On a generalization of quantifiers

by

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In this paper I shall deal with operators which represent a natural generalization of the logical quantifiers [1]). I shall formulate, for the generalized quantifiers, problems which correspond to the classical problems of the first-order logic. Some of these problems will be solved in the present paper, other more interesting ones are left open.

Most of our discussion centers around the problem whether it is possible to set up a formal calculus which would enable us to prove all true propositions involving the new quantifiers. Although this problem is not solved in its full generality, yet it is clear from the partial results which will be discussed below that the answer to the problem is essentially negative. In spite of this negative result we believe that some at least of the generalized quantifiers deserve a closer study and some deserve even to be included into systematic expositions of symbolic logic. This belief is based on the conviction that the construction of formal calculi is not the unique and even not the most important goal of symbolic logic.

1. Propositional functions and quantifiers. Let I be an arbitrary set and $I^* = I \times I \times \ldots$ its infinite Cartesian power, *i. e.*, the set of infinite sequences $(x_1, x_2, \ldots)$ with $x_j \in I$ for $j = 1, 2, \ldots$ We denote by $\vee$ and $\wedge$ the truth-values "truth" and "falsity". The Boolean operations of join, meet, and complementation are denoted by $\vee$, $\wedge$ and $\sim$; we use these symbols for all Boolean algebras which we shall have to consider and, in particular, for the two-element algebra consisting of the truth-values $\wedge$ and $\vee$.

A mapping F of I^* into $\{\vee, \wedge\}$ is called a *propositional function* on I provided that it satisfies the following condition: there is a finite set K of integers such that if

$$x = (x_1, x_2, \ldots) \in I^*, \quad y = (y_1, y_2, \ldots) \in I^*, \quad \text{and} \quad x_j = y_j \text{ for } j \in K,$$

then $F(x) = F(y)$.

[1]) Parts of the results contained in this paper were presented to the Toruń Section of the Polish Mathematical Society in January 1954. Other parts were included in my paper [5], which, however, contains no proofs.

This condition says, of course, that F depends essentially on a finite number of arguments. The smallest set K with the property stated above is called the *support* of F; if it has only one element, then F is a function of one argument and can be identified with a subset of I [2]).

Let φ be a one-one mapping of I onto a set I' not necessarily different from I. If $x = (x_1, x_2, \ldots) \in I^*$, then we denote by $\varphi(x)$ the sequence $(\varphi(x_1), \varphi(x_2), \ldots)$; if F is a propositional function on I, then we denote by F_φ the propositional function on I' such that $F_\varphi(\varphi(x)) = F(x)$.

A quantifier limited to I is a function $\boldsymbol{Q}$ which assigns one of the elements $\vee$, $\wedge$ to each propositional function F on I with one argument and which satisfies the invariance condition

$$\boldsymbol{Q}(F) = \boldsymbol{Q}(F_\varphi)$$

for each F and each permutation φ of I.

The first part of this definition generalizes the elementary fact that quantifiers enable us to construct propositions from propositional functions with one argument. The second part expresses the requirement that quantifiers should not allow us to distinguish between different elements of I [3]).

Let $(\mathfrak{m}_\xi, \mathfrak{n}_\xi)$ be the (finite or transfinite) sequence of all pairs of cardinal numbers satisfying the equation $\mathfrak{m}_\xi + \mathfrak{n}_\xi = \overline{\overline{I}}$ [4]). For each function T which assigns one of the truth-values to each pair $(\mathfrak{m}_\xi, \mathfrak{n}_\xi)$ we put

$$\boldsymbol{Q}_T(F) = T\left(\overline{\overline{F^{-1}(\vee)}}, \overline{\overline{F^{-1}(\wedge)}}\right) \text{ [4]) }.$$

The following theorem is easily provable:

THEOREM 1. *$\boldsymbol{Q}_T$ is a quantifier limited to I; for each quantifier limited to I there is a T such that $\boldsymbol{Q}_T = \boldsymbol{Q}$.*

If $\boldsymbol{Q} = \boldsymbol{Q}_T$, then we shall say that the function T *determines the quantifier* $\boldsymbol{Q}$; there is evidently exactly one such function for each $\boldsymbol{Q}$.

Let us put $T^*(\mathfrak{m}_\xi, \mathfrak{n}_\xi) = \sim T(\mathfrak{n}_\xi, \mathfrak{m}_\xi)$. The quantifier determined by T^* is said to be a *dual* of $\boldsymbol{Q}_T$ and is denoted by $\boldsymbol{Q}_T^*$.

