On a generalization of Mycielski's and Znám's conjectures about coset decomposition of Abelian groups

by

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Abstract. An exact lower bound for the cardinality of a partition of a group into cosets by its invariant subgroups is given. This lower bound is a function of the index of the intersection of all the subgroups, cosets by which occur in the partition. For Abelian groups the present theorem gives the bound conjectured by Mycielski and Sierpiński, and for the infinite cyclic group we obtain the bound conjectured by Znám.

In this paper groups will be considered as multiplicative groups and their neutral element will be denoted by e. The order of a finite group G will be denoted by $|G|$. If H is a subgroup of G and $a \in G$, then $[G:H]$, aH will denote the index of H in G and the set $\{ah|\ h \in H\}$, respectively. If H is an invariant subgroup of G, then G/H will denote the factor group of G by H. In the notation of the form $\{x_1, \ldots, x_n\}$ we always suppose $x_1, \ldots, x_n$ to be pairwise different.

J. Mycielski and W. Sierpiński [2] made the following conjecture: Let

(1) $$\{a_1G_1, a_2G_2, \ldots, a_kG_k\}$$

be a coset decomposition of an Abelian group G (i.e., the elements of (1) are pairwise disjoint cosets of G and its set-theoretical union is G) and let $n = [G:G_1]$ be finite. If

(2) $$n = \prod_{i=1}^{r} p_i^{a_i}$$

is the standard form of n, then

(3) $$k \geqslant 1 + \sum_{i=1} a_i(p_i - 1) .$$

Š. Znám made a similar conjecture in which G is the additive group of integers and the condition $n = [G:G_1]$ is replaced by $n = [G:\bigcap_{i=1}^{k} G_i]$, but the proof was not published.

Theorem 1 of the present paper is a generalization of both conjectures above.

For every (non-zero) natural n with the standard form (2) put

$$\mathcal{F}(n) = \sum_{i=1}^{r} a_i(p_i - 1)\,. \tag{4}$$

Lemma 1. *For arbitrary non-zero naturals m, n*

$$\mathcal{F}(mn) = \mathcal{F}(m) + \mathcal{F}(n)\,, \tag{5}$$

$$\mathcal{F}(n) < n\,. \tag{6}$$

Proof. (5) is obviously true. If n is a prime or $n = 1$, then $\mathcal{F}(n) = n-1 < n$. Let (6) hold for all n which are products of at most k primes, and let m be a product of $k+1$ primes. Suppose $m = pn$ where p is a prime. Then

$$\mathcal{F}(m) = \mathcal{F}(p) + \mathcal{F}(n) \leqslant (p-1) + (n-1) \leqslant (p-1)n + (n-1) < m\,, \quad \text{q.e.d.}$$

Definition. Let G be a (not necessary Abelian) group. (1) will be called an *invariant coset decomposition* (ICD) of the group G if all the groups $G_1, G_2, \ldots, G_k$ are invariant subgroups of G and if every element of G belongs to exactly one element of (1).

The main result of this paper is:

Theorem 1. *Let G be a group and* (1) *its* ICD. *Then*

$$k \geqslant 1 + \mathcal{F}\Big(\Big[G : \bigcap_{i=1}^{k} G_i\Big]\Big)\,. \tag{7}$$

Remark. In the following we show the finiteness of $[G : \bigcap_{i=1}^{k} G_i]$. Theorem 1 is obviously a generalization of Znám's conjecture. It is also a generalization of Mycielski's conjecture because every coset decomposition of an Abelian group G is its ICD, and $F([G : \bigcap_{i=1}^{k} G_i]) \geqslant F([G : G_1])$.

Lemma 2. *Let G be a group and a_1G_1, a_2G_2 its cosets. Then either $a_1G_1 \cap a_2G_2 = \emptyset$ or $a_1G_1 \cap a_2G_2$ is a coset by $G_1 \cap G_2$.*

Proof. Suppose $a_1G_1 \cap a_2G_2 \neq \emptyset$ and $b \in a_1G_1 \cap a_2G_2$. Then $a_1G_1 = bG_1$, $a_2G_2 = bG_2$ and hence $a_1G_1 \cap a_2G_2 = b(G_1 \cap G_2)$, q.e.d.

Lemma 3. *If G is a group, $x \in G$ and* (1) *is an* ICD *of G, then*

$$\{(xa_1)G_1, (xa_2)G_2, \ldots, (xa_k)G_k\} \tag{8}$$

is also an ICD *of the group G.*

The proof is obvious.

