

On a Generalization of Normal, Almost Normal and Mildly Normal Spaces–I

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ABSTRACT. In this paper, we introduce the notions of δp -normal spaces, almost δp -normal spaces, mildly δp -normal spaces, $g\delta p$ -closed sets and the forms of generalized δ -preclosed functions. We obtain characterizations and the relationships of such normal spaces, properties of the forms of generalized δ -preclosed functions and preservation theorems.

1. INTRODUCTION

Levine [8] initiated the investigation of so-called g -closed sets in topological spaces, since then many modifications of g -closed sets were defined and investigated by a large number of topologists [1, 5, 14]. In 1996, Maki et al. [9] introduced the concepts of gp -closed sets. On the other hand, the notions of p -normal spaces, almost p -normal spaces and mildly p -normal spaces were introduced by Paul and Bhattacharyya [15]; Navalagi [11]; Navalagi [11], respectively.

In this paper we introduce five sections. In the first section, second section and third section, we introduce the notions of δp -normal spaces, almost δp -normal spaces and mildly δp -normal spaces which are the generalized forms of p -normal spaces, almost p -normal spaces and mildly p -normal spaces, respectively. Also, we obtain characterizations and properties of such generalizations of normal spaces. In fourth section, we introduce and study the concepts of $g\delta p$ -closed sets and some new forms of generalized δ -preclosed functions. In the last section, we obtain the relationships between δp -normal spaces and generalized δ -preclosed functions.

2. PRELIMINARIES

In this paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \tau) \rightarrow (Y, \sigma)$ (or simply $f : X \rightarrow Y$) denotes a function f of a space (X, τ) into a space (Y, σ) .

Let A be a subset of a space X . The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

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Definition 1. A subset A of a space X is said to be:

- (1) regular open [19] if $A = \text{int}(cl(A))$,
- (2) α -open [12] if $A \subset \text{int}(cl(\text{int}(A)))$,
- (3) preopen [10] or nearly open [7] if $A \subset \text{int}(cl(A))$.

The complement of an α -open (resp. preopen, regular open) set is called α -closed [12] (resp. preclosed [10], regular closed [19]). The intersection of all α -closed (resp. preclosed) sets containing A is called the α -closure (resp. preclosure) of A and is denoted by $\alpha\text{-}cl(A)$ (resp. $pcl(A)$). The α -interior (resp. preinterior) of A , denoted by $\alpha\text{-}int(A)$ (resp. $pint(A)$) is defined to be the union of all α -open (resp. preopen) sets contained in A .

The δ -interior [20] of a subset A of X is defined by the union of all regular open sets of X contained in A and is denoted by $\delta\text{-}int(A)$. A subset A is called δ -open [20] if $A = \delta\text{-}int(A)$, i.e. a set is δ -open if it is the union of regular open sets. The complement of a δ -open set is called δ -closed. Alternatively, a set A of (X, τ) is called δ -closed [20] if $A = \delta\text{-}cl(A)$, where $\delta\text{-}cl(A) = \{x \in X : A \cap \text{int}(cl(U)) \neq \emptyset, U \in \tau \text{ and } x \in U\}$. A subset A of a space X is said to be δ -preopen [16] if $A \subset \text{int}(\delta\text{-}cl(A))$.

The complement of a δ -preopen set is said to be δ -preclosed. The intersection of all δ -preclosed sets of X containing A is called the δ -preclosure [16] of A and is denoted by $\delta\text{-}pcl(A)$. The union of all δ -preopen sets of X contained in A is called the δ -preinterior of A and is denoted by $\delta\text{-}pint(A)$ [16]. A subset U of X is called a δ -preneighborhood of a point $x \in X$ if there exists a δ -preopen set V such that $x \in V \subset U$. Note that $\delta\text{-}pcl(A) = A \cup cl(\delta\text{-}int(A))$ and $\delta\text{-}pint(A) = A \cap \text{int}(\delta\text{-}cl(A))$.

The family of all δ -preopen (δ -preclosed, α -open, regular open, regular closed, δ -open, δ -closed, preopen) sets of a space X is denoted by $\delta PO(X)$ (resp. $\delta PC(X)$, $\alpha O(X)$, $RO(X)$, $RC(X)$, $\delta O(X)$, $\delta C(X)$, $PO(X)$). The family of all δ -preopen sets containing a point x is denoted by $\delta PO(X, x)$. It is shown in [13] that $\alpha O(X)$ is a topology and it is stronger than given topology on X .