An *unlimited quantifier* (or simply a quantifier) is a function which assigns a quantifier $\boldsymbol{Q}_I$ limited to I to each set I and which satisfies the equation $\boldsymbol{Q}_I(F) = \boldsymbol{Q}_{I'}(F_\varphi)$ for each propositional function F on I with one argument and for each one-one mapping of I onto I'.

[2]) In connection with these definitions compare Halmos [1].

[3]) Cf. in this connection Lindenbaum-Tarski [3] and Mautner [4].

[4]) $\overline{\overline{X}}$ denotes the cardinal number of X and $\underset{x \in X}{E}[W(x)]$ the set of elements x in X satisfying the condition $W(x)$. If f is a mapping of X into Y, then $f^{-1}(y)$ denotes the set $\underset{x \in X}{E}[f(x) = y]$.

It is clear how to define Boolean operations on limited and unlimited quantifiers; they will be denoted by the usual symbols $\vee$, $\wedge$, $\sim$. Thus *e.g.* $Q'_I \vee Q''_I$ is a function Q_I such that $Q_I(F) = Q'_I(F) \vee Q''_I(F)$ for each propositional function F.

2. Examples of quantifiers. (a) If $\{T(\mathfrak{m}_\xi, \mathfrak{n}_\xi) = \bigvee\} \equiv \{\mathfrak{m}_\xi \neq 0\}$, then Q_T is the existential quantifier $\boldsymbol{\exists}$ limited to I; the dual of Q_T is the general quantifier $\boldsymbol{\forall}$ limited to I.

(b) Let m, n be non-negative integers and T', T'' functions such that

$$\{T'(\mathfrak{m}_\xi, \mathfrak{n}_\xi) = \bigvee\} \equiv \{\mathfrak{m}_\xi = m\}, \qquad \{T''(\mathfrak{m}_\xi, \mathfrak{n}_\xi) = \bigvee\} \equiv (\mathfrak{n}_\xi = n).$$

Quantifiers $Q_{T'}$ and $Q_{T''}$ will be denoted by $\overset{(m)}{\underset{I}{\sum}}$ and $\overset{(n)}{\underset{I}{\prod}}$. The unlimited quantifiers which assign $\overset{(m)}{\underset{I}{\sum}}$ and $\overset{(n)}{\underset{I}{\prod}}$ to I will be denoted by $\overset{(m)}{\sum}$ and $\overset{(n)}{\prod}$. Boolean polynomials of quantifiers $\overset{(m)}{\sum}$, $\overset{(n)}{\prod}$ $(m, n = 0, 1, 2, \ldots)$ are called *numerical quantifiers* (cf. Tarski [8], p. 63).

Examples of such quantifiers are

$$Q^{(1)} = \overset{(m_1)}{\sum} \vee \overset{(m_2)}{\sum} \vee \ldots \vee \overset{(m_k)}{\sum}, \qquad Q^{(2)} = \overset{(m_1)}{\prod} \vee \overset{(m_2)}{\prod} \vee \ldots \vee \overset{(m_k)}{\prod}$$

If I is an infinite set and F a propositional function on [illegible] th the support $\{1\}$, then the formula $Q_I^{(1)}(F) = \bigvee$ (or the formula $Q_I^{(2)}(F) = \bigvee$) is equivalent to the statement: the set of elements x in I such that $F(x, \ldots) = \bigvee$ [5]) (or such that $F(x, \ldots) = \bigwedge$) has exactly m_1 or exactly m_2 or ... or exactly m_k elements.

(c) Let T_1 and T_2 be functions such that

$$\{T_1(\mathfrak{m}_\xi, \mathfrak{n}_\xi) = \bigvee\} \equiv (\mathfrak{m}_\xi < \aleph_0), \; \{T_2(\mathfrak{m}_\xi, \mathfrak{n}_\xi) = \bigwedge\} \equiv (\mathfrak{m}_\xi < \aleph_0 \text{ or } \mathfrak{n}_\xi < \aleph_0).$$

We denote by S_I and S_I^0 the quantifiers Q_{T_1} and Q_{T_2} and by S and S^0 the unlimited quantifiers assigning S_I and S_I^0 to each I. If F is a propositional function on I with the support $\{1\}$, then the formula $S_I(F) = \bigvee$ (or the formula $S_I^0(F) = \bigvee$) is equivalent to the statement: there are at most finitely many elements $x \in I$ such that $F(x, \ldots) = \bigvee$ (or to the statement: there are at most finitely many elements x in I such that $F(x, \ldots) = \bigvee$ or at most finitely many elements x in I such that $F(x, \ldots) = \bigwedge$).

