\,|F(z)|\,d\theta,$$

and by using lemma 1 and (2.2), we obtain

$$|(n+1)^2\xi a_{n+1}-n^2a_n| \le \frac{1}{2\pi r^{n+1}}\int_0^{2\pi}|z-\xi|\frac{|s_1(z)|^{\frac{1}{4}k+\frac{1}{2}}}{|s_2(z)|^{\frac{1}{4}k-\frac{1}{2}}}\,|H(z)h(z)+zh'(z)|\,d\theta, \tag{2.3}$$

where s_1, s_2 are starlike functions.

It is well-known [1] that for starlike function $s\varepsilon S$,

$$\frac{r}{(1+r)^2} \le |s(z)| \le \frac{r}{(1-r)^2} \tag{2.4}$$

Let $0<r<1$. Then by a result of Golusin [6,p162], there exists a z_1 with $|z_1| = r$ such that for all z, $|z| = r$,

$$|z-z_1|\,|s_1(z)| \le \frac{2r^2}{1-r^2}. \tag{2.5}$$

From (2.3)-(2.5), we have

$$|(n+1)^2\xi a_{n+1}-n^2a_n| \le \frac{1}{2\pi r^{n+1}}\left(\frac{4}{r}\right)^{\frac{1}{4}k-\frac{1}{2}}\left(\frac{2r^2}{1-r^2}\right)\left(\frac{r}{(1-r)^2}\right)^{\frac{1}{4}k-\frac{1}{2}}\int^{2\pi}|H(z)h(z)+zh'(z)|\,d\theta \tag{2.6}$$

Now as in [7], we have with $z = re^{i\theta}$

$$\left.\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}|h(z)|^2d\theta \le \frac{1+3r^2}{1-r^2}\\ &\text{and}\\ &\frac{1}{2\pi}\int_0^{2\pi}|zh'(z)|\,d\theta \le \frac{2r}{1-r^2}, \quad \text{where } \mathrm{Re}\, h(z) > 0.\end{aligned}\right\} \tag{2.7}$$

Also

$$\frac{1}{2\pi}\int_0^{2\pi}|H(z)h(z)+zh'(z)|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi}|H(z)h(z)|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}|zh'(z)|\,d\theta$$

$$\le \frac{(1+(k^2-1)r^2)^{\frac{1}{2}}(1+3r^2)^{\frac{1}{2}}}{1-r^2} + \frac{2r}{1-r^2} \tag{2.8}$$

by using Schwarz's inequality, lemma 2 and (2.7).

Hence from (2.6) and (2.8), we have

$$|(n+1)^2\xi a_{n+1}-n^2a_n| \le \frac{1}{r^{n+1}}\,2^{\frac{1}{2}k}\left[(1+(k^2-1)r^2)^{\frac{1}{2}}+1\right]\frac{1}{(1-r)^{\frac{1}{2}k+1}},$$

and so choosing $|\xi| = r = \left(\frac{n}{n+1}\right)^2$, we obtain for $n \geq 1$

$$n^2 \left| |a_{n+1}| - |a_n| \right| \leq \left[(1+(k^2-1)r^2)^{\frac{1}{2}} + 1\right] e^2 \, 2^{\frac{1}{2}k+2} \left(\frac{4}{3}\right)^{\frac{1}{2}k+1} n^{\frac{1}{2}k+1}$$

Thus

$$\left| |a_{n+1}| - |a_n| \right| \leq c(k) n^{\frac{1}{2}k-1} .$$

The function F_k defined by (2.1) shows that the index $\left(\frac{k}{2} - 1\right)$ is best possible.

We now evaluate the radius of convexity for the class T_k.

THEOREM 5: Let $f \in T_k$. Then the radius R of the circle which f maps onto a convex domain is given by

$$R = \frac{1}{2}\left[(k+2) - \sqrt{(k^2+4k)}\right].$$

The function F_k defined by (2.1) shows that this result is best possible. In particular, when k = 2, $R = 2-\sqrt{3}$, which is well known. This result also follows from the remark in [3,p.23].

PROOF: By definition

$$zf'(z) = ag'(z)h(z) \qquad g \in V_k;\ \mathrm{Re}\ h(z) > 0.$$

Thus

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zh'(z)}{h(z)}$$

and so

$$\mathrm{Re}\ \frac{(zf'(z))'}{f'(z)} \geq \mathrm{Re}\ \frac{(zg'(z))'}{g'(z)} - \left|\frac{zh'(z)}{h(z)}\right|$$

For $g \in V_k$, it is well known [9] that, for $z = re^{i\theta}$, $0 \leq r \leq 1$,

$$\mathrm{Re}\ \frac{(zg'(z))'}{g'(z)} \geq \frac{r^2-kr+1}{1-r^2}$$

Hence

$$\mathrm{Re}\ \frac{(zf'(z))'}{f'(z)} \geq \frac{r^2-kr+1}{1-r^2} - \frac{2r}{1-r^2} = \frac{r^2-(k+2)r+1}{1-r^2}$$

This gives the required result.

REMARKS 3.

(i). We also note that the extremal function $F_k(z)$ defined by (2.1) is the same function as $F_\beta(z)$ defined by equation (2.6) in [3]. As A. W. Goodman has pointed out that this function is sometime referred to as the generalized Koebe function.

(ii). We conjecture that the class T_k is a proper subclass of the class $K(\beta)$ as defined in [3], since in the definition of T_k, $g \varepsilon V_k$ and we know that $g \varepsilon V_k$, $2 \leq k \leq 4$, is convex in one direction and all the functions in one direction form a proper subclass of the class of close-to-convex functions.

(iii). It remains open whether T_k is a linear in variant family.

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REFERENCES

1. HAYMAN, W. K. Multivalent Functions, Cambridge University press, 1967.

2. KAPLAN, W. Close-to-convex Schilcht Functions, Mich. Math. J. 1 (1952), 169-185.

3. GOODMAN, A. W. On Close-to-convex Functions of Higher Order, Univ. Sci. Budapest. Eȏtvos. Sect. Mathematica, 15 (1972), 17-30.

4. BRANNAN, D. A. On Functions of Bounded Boundary Rotation, Proc. Edin. Math. Soc. 2 (1968/69), 330-347.

5. PAATERO, V. Uber Gebiete von beschrankter Randdrehung Annal/Acad. Sci. Fenn. Ser A 37 (1933), 20pp.

6. GOLUSIN, G. M. Geometrische Functionstheorie, Berlin, 1957.

7. POMMERENKE, Ch. On Starlike and Close-to-convex Functions, Proc. Lond. Math. Soc. 3 (1963), 290-304.