Lemma 4. *Let* (1) *be an* ICD *of a group G. Let F be an invariant subgroup of G contained in $\bigcap_{i=1}^{k} G_i$. Write $\bar{G} = G/F$, $\bar{G}_i = G_i/F$ and $\bar{a}_i = a_iF$ for every $i = 1, 2, \ldots, k$. Then*

$$\{\bar{a}_1\bar{G}_1, \bar{a}_2\bar{G}_2, \ldots, \bar{a}_k\bar{G}_k\} \tag{9}$$

is an ICD *of the group $\bar{G}$ and $[\bar{G} : \bigcap_{i=1}^{k} \bar{G}_i] = [G : \bigcap_{i=1}^{k} G_i]$.*

Proof. By Theorem 2.3.2 and Theorem 2.3.4 of [1] $\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_k$ are invariant subgroups of $\bar{G}$. Denote by φ the canonical homomorphism from G onto $\bar{G}$. Since, for every $i = 1, 2, \ldots, k$, $\varphi^{-1}(\bar{a}_i\bar{G}_i) = a_iG_i$ and (1) is a partition of G, (9) is a partition of $\bar{G}$. Since $\varphi^{-1}(\bigcap_{i=1}^{k} \bar{G}_i) = \bigcap_{i=1}^{k} G_i$, we have $[\bar{G} : \bigcap_{i=1}^{k} G_i] = [G : \bigcap_{i=1}^{k} G_i]$.

Lemma 5. *Let G be a group and* (1) *its* ICD. *Then $[G : \bigcap_{i=1}^{k} G_i] < \infty$ and $[G : G_i] < \infty$ for every $i = 1, 2, \ldots, k$.*

Proof. Since every coset by $\bigcap_{i=1}^{k} G_i$ is an intersection of some cosets by G_i, we have $[G : \bigcap_{i=1}^{k} G_i] \leqslant \prod_{i=1}^{k} [G : G_i]$. Hence it is sufficient to prove $[G : G_i] < \infty$ for $i = 1, 2, \ldots, k$. Let k be the smallest natural number for which there exists a group G and an ICD (1) of G such that $[G : G_r]$ is infinite for some $r = 1, 2, \ldots, k$.

Suppose at first that $[G_i : G_i \cap G_j]$ is infinite for some $i, j = 1, 2, \ldots, k$. By Lemma 3 without loss of generality $a_j = e$ can be assumed. Then the non-empty elements of the sequence

$$G_i \cap a_1G_1,\; G_i \cap a_2G_2,\; \ldots,\; G_i \cap a_kG_k \tag{10}$$

form an ICD of G_i with less than k elements (because $G_i \cap a_iG_i = \emptyset$) and with at least one infinite index $[G_i : G_i \cap G_j]$. This contradiction implies that all the indices $[G_i : G_i \cap G_j]$ are finite.

Denote $\bigcap_{i=1}^{k} G_i$ by F. For every $s = 1, 2, \ldots, k$ we have

$$[G_s : F] = \Big[G_s : \bigcap_{i=1}^{k} G_s \cap G_i\Big] \leqslant \prod_{i=1}^{k} [G_s : (G_s \cap G_i)]$$

and hence all the indices $[G_s : F]$ are finite. Every coset a_iG_i consists of $[G_i : F]$ cosets by F and hence the group G consists of $[G : F] = \sum_{i=1}^{k} [G_i : F]$ cosets by F. However, $[G : G_i] \leqslant [G : F]$ for every $i = 1, 2, \ldots, k$, and hence all the indices $[G : G_i]$ are finite, which is a contradiction.

Remark. Lemma 5 is a corollary of Theorem 1 of [4]. That paper has not been published yet.

Lemma 6. *Let G be a group, let (1) be ICD of G and let H be a (proper) maximal invariant subgroup of G. Then*

a) *every element of (1) either is contained in some member of G/H or has a non-empty intersection with every element of G/H.*

b) *if an element of (1) is contained in some element of G/H, then every element of G/H contains a number of (1).*

Proof. a) It is sufficient to consider a_1G_1, and without loss of generality $a_1 = e$ can be supposed. Consider the set-theoretical union of all members of G/H which have a non-empty intersection with G_1, and denote it by F. Obviously $F = G_1H = \{gh \mid g \in G_1, h \in H\}$. By [1], Theorem 2.3.3, F is an invariant subgroup of G. Since $H \subseteq F \subseteq G$, we have either $F = H$ or $F = G$. If $F = H$, i.e. $G_1H = H$, then $G_1 \subseteq H$. Let $G_1H = G$ and let x be an arbitrary element of G. There exist $y \in G_1$, $z \in H$ such that $x = yz$. Hence $G_1 \cap xH = G_1 \cap yzH = yG_1 \cap yH = y(G_1 \cap H) \neq \emptyset$, q.e.d.