Definition 2. A space X is said to be prenormal [13] or p -normal [15] if for any pair of disjoint closed sets A and B , there exist disjoint preopen sets U and V such that $A \subset U$ and $B \subset V$.

Definition 3. A space X is said to be almost normal [17] (resp. almost p -normal [11]) if for each closed set A and each regular closed set B such that $A \cap B = \emptyset$, there exist disjoint open (resp. preopen) sets U and V such that $A \subset U$ and $B \subset V$.

Definition 4. A space X is said to be mildly normal [18] (resp. mildly p -normal [11]) if for every pair of disjoint regular closed sets A and B of X , there exist disjoint open (resp. preopen) sets U and V such that $A \subset U$ and $B \subset V$.

Definition 5. A function $f : X \rightarrow Y$ is called

- (1) R -map [4] if $f^{-1}(V)$ is regular open in X for every regular open set V of Y ,
- (2) completely continuous [2] if $f^{-1}(V)$ is regular open in X for every open set V of Y .

3. δp -NORMAL SPACES

Definition 6. A space X is said to be δp -normal if for any pair of disjoint closed sets A and B , there exist disjoint δ -preopen sets U and V such that $A \subset U$ and $B \subset V$.

Remark 1. The following implication holds for a topological space (X, τ) :

$$\text{normal} \Rightarrow p\text{-normal} \Rightarrow \delta p\text{-normal}$$

None of these implications is reversible as shown by the following example.

Example 1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the space (X, τ) is δp -normal but not p -normal.

For the other implication the example can be seen in [13, 15].

Theorem 1. For a space X the following are equivalent:

- (1) X is δp -normal,
- (2) for every pair of open sets U and V whose union is X , there exist δ -preclosed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,
- (3) for every closed set F and every open set D containing F , there exists a δ -preopen set U such that $C \subset U \subset \delta\text{-pcl}(U) \subset D$.

Proof. (1) \implies (2) : Let U and V be a pair of open sets in a δp -normal space X such that $X = U \cup V$. Then $X \setminus U$, $X \setminus V$ are disjoint closed sets. Since X is δp -normal, there exist disjoint δ -preopen sets U_1 and V_1 such that $X \setminus U \subset U_1$ and $X \setminus V \subset V_1$. Let $A = X \setminus U_1$, $B = X \setminus V_1$. Then A and B are δ -preclosed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2) \implies (3) : Let C be a closed set and D be an open set containing C . Then $X \setminus C$ and D are open sets whose union is X . Then by (2), there exist δ -preclosed sets M_1 and M_2 such that $M_1 \subset X \setminus C$ and $M_2 \subset D$ and $M_1 \cup M_2 = X$. Then $C \subset X \setminus M_1$, $X \setminus D \subset X \setminus M_2$ and $(X \setminus M_1) \cap (X \setminus M_2) = \emptyset$. Let

$$U = X \setminus M_1 \text{ and } V = X \setminus M_2.$$

Then U and V are disjoint δ -preopen sets such that $C \subset U \subset X \setminus V \subset D$. As $X \setminus V$ is δ -preclosed set, we have $\delta\text{-pcl}(U) \subset X \setminus V$ and $C \subset U \subset \delta\text{-pcl}(U) \subset D$.

(3) \implies (1) : Let C_1 and C_2 be any two disjoint closed sets of X . Put $D = X \setminus C_2$, then $C_2 \cap D = \emptyset$. $C_1 \subset D$ where D is an open set. Then by (3), there exists a δ -preopen set U of X such that

$$C_1 \subset U \subset \delta\text{-pcl}(U) \subset D.$$

It follows that

$$C_2 \subset X \setminus \delta - pcl(U) = V,$$

say, then V is δ -preopen and $U \cap V = \emptyset$. Hence, C_1 and C_2 are separated by δ -preopen sets U and V . Therefore X is δp -normal. \square

Definition 7. A function $f : X \rightarrow Y$ is called strongly δ -preopen if $f(U) \in \delta PO(Y)$ for each $U \in \delta PO(X)$.

Definition 8. A function $f : X \rightarrow Y$ is called strongly δ -preclosed if $f(U) \in \delta PC(Y)$ for each $U \in \delta PC(X)$.

Theorem 2. A function $f : X \rightarrow Y$ is strongly δ -preclosed if and only if for each subset B in Y and for each δ -preopen set U in X containing $f^{-1}(B)$, there exists a δ -preopen set V containing B such that $f^{-1}(V) \subset U$.