(d) Let T be a function such that $\{T(\mathfrak{m}_\xi, \mathfrak{n}_\xi) = \bigvee\} \equiv (\mathfrak{m}_\xi \leqslant \aleph_0)$. The quantifier Q_T will be denoted by P_I and the corresponding unlimited quantifier by P. If F is again a propositional function on I with the support $\{1\}$, then the formula $P_I(F) = \bigvee$ is equivalent to the statement: there are at most denumerably many elements $x \in I$ such that $F(x, \ldots) = \bigvee$.

[5]) $(x, \ldots)$ denotes a sequence whose first term is x while the remaining terms are arbitrary elements of I.

(e) Let I be a denumerable set. A quantifier Q_T limited to I is wholly characterized by the values $T(n, \aleph_0)$, $T(\aleph_0, n)$, $\gamma_T = T(\aleph_0, \aleph_0)$, $n = 0, 1, 2, \ldots$

We are going to discuss the Boolean algebra A of quantifiers Q_T such that $T(n, \aleph_0) = \text{const}$ and $T(\aleph_0, n) = \text{const}$ from certain n, m on.

For the purpose of this discussion we consider the ideal $\mathfrak{J}$ of finite subsets of N where N is the set of non-negative integers. Let $\mathfrak{A}$ be the Boolean algebra generated by $\mathfrak{J}$ and let $\mathfrak{C}$ be the Cartesian product $\mathfrak{A} \times \mathfrak{A} \times \{\bigwedge, \bigvee\}$. The algebra $\mathfrak{C}$ is an isomorphic image of A under the mapping $Q_T \rightleftarrows \langle A_T, B_T, \gamma_T \rangle$ where $A_T = \underset{n}{E}[T(n, \aleph_0) = \bigvee]$ and $B_T = \underset{n}{E}[T(\aleph_0, n) = \bigvee]$. The quantifiers S_I, S_I^*, S_I^0 correspond to the elements

$$X_1 = \langle N, 0, \wedge \rangle, \qquad X_2 = \langle N, 0, \vee \rangle, \qquad X_3 = \langle N, N, \wedge \rangle$$

of $\mathfrak{C}$ (the symbol 0 denotes here the void set).

The product $\mathfrak{J}_0 = \mathfrak{J} \times \mathfrak{J} \times \{\bigwedge\}$ is obviously an ideal in $\mathfrak{C}$. The quantifiers Q_T whose images are in $\mathfrak{J}_0$ form an ideal A_0 of A; it is easy to see that all these quantifiers are numerical.

Each element of $\mathfrak{C}$ is congruent mod $\mathfrak{J}_0$ to one of the 8 elements $X_0 = \langle 0, 0, \bigwedge \rangle$, X_1, X_2, X_3, $\sim X_0$, $\sim X_1$, $\sim X_2$, $\sim X_3$. For quantifiers Q_T of A this result means that Q_T is either numerical or congruent mod A_0 to one of the quantifiers S_I, S_I^*, S_I^0, $\sim S_I$, $\sim S_I^*$, $\sim S_I^0$. Hence we obtain the result that *if a quantifier Q_T of A is not numerical, then one of the quantifiers S_I, S_I^*, S_I^0 is definable in* [illegible] *of Q_T and of the numerical quantifiers* [6]).

3. A formal calculus and its interpretation. Let (S) be a formal logical calculus whose structure differs from the usual functional calculus of first order (with identity) only in the following: Instead of the usual symbols for the existential and the general quantifiers (S) contains s symbols $\mathbf{Q}^1, \mathbf{Q}^2, \ldots, \mathbf{Q}^s$. The rules of building formulas by means of the symbols $\boldsymbol{\exists}$, $\boldsymbol{\forall}$ are replaced by the rule: if F is a formula and x a variable, then $(\mathbf{Q}^j\mathrm{x})\mathrm{F}$ is a formula $(j = 1, 2, \ldots, s)$.

The variable x is bound in the formula $(\mathbf{Q}^j\mathrm{x})\mathrm{F}$; a formula containing exclusively bound variables is called *closed*.

We shall define in the usual way the notion of satisfaction for formulas of (S) [7]).

[6]) The simple proof given above is due to J. Łoś; my former proof of this theorem was much more complicated.

[7]) For the subsequent definitions see Tarski [7].