b) Let an element of (1) be contained in some element of G/H. Denote $\bigcap_{i=1}^{k} G_i$ by F. Obviously $F \subseteq H$ holds. By Lemma 5, $[G:F]$ is finite. Suppose that $\{a_1G_1, a_2G_2, \ldots, a_rG_r\}$ is the set of all elements of (1) which have a non-empty intersection with every member of G/H. Every intersection $a_iG_i \cap xH$ $(i = 1, 2, \ldots, r,\ x \in G)$ consists of $[H \cap G_i:F]$ elements of G/F. Hence $(\bigcup_{i=1}^{r} a_iG_i) \cap xH$ consists of $\sum_{i=1}^{r} [H \cap G_i:F]$ elements of G/H, and then $xH - (\bigcup_{i=1}^{r} a_iG_i)$ consists of $[H:F] - \sum_{i=1}^{r} [H \cap G_i:F]$ elements of G/F. This number does not depend on x. Since it is non-zero for some $x \in G$, it is non-zero for every $x \in G$, i.e. every set $xH - (\bigcup_{i=1}^{r} a_iG_i)$ contains at least one coset by F (as a subset). This coset must be contained in some a_jG_j, $j > r$. By a), a_jG_j is a subset of xH, q.e.d.

Proof of Theorem 1. In the sequel, a "group" means an invariant subgroup of G and a "coset" means a coset of G by an invariant subgroup of G. By Lemma 4 and Lemma 5 it is sufficient to consider the case $|G| < \infty$ and $|\bigcap_{i=1}^{k} G_i| = 1$. Now we can prove Theorem 1 by induction with respect to $|G|$.

If $|G| = 1$ then $k = 1$, $\bigcap_{i=1}^{k} G_i = G$, and (7) obviously holds.

Let $|G| > 1$. Denote by $\mathfrak{M}$ the set of all such non-empty subsets M of $\{1, 2, \ldots, k\}$ that there exists a partition

$$\{X_1, X_2, \ldots, X_m\} \tag{11}$$

of the group G so that every element of (1) is contained in a member of (11), every coset by $\bigcap_{i \in M} G_i$ is also contained in a member of (11) and

$$m \geq 1 + \mathcal{F}([G:\bigcap_{i \in M} G_i]). \tag{12}$$

Suppose now $\{1, 2, \ldots, k\} \in \mathfrak{M}$. Then there exists such a partition (11) of G that $m \geq 1 + \mathcal{F}([G:\bigcap_{i=1}^{k} G_i])$. However, $k \geq m$ and hence (7) holds.

Therefore to finish the proof of Theorem 1 it suffices to show that $\{1, 2, \ldots, k\} \in \mathfrak{M}$.

Proposition A. *The set $\mathfrak{M}$ is non-empty.*

Without loss of generality $a_1 = e$ can be put. Since G_1 is a proper subgroup of G, there exists a maximal proper subgroup H of G containing G_1. The non-empty members of the sequence

$$H \cap a_1G_1,\ H \cap a_2G_2,\ \ldots,\ H \cap a_kG_k \tag{13}$$

form an ICD of the group H because $H \cap G_1, H \cap G_2, \ldots, H \cap G_k$ are invariant subgroups of H. We may suppose that

$$H \cap a_1G_1,\ H \cap a_2G_2,\ \ldots,\ H \cap a_sG_s \tag{14}$$

are non-empty and that the other members of (13) are empty. We have $|H| < |G|$ and hence, by the induction hypothesis,

$$s \geq 1 + \mathcal{F}\big([H:\bigcap_{i=1}^{s} (H \cap G_i)]\big) = 1 + \mathcal{F}([H:\bigcap_{i=1}^{s} G_i]). \tag{15}$$

Let $[G:H] = h$ and $G/H = \{H^1, H^2, \ldots, H^h\}$, $H^1 = H$. Consider the set

$$\{a_1G_1, a_2G_2, \ldots, a_sG_s, H^2 - \bigcup_{i=1}^{s} a_iG_i, \ldots, H^h - \bigcup_{i=1}^{s} a_iG_i\}. \tag{16}$$