Proof. (\Rightarrow): Suppose that f is strongly δ -preclosed. Let B be a subset of Y and $U \in \delta PO(X)$ containing $f^{-1}(B)$. Put $V = Y \setminus f(X \setminus U)$, then V is a δ -preopen set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

(\Leftarrow): Let K be any δ -preclosed set of X . Then $f^{-1}(Y \setminus f(K)) \subset X \setminus K$ and $X \setminus K \in \delta PO(X)$. There exists a δ -preopen set V of Y such that $Y \setminus f(K) \subset V$ and $f^{-1}(V) \subset X \setminus K$. Therefore, we have $f(K) \supset Y \setminus V$ and $K \subset f^{-1}(Y \setminus V)$. Hence, we obtain $f(K) = Y \setminus V$ and $f(K)$ is δ -preclosed in Y . This shows that f is strongly δ -preclosed. \square

Theorem 3. If $f : X \rightarrow Y$ is a strongly δ -preclosed continuous function from a δp -normal space X onto a space Y , then Y is δp -normal.

Proof. Let K_1 and K_2 be disjoint closed sets in Y . Then $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are closed sets. Since X is δp -normal, then there exist disjoint δ -preopen sets U and V such that $f^{-1}(K_1) \subset U$ and $f^{-1}(K_2) \subset V$. By the previous theorem, there exist δ -preopen sets A and B such that $K_1 \subset A$, $K_2 \subset B$, $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Also, A and B are disjoint. Thus, Y is δp -normal. \square

Definition 9. A function $f : X \rightarrow Y$ is said to be almost δ -preirresolute if for each x in X and each δ -preneighborhood V of $f(x)$, $\delta - pcl(f^{-1}(V))$ is a δ -preneighborhood of x .

Lemma 1. Let $f : X \rightarrow Y$ be a function. Then f is almost δ -preirresolute if and only if $f^{-1}(V) \subset \delta - pint(\delta - pcl(f^{-1}(V)))$ for every $V \in \delta PO(Y)$.

Theorem 4. A function $f : X \rightarrow Y$ is almost δ -preirresolute if and only if $f(\delta - pcl(U)) \subset \delta - pcl(f(U))$ for every $U \in \delta PO(X)$.

Proof. (\Rightarrow): Let $U \in \delta PO(X)$. Suppose $y \notin \delta - pcl(f(U))$. Then there exists $V \in \delta PO(Y, y)$ such that $V \cap f(U) = \emptyset$. Hence $f^{-1}(V) \cap U = \emptyset$. Since $U \in \delta PO(X)$, we have $\delta - pint(\delta - pcl(f^{-1}(V))) \cap \delta - pcl(U) = \emptyset$. Then by Lemma 1, $f^{-1}(V) \cap \delta - pcl(U) = \emptyset$ and hence $V \cap f(\delta - pcl(U)) = \emptyset$. This implies that $y \notin f(\delta - pcl(U))$.

(\Leftarrow) : If $V \in \delta PO(Y)$, then $M = X \setminus \delta - pcl(f^{-1}(V)) \in \delta PO(X)$. By hypothesis, $f(\delta - pcl(M)) \subset \delta - pcl(f(M))$ and hence $X \setminus \delta - pint(\delta - pcl(f^{-1}(V))) = \delta - pcl(M) \subset f^{-1}(\delta - pcl(f(M))) \subset f^{-1}(\delta - pcl(f(X \setminus f^{-1}(V)))) \subset f^{-1}(\delta - pcl(Y \setminus V)) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Therefore, $f^{-1}(V) \subset \delta - pint(\delta - pcl(f^{-1}(V)))$. By Lemma 1, f is almost δ -preirresolute. \square

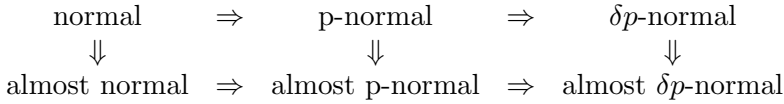
Theorem 5. *If $f : X \rightarrow Y$ is a strongly δ -preopen continuous almost δ -preirresolute function from a δp -normal space X onto a space Y , then Y is δp -normal.*

Proof. Let A be a closed subset of Y and B be an open set containing A . Then by continuity of f , $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is δp -normal, there exists a δ -preopen set U in X such that $f^{-1}(A) \subset U \subset \delta - pcl(U) \subset f^{-1}(B)$ by Theorem 1. Then, $f(f^{-1}(A)) \subset f(U) \subset f(\delta - pcl(U)) \subset f(f^{-1}(B))$. Since f is strongly δ -preopen almost δ -preirresolute surjection, we obtain $A \subset f(U) \subset \delta - pcl(f(U)) \subset B$. Then again by Theorem 1 the space Y is δp -normal. \square

4. ALMOST δp -NORMAL SPACES

Definition 10. A space X is said to be almost δp -normal if for each closed set A and each regular closed set B such that $A \cap B = \emptyset$, there exist disjoint δ -preopen sets U and V such that $A \subset U$ and $B \subset V$.