The elements of (16) are obviously pairwise disjoint and their union is the set G. Clearly $a_1G_1 \subseteq H$ and hence by Lemma 6 every coset H^j, $j = 2, \ldots, h$, also contains an element of (1). Therefore the sets $H^j - \bigcup_{i=1}^{s} a_iG_i$, $j = 2, \ldots, k$, are non-empty, and hence (16) is a partition of the group G. By Lemma 6 every member of (1) either has a non-empty intersection with every element of G/H or is a subset of some H^j, $j = 1, 2, \ldots, h$. Hence every element of (1) is contained in an element of (16). It is easy to see that every coset by $\bigcap_{i \in M} G_i$ is also contained in some element of (16). To prove $\{1, 2, \ldots, s\} \in \mathfrak{M}$ it remains to verify that $s + h - 1 \geq 1 + \mathcal{F}([G:\bigcap_{i=1}^{s} G_i])$.

By Lemma 1 we have

$$s+h-1 \geq 1+\mathcal{F}([H:\bigcap_{i=1}^{s} G_i])+[G:H]-1 \geq 1+\mathcal{F}([H:\bigcap_{i=1}^{s} G_i])+\mathcal{F}([G:H])$$
$$= 1+\mathcal{F}([G:H][H:\bigcap_{i=1}^{s} G_i]) = 1+\mathcal{F}([G:\bigcap_{i=1}^{s} G_i]), \quad \text{q.e.d.}$$

PROPOSITION B. *No proper subset of* $\{1, 2, \ldots, k\}$ *is a maximal element of* $\mathfrak{M}$.

Assume on the contrary that $M \neq \{1, 2, \ldots, k\}$ is a maximal element of $\mathfrak{M}$. Without loss of generality $M = \{s, s+1, \ldots, k\}$ and $a_1 = e$ can be assumed. Let (11) be a partition of G such that every element of (1) is contained in some element of (11), every coset by $\bigcap_{i\in M} G_i$ is contained in some element of (11) and let (12) hold. Put $F = \bigcap_{i\in M} G_i$. Obviously $F \cap a_iG_i = \emptyset$ for $i \in M$ and hence the non-empty members of the sequence

$$(17) \qquad F\cap a_1G_1,\ F\cap a_2G_2,\ \ldots,\ F\cap a_{s-1}G_{s-1}$$

form a partition of the group F. We may assume that

$$(18) \qquad F\cap a_1G_1,\ F\cap a_2G_2,\ \ldots,\ F\cap a_rG_r$$

are non-empty and that the other members of (17) are empty. Clearly $|F| < |G|$ and hence by Lemma 2 and the inductive hypothesis

$$(19) \qquad r \geq 1+\mathcal{F}([F:\bigcap_{i=1}^{r} (F\cap G_i)]).$$

The group F is contained in some member of (11); suppose $F \subseteq X_m$. Then every coset a_iG_i, $i = 1, 2, \ldots, r$, is also contained in X_m because it is a subset of some member of (11) and has a non-empty intersection with F. Consider now the set

$$(20) \qquad \{a_1G_1, a_2G_2, \ldots, a_rG_r, X_1, X_2, \ldots, X_{m-1}, X_m - \bigcup_{i=1}^{r} a_iG_i\}.$$

Let $N = M \cup \{1, 2, \ldots, r\}$. Every coset by $\bigcap_{i\in N} G_i = (\bigcap_{i\in M} G_i)\cap F$ is contained in some element of (20). Every element of (1) is also contained in some element of (20). The number of non-empty elements of (20) is at least $r+m-1$ and we have

$$r+m-1 \geq 1+\mathcal{F}([F:\bigcap_{i=1}^{r} (F\cap G_i)])+\mathcal{F}([G:F]) = 1+\mathcal{F}([G:\bigcap_{i=1}^{r} (F\cap G_i)])$$
$$= 1+\mathcal{F}([G:(\bigcap_{i=1}^{r} G_i \cap \bigcap_{i\in M} G_i)]) = 1+\mathcal{F}([G:\bigcap_{i\in N} G_i]).$$

Hence $N \in \mathfrak{M}$, contrary to the maximality of M.

Propositions A and B imply $\{1, 2, \ldots, k\} \in \mathfrak{M}$. Therefore (7) holds, q.e.d.