Remark 2. The following diagram holds for a topological space (X, τ) :



None of these implications is reversible as shown by the following example.

Example 2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the space (X, τ) is almost δp -normal but not almost p -normal.

For the other implication the example can be seen in the related papers.

Question: Clearly every δp -normal space is almost δp -normal. Does there exist an almost δp -normal space which is not δp -normal?

Theorem 6. *For a space X the following statements are equivalent:*

- (1) X is almost δp -normal,
- (2) For every pair of sets U and V , one of which is open and the other is regular open whose union is X , there exist δ -preclosed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,
- (3) For every closed set A and every regular open set B containing A , there exists a δ -preopen set V such that $A \subset V \subset \delta - pcl(V) \subset B$.

Proof. (1) \implies (2) : Let U be an open set and V be a regular open set in an almost δp -normal space X such that $U \cup V = X$. Then $(X \setminus U)$ is a closed set and $(X \setminus V)$ is a regular closed set with $(X \setminus U) \cap (X \setminus V) = \emptyset$. By almost δp -normality

of X , there exist disjoint δ -preopen sets U_1 and V_1 such that $X \setminus U \subset U_1$ and $X \setminus V \subset V_1$. Let $A = X \setminus U_1$ and $B = X \setminus V_1$. Then A and B are δ -preclosed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2) \implies (3) : Let A be a closed set and B be a regular open set containing A . Then $X \setminus A$ is open and B is regular open sets whose union is X . Then by (2), there exist δ -preclosed sets M_1 and M_2 such that $M_1 \subset X \setminus A$ and $M_2 \subset B$ and $M_1 \cup M_2 = X$. Then $A \subset X \setminus M_1$, $X \setminus B \subset X \setminus M_2$ and $(X \setminus M_1) \cap (X \setminus M_2) = \emptyset$. Let $U = X \setminus M_1$ and $V = X \setminus M_2$. Then U and V are disjoint δ -preopen sets such that $A \subset U \subset X \setminus V \subset B$. As $X \setminus V$ is δ -preclosed set, we have $\delta\text{-}pcl(U) \subset X \setminus V$ and $A \subset U \subset \delta\text{-}pcl(U) \subset B$.

(3) \implies (1) : Let A_1 and A_2 be any two disjoint closed and regular closed sets, respectively. Put $D = X \setminus A_2$, then $A_2 \cap D = \emptyset$. $A_1 \subset D$ where D is a regular open set. Then by (3), there exists a δ -preopen set U of X such that $A_1 \subset U \subset \delta\text{-}pcl(U) \subset D$. It follows that $A_2 \subset X \setminus \delta\text{-}pcl(U) = V$, say, then V is δ -preopen and $U \cap V = \emptyset$. Hence, A_1 and A_2 are separated by δ -preopen sets U and V . Therefore X is almost δp -normal. \square

Theorem 7. *If $f : X \rightarrow Y$ is a continuous strongly δ -preopen R -map and almost δ -preirresolute surjection from an almost δp -normal space X onto a space Y , then Y is almost δp -normal.*

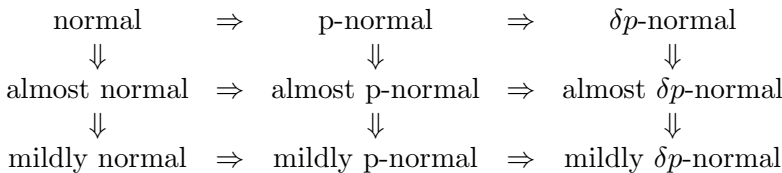
Proof. Similar to Theorem 5. \square

Corollary 1. *If $f : X \rightarrow Y$ is a completely continuous strongly δ -preopen and almost δ -preirresolute surjection from an almost δp -normal space X onto a space Y , then Y is almost δp -normal.*

5. MILDLY δp -NORMAL SPACES

Definition 11. A space X is said to be mildly δp -normal if for every pair of disjoint regular closed sets A and B of X , there exist disjoint δ -preopen sets U and V such that $A \subset U$ and $B \subset V$.

Remark 3. The following diagram holds for a topological space (X, τ) :



Question: Clearly every almost δp -normal space is mildly δp -normal. Does there exist a mildly δp -normal (resp. mildly δp -normal) space which is not mildly p -normal (resp. almost δp -normal)?

Theorem 8. *For a space X the following are equivalent:*

- (1) X is mildly δp -normal,

- (2) For every pair of regular open sets U and V whose union is X , there exist δ -preclosed sets G and H such that $G \subset U$, $H \subset V$ and $G \cup H = X$,
- (3) For any regular closed set A and every regular open set B containing A , there exists a δ -preopen set U such that $A \subset U \subset \delta - pcl(U) \subset B$,
- (4) For every pair of disjoint regular closed sets A and B , there exist δ -preopen sets U and V such that $A \subset U$, $B \subset V$ and $\delta - pcl(U) \cap \delta - pcl(V) = \emptyset$.

Proof. Similar to Theorem 1. □

Theorem 9. *If $f : X \rightarrow Y$ is an strongly δ -preopen R -map and almost δ -preirresolute function from a mildly δp -normal space X onto a space Y , then Y is mildly δp -normal.*

Proof. Let A be a regular closed set and B be a regular open set containing A . Then by R -map of f , $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$. Since X is mildly δp -normal, there exists a δ -preopen set V such that

$$f^{-1}(A) \subset V \subset \delta - pcl(V) \subset f^{-1}(B)$$

by Theorem 8. As f is strongly δ -preopen and an almost δ -preirresolute surjection, it follows that $f(V) \in \delta PO(Y)$ and $A \subset f(V) \subset \delta - pcl(f(V)) \subset B$. Hence Y is mildly δp -normal. □

Theorem 10. *If $f : X \rightarrow Y$ is R -map, strongly δ -preclosed function from a mildly δp -normal space X onto a space Y , then Y is mildly δp -normal.*

Proof. Similar to Theorem 3. □

6. $g\delta p$ -CLOSED SETS AND GENERALIZED FUNCTIONS

Definition 12. A subset A of a space (X, τ) is said to be g -closed [8] (resp. gp -closed [9]) if $cl(A) \subset U$ (resp. $p-cl(A) \subset U$) whenever $A \subset U$ and $U \in \tau$. The complement of g -closed (resp. gp -closed) set is said to be g -open (resp. gp -open).

Definition 13. A subset A of a space (X, τ) is said to be $g\delta p$ -closed if $\delta - pcl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$. The complement of $g\delta p$ -closed set is said to be $g\delta p$ -open.

Remark 4. The following diagram holds for any subset of a topological space (X, τ) :

$$\begin{array}{ccc}
 \delta\text{-preclosed} & \Rightarrow & g\delta p\text{-closed} \\
 \uparrow & & \uparrow \\
 \text{preclosed} & \Rightarrow & gp\text{-closed} \\
 \uparrow & & \uparrow \\
 \text{closed} & & g\text{-closed}
 \end{array}$$

None of these implications is reversible as shown by the following examples.

Example 3. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $\{a, d\}$ is $g\delta p$ -closed but it is not gp -closed. Let $\tau = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}\}$. Then the set $\{b, c\}$ is $g\delta p$ -closed but it is not δ -preclosed.

For the other implications the examples can be seen in [3, 9, 10, 16].

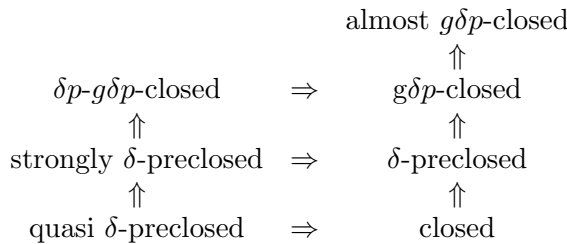
Definition 14. A function $f : X \rightarrow Y$ is said to be

- (1) δ -preclosed if $f(K)$ is δ -preclosed in Y for each closed set K of X ,
- (2) $g\delta p$ -closed if $f(K)$ is $g\delta p$ -closed in Y for each closed set K of X .

Definition 15. A function $f : X \rightarrow Y$ is said to be

- (1) quasi δ -preclosed if $f(K)$ is closed in Y for each $K \in \delta PC(X)$,
- (2) δp - $g\delta p$ -closed if $f(K)$ is $g\delta p$ -closed in Y for each $K \in \delta PC(X)$,
- (3) almost $g\delta p$ -closed if $f(K)$ is $g\delta p$ -closed in Y for each $K \in RC(X)$.