Actually we have proved more than Theorem 1. For an arbitrary group G and its invariant subgroup F such that $[G:F] < \infty$ define

$$(21) \qquad \mathcal{G}(G, F) = \sum_{i=1}^{s} ([H_i:H_{i-1}]-1),$$

where $H_0 = F, H_1, \ldots, H_s = G$ is a maximal chain of subgroups of G such that H_i is invariant in H_{i+1} for all $i = 0, 1, \ldots, s-1$. By [1], Theorem 8.4.4, $\mathcal{G}(G, F)$ does not depend on the choice of the chain $H_0, H_1, \ldots, H_s$. The following theorem can be proved in the same way as Theorem 1:

THEOREM 2. *Let G be a group and* (1) *its* ICD. *Then*

$$(22) \qquad k \geq 1+\mathcal{G}(G, \bigcap_{i=1}^{k} G_i).$$

If the group G is Abelian, then the right sides of (7) and (22) are equal. However, there are (non-Abelian) groups and their ICD (1) for which the right side of (22) is greater than that of (7).

Remark. Theorem 1 does not hold if we replace the words "invariant coset decomposition" by the "left coset decomposition" or "right coset decomposition". We give some examples. Let $G = S_3$ be the symmetric group of degree 3, S_2 one of its two-elements subgroups and A_1, A_2, A_3 (resp. B_1, B_2, B_3) right (resp. left) cosets of S_3 by S_2. If we consider the set $\{A_1, A_2, A_3\}$ (resp. $\{B_1, B_2, B_3\}$) as a left (resp. right) coset decomposition $\{a_1G_1, a_2G_2, a_3G_3\}$ (resp. $\{G_1a_1, G_2a_2, G_3a_3\}$) of the group G, then $G_1\cap G_2\cap G_3 = \{e\}$. We have $k = 3$ and $1+\mathcal{F}([G:\{e\}]) = 1+\mathcal{F}(6) = 4$. Hence (7) does not hold.

We give one more example. If we consider

$$(23) \qquad \{A_1\times B_1, A_2\times B_1, A_3\times B_1, S_3\times B_2, S_3\times B_3\}$$

as a left (resp. right) coset decomposition $\{a_1G_1, \ldots, a_5G_5\}$ (resp. $\{G_1a_1, \ldots, G_5a_5\}$) of the group $G = S_3\times S_3$, then $G_1\cap G_2\cap \ldots \cap G_5 = S_2\times\{e\}$ (resp. $\{e\}\times S_2$). In both cases $k = 5$ and the right side of (7) is $1+\mathcal{F}(18) = 6$. Hence (7) holds neither if (23) is considered as a left coset decomposition nor if it is considered as a right coset decomposition.

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Reçu par la Rédaction le 6. 2. 1973

A new definition of the circle by the use of bisectors

by

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Abstract. The subset $B(x, y) = \{q \in X\colon \varrho(x, q) = \varrho(y, q)\}$ in a metric space (X, ϱ) is called the *bisector* of a pair x, y. It is known that any connected metric space in which each bisector is a unique point, is topologically an interval of the real line R.

If each bisector consists of exactly two points, then X has DBP property.

The question whether every connected metric space with DBP is homeomorphic to the one-sphere S^1 is still open.

A metric space is segment-convex if for each pair p, r of its points it contains an arc joining p to r which is isometric to a line segment.

We show that any segment-convex metric space with DBP is isometric to a metric one-sphere with its natural geodesic metric.

1. Introduction. For any pair of distinct points x and y in a non-trivial metric space (X, ϱ) the subset $B(x, y) = \{q \in X|\ \varrho(x, q) = \varrho(y, q)\}$ will be called the *bisector* ([3], see also [1] where it is called the midset). If each bisector is a unique point, then X has [1] the *unique bisector property* (UBP). If each bisector consists of exactly two points, then X has the *double bisector property* (DBP).

It is known [1] that any connected metric space with UBP is homeomorphic to a subset of the real line R, and is tnerefore an interval.

The question whether every connected metric space with DBP is homeomorphic to the one-sphere S^1 is still open.

The aim of the present paper is the following result: If (X, ϱ) is a segment-convex metric space with DBP, then X is isometric to a metric one-sphere.

The proof will be based on the following three auxiliary propositions:

Let a_1 and a_2 be two distinct points of X, and let $B(a_1, a_2) = \{x_1, x_2\}$, then

1° $L_1 = \overline{x_1 a_1} \cup \overline{a_1 x_2}$ and $L_2 = \overline{x_1 a_2} \cup \overline{a_2 x_2}$ are two simple arcs joining x_1 to x_2, and $L_1 \cap L_2 = B(a_1, a_2)$.

2° More precisely, L_1 and L_2 are two metric segments joining x_1 to x_2.

3° $L_1 \cup L_2 = X$.