Remark 5. The following diagram holds for a function $f : (X, \tau) \rightarrow (Y, \sigma)$:



None of these implications is reversible as shown by the following examples.

Example 4. Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined as follows: $f(a) = a, f(b) = c, f(c) = c$ and $f(d) = d$. Then f is almost $g\delta p$ -closed but it is not $g\delta p$ -closed.

If we define the function $f : (X, \tau) \rightarrow (Y, \sigma)$ as follows: $f(a) = a, f(b) = a, f(c) = c$ and $f(d) = d$, then f is δ -preclosed but it is not closed. If we define the function f as follows: $f(a) = c, f(b) = a, f(c) = c$ and $f(d) = d$, then f is δ -preclosed but it is not strongly δ -preclosed. If we define the function f as an identity function, then f is closed and strongly δ -preclosed but it is not quasi δ -preclosed.

Example 5. Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. If we define the function $f : (X, \tau) \rightarrow (Y, \sigma)$ as follows: $f(a) = c, f(b) = d, f(c) = c$ and $f(d) = d$, then f is δp - $g\delta p$ -closed but it is not δp - δpg -closed. If we define the function f as follows: $f(a) = a, f(b) = a, f(c) = d$ and $f(d) = d$, then f is $g\delta p$ -closed but it is neither δpg -closed nor δp - $g\delta p$ -closed.

Definition 16. A function $f : X \rightarrow Y$ is said to be δp - $g\delta p$ -continuous if $f^{-1}(K)$ is $g\delta p$ -closed in X for every $K \in \delta PC(Y)$.

Theorem 11. A function $f : X \rightarrow Y$ is δp - $g\delta p$ -continuous if and only if $f^{-1}(V)$ is $g\delta p$ -open in X for every $V \in \delta PO(Y)$.

Theorem 12. If $f : X \rightarrow Y$ is closed δp - $g\delta p$ -continuous, then $f^{-1}(K)$ is $g\delta p$ -closed in X for each $g\delta p$ -closed set K of Y .

Proof. Let K be a $g\delta p$ -closed set of Y and U an open set of X containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then V is open in Y , $K \subset V$, and $f^{-1}(V) \subset U$. Therefore, we have $\delta\text{-}pcl(K) \subset V$ and hence

$$f^{-1}(K) \subset f^{-1}(\delta\text{-}pcl(K)) \subset f^{-1}(V) \subset U.$$

Since f is $\delta p\text{-}g\delta p$ -continuous, $f^{-1}(\delta\text{-}pcl(K))$ is $g\delta p$ -closed in X and hence $\delta\text{-}pcl(f^{-1}(K)) \subset \delta\text{-}pcl(f^{-1}(\delta\text{-}pcl(K))) \subset U$. This shows that $f^{-1}(K)$ is $g\delta p$ -closed in X . \square

Theorem 13. *A function $f : X \rightarrow Y$ is $\delta p\text{-}g\delta p$ -closed if and only if for each subset B of Y and each $U \in \delta PO(X)$ containing $f^{-1}(B)$, there exists a $g\delta p$ -open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Proof. (\Rightarrow) : Suppose that f is $\delta p\text{-}g\delta p$ -closed. Let B be a subset of Y and $U \in \delta PO(X)$ containing $f^{-1}(B)$. Put $V = Y \setminus f(X \setminus U)$, then V is a $g\delta p$ -open set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

(\Leftarrow) : Let K be any δ -preclosed set of X . Then $f^{-1}(Y \setminus f(K)) \subset X \setminus K$ and $X \setminus K \in \delta PO(X)$. There exists a $g\delta p$ -open set V of Y such that $Y \setminus f(K) \subset V$ and $f^{-1}(V) \subset X \setminus K$. Therefore, we have $f(K) \supset Y \setminus V$ and $K \subset f^{-1}(Y \setminus V)$. Hence, we obtain $f(K) = Y \setminus V$ and $f(K)$ is $g\delta p$ -closed in Y . This shows that f is $\delta p\text{-}g\delta p$ -closed. \square

Theorem 14. *If $f : X \rightarrow Y$ is continuous $\delta p\text{-}g\delta p$ -closed, then $f(H)$ is $g\delta p$ -closed in Y for each $g\delta p$ -closed set H of X .*

Proof. Let H be any $g\delta p$ -closed set of X and V an open set of Y containing $f(H)$. Since $f^{-1}(V)$ is an open set of X containing H , $\delta\text{-}pcl(H) \subset f^{-1}(V)$ and hence $f(\delta\text{-}pcl(H)) \subset V$. Since f is $\delta p\text{-}g\delta p$ -closed and $\delta\text{-}pcl(H) \in \delta PC(X)$, we have $\delta\text{-}pcl(f(H)) \subset \delta\text{-}pcl(f(\delta\text{-}pcl(H))) \subset V$. Therefore, $f(H)$ is $g\delta p$ -closed in Y . \square

Definition 17. A function $f : X \rightarrow Y$ is said to be δ -preirresolute [6] if $f^{-1}(V) \in \delta PO(X)$ for every $V \in \delta PO(Y)$.

Remark 6. A δ -preirresolute function is $\delta p\text{-}g\delta p$ -continuous but not conversely.

Example 6. Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined as follows: $f(a) = b$, $f(b) = d$, $f(c) = c$ and $f(d) = d$. Then f is $\delta p\text{-}g\delta p$ -continuous but it is not δ -preirresolute.

Corollary 2. *If $f : X \rightarrow Y$ is closed δ -preirresolute, then $f^{-1}(K)$ is $g\delta p$ -closed in X for each $g\delta p$ -closed set K of Y .*

Theorem 15. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Then*

- (1) *the composition $g \circ f : X \rightarrow Z$ is $\delta p\text{-}g\delta p$ -closed if f is $\delta p\text{-}g\delta p$ -closed and g is continuous $\delta p\text{-}g\delta p$ -closed,*
- (2) *the composition $g \circ f : X \rightarrow Z$ is $\delta p\text{-}g\delta p$ -closed if f is strongly δ -preclosed and g is $\delta p\text{-}g\delta p$ -closed,*
- (3) *the composition $g \circ f : X \rightarrow Z$ is $\delta p\text{-}g\delta p$ -closed if f is quasi δ -preclosed and g is $g\delta p$ -closed.*

Theorem 16. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and let the composition $gof : X \rightarrow Z$ be δp - $g\delta p$ -closed. Then, if f is an δ -preirresolute surjection, then g is δp - $g\delta p$ -closed.*

Proof. Let $K \in \delta PC(Y)$. Since f is δ -preirresolute and surjective, $f^{-1}(K) \in \delta PC(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Therefore, $g(K)$ is $g\delta p$ -closed in Z and hence g is δp - $g\delta p$ -closed. \square

Theorem 17. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and let the composition $gof : X \rightarrow Z$ be δp - $g\delta p$ -closed. Then, if g is a closed δp - $g\delta p$ -continuous injection, then f is δp - $g\delta p$ -closed.*

Proof. Let $H \in \delta PC(X)$. Then $(gof)(H)$ is $g\delta p$ -closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 12, $f(H)$ is $g\delta p$ -closed in Y and hence f is δp - $g\delta p$ -closed. \square

7. FURTHER CHARACTERIZATIONS OF δp -NORMAL SPACES

Lemma 2. *A subset A of a space X is $g\delta p$ -open if and only if $F \subset \delta\text{-pint}(A)$ whenever F is closed and $F \subset A$.*

Theorem 18. *For a topological space X , the following are equivalent:*

- (a) X is δp -normal,
- (b) for any pair of disjoint closed sets A and B of X , there exist disjoint $g\delta p$ -open sets U and V of X such that $A \subset U$ and $B \subset V$,
- (c) for each closed set A and each open set B containing A , there exists a $g\delta p$ -open set U such that $A \subset U \subset \delta\text{-pcl}(U) \subset B$,
- (d) for each closed set A and each g -open set B containing A , there exists a δ -preopen set U such that $A \subset U \subset \delta\text{-pcl}(U) \subset \text{int}(B)$,
- (e) for each closed set A and each g -open set B containing A , there exists a $g\delta p$ -open set G such that $A \subset G \subset \delta\text{-pcl}(G) \subset \text{int}(B)$,
- (f) for each g -closed set A and each open set B containing A , there exists a δ -preopen set U such that $\text{cl}(A) \subset U \subset \delta\text{-pcl}(U) \subset B$,
- (g) for each g -closed set A and each open set B containing A , there exists a $g\delta p$ -open set G such that $\text{cl}(A) \subset G \subset \delta\text{-pcl}(G) \subset B$.

Proof. (a) \Leftrightarrow (b) \Leftrightarrow (c) : Since every δ -preopen set is $g\delta p$ -open, it is obvious.

(d) \Rightarrow (e) and (f) \Rightarrow (g) \Rightarrow (c) : Since every closed (resp. open) set is g -closed (resp. g -open), it is obvious.

(c) \Rightarrow (d) : Let A be any closed subset of X and B be a g -open set containing A . We have $A \subset \text{int}(B)$. Then there exists a $g\delta p$ -open set G such that $A \subset G \subset \delta\text{-pcl}(G) \subset \text{int}(B)$. Since G is $g\delta p$ -open, by Lemma 2 $A \subset \delta\text{-pint}(G)$. Put $U = \delta\text{-pint}(G)$, then U is δ -preopen and $A \subset U \subset \delta\text{-pcl}(U) \subset \text{int}(B)$.

(e) \Rightarrow (f) : Let A be any g -closed subset of X and B be an open set containing A . We have $\text{cl}(A) \subset B$. Then there exists a $g\delta p$ -open set G such that $\text{cl}(A) \subset G \subset \delta\text{-pcl}(G) \subset B$. Since G is $g\delta p$ -open and $\text{cl}(A) \subset G$, by Lemma 2 we have $\text{cl}(A) \subset \delta\text{-pint}(G)$, put $U = \delta\text{-pint}(G)$, then U is δ -preopen and $\text{cl}(A) \subset U \subset \delta\text{-pcl}(U) \subset B$. \square

Theorem 19. *If $f : X \rightarrow Y$ is a continuous quasi δ -preclosed surjection and X is δp -normal, then Y is normal.*

Proof. Let M_1 and M_2 be any disjoint closed sets of Y . Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X . Since X is δp -normal, there exist disjoint $U_1, U_2 \in \delta PO(X)$ such that $f^{-1}(M_i) \subset U_i$ for $i = 1, 2$. Put $V_i = Y - f(X - U_i)$, then V_i is open in Y , $M_i \subset V_i$ and $f^{-1}(V_i) \subset U_i$ for $i = 1, 2$. Since $U_1 \cap U_2 = \emptyset$ and f is surjective, we have $V_1 \cap V_2 = \emptyset$. This shows that Y is normal. \square

Theorem 20. *Let $f : X \rightarrow Y$ be a closed δp - $g\delta p$ -continuous injection. If Y is δp -normal, then X is δp -normal.*

Proof. Let N_1 and N_2 be disjoint closed sets of X . Since f is a closed injection, $f(N_1)$ and $f(N_2)$ are disjoint closed sets of Y . By the δp -normality of Y , there exist disjoint $V_1, V_2 \in \delta PO(Y)$ such that $f(N_i) \subset V_i$ for $i = 1, 2$. Since f is δp - $g\delta p$ -continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint $g\delta p$ -open sets of X and $N_i \subset f^{-1}(V_i)$ for $i = 1, 2$. Now, put $U_i = \delta\text{-pint}(f^{-1}(V_i))$ for $i = 1, 2$. Then, $U_i \in \delta PO(X)$, $N_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$. This shows that X is δp -normal. \square

Corollary 3. *If $f : X \rightarrow Y$ is a closed δ -preirresolute injection and Y is δp -normal, then X is δp -normal.*

Lemma 3. *A function $f : X \rightarrow Y$ is almost $g\delta p$ -closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a $g\delta p$ -open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Lemma 4. *If $f : X \rightarrow Y$ is almost $g\delta p$ -closed, then for each closed set M of Y and each $U \in RO(X)$ containing $f^{-1}(M)$, there exists $V \in \delta PO(Y)$ such that $M \subset V$ and $f^{-1}(V) \subset U$.*

Theorem 21. *Let $f : X \rightarrow Y$ be a continuous almost $g\delta p$ -closed surjection. If X is normal, then Y is δp -normal.*

Proof. Let M_1 and M_2 be any disjoint, closed sets of Y . Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X . By the normality of X , there exist disjoint open sets U_1 and U_2 such that $f^{-1}(M_i) \subset U_i$, where $i = 1, 2$. Now, put $G_i = \text{int}(cl(U_i))$ for $i = 1, 2$, then $G_i \in RO(X)$, $f^{-1}(M_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \emptyset$. By Lemma 4, there exists $V_i \in \delta PO(Y)$ such that $M_i \subset V_i$ and $f^{-1}(V_i) \subset G_i$, where $i = 1, 2$. Since $G_1 \cap G_2 = \emptyset$ and f is surjective, we have $V_1 \cap V_2 = \emptyset$. This shows that Y is δp -normal. \square

Corollary 4. *If $f : X \rightarrow Y$ is a continuous δ -preclosed surjection and X is normal, then Y is δp -normal.*

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