Let $Q^1,\ldots,Q^s$ be s unlimited quantifiers and I a set. We shall consider functions which assign an element of I to each individual variable of (S) and a propositional function on I with the support $\{1,2,\ldots,k\}$ to each functional variable of degree k of (S). Functions of this kind will be called I-valuations. If M is an I-valuation, then $[\mathrm{x}_i]_M$ and $[\mathrm{F}_j]_M$ denote the element of I and the propositional function assigned by M to x_i and F_j.

Each I-valuation M determines a mapping val_{MI} of formulas of (S) into the set $\{\vee, \wedge\}$. If Z is the formula $\mathrm{F}(\mathrm{x}_{i_1},\ldots,\mathrm{x}_{i_k})$, then we put

$$val_{MI}(\mathrm{Z}) = [\mathrm{F}]_M([\mathrm{x}_{i_1}]_M, [\mathrm{x}_{i_2}]_M, \ldots, [\mathrm{x}_{i_k}]_M, [\mathrm{x}_{i_k}]_M, [\mathrm{x}_{i_k}]_M, \ldots)\,.$$

If Z is the formula $\mathrm{x}_i = \mathrm{x}_j$, then $\{val_{MI}(\mathrm{Z}) = \vee\} \equiv \{[\mathrm{x}_i]_M = [\mathrm{x}_j]_M\}$. If Z is the formula $\mathrm{Z}_1|\mathrm{Z}_2$, then $val_{MI}(\mathrm{Z}) = \sim val_{MI}(\mathrm{Z}_1) \vee \sim val_{MI}(\mathrm{Z}_2)$.

Now let us assume that Z is the formula $(\mathbf{Q}^j\mathrm{x}_i)\mathrm{Z}_1$. Let $M(i,x)$ be an I-valuation differing from M only by assigning x to the variable x_i and let F be a propositional function on I with the support $\{i\}$ such that

$$F(y_1, y_2, \ldots) = val_{M(i,y_i),I}(\mathrm{Z}_1)\,.$$

We put $val_{MI}\big((\mathbf{Q}^j\mathrm{x}_i)\mathrm{Z}_1\big) = Q^j_I(F)$. The mapping val_{MI} is thus defined by induction.

It is easy to prove that $val_{MI}(\mathrm{Z})$ depends not on the whole M but only on those values $[\mathrm{x}_i]_M$ and $[\mathrm{F}_j]_M$ for which x_i and F_j are free in Z. If $\mathrm{x}_{i_1},\ldots,\mathrm{x}_{i_k}, \mathrm{F}_{j_1},\ldots,\mathrm{F}_{j_l}$ are all the variables which are free in Z, then instead of saying that $val_{MI}(\mathrm{Z}) = \vee$ we shall sometimes say that the elements $[\mathrm{x}_{i_1}]_M, \ldots, [\mathrm{x}_{i_k}]_M$, and the propositional functions $[\mathrm{F}_{j_1}]_M, \ldots, [\mathrm{F}_{j_l}]_M$ satisfy Z in I.

A formula Z is

(a) true in I, (b) satisfiable in I

if the equation $val_{MI}(\mathrm{Z}) = \vee$ holds

(a) for each M, (b) for at least one M.

The formula Z is

(c) true, (d) satisfiable

if it is

(c) true in I for each I, (d) satisfiable in I for at least one I.

All these notions depend, of course, on the quantifiers $Q^1, Q^2, \ldots, Q^s$. Our expression "Z is true (or satisfiable) in I" is therefore not exact and ought to be replaced by a more complex one "Z is true (or satisfiable) in I with $\mathbf{Q}^j$ interpreted as Q^j, $j = 1,2,\ldots,s$". Since, however, the quantifiers $Q^1,\ldots,Q^s$ will be held constant throughout the rest of the paper, there will be no occasion to use this complex expression.

Many other semantical notions can be defined with the help of the notions of truth and satisfaction. In what follows we shall need the notion of definability of quantifiers.

The quantifier Q^1_I is definable in I in terms of the quantifiers $Q^2_I,\ldots,Q^s_I$ if there is a formula Z of (S) not containing the symbol $\mathbf{Q}^1$ having F as its unique free variable and such that the equivalence $(\mathbf{Q}^1\mathrm{x})\mathrm{F}(\mathrm{x}) \equiv \mathrm{Z}$ is true in I. If this condition is satisfied for each I and Z is the same for all I, then we say that the unlimited quantifier Q^1 is definable in terms of $Q^2,\ldots,Q^s$.

4. The completeness problem. The completeness problem for quantifiers $Q^1,\ldots,Q^s$ (or the completeness problem for quantifiers $Q^1_I,\ldots,Q^s_I$ limited to I) consists in answering the question: is the set of true formulas (or the set of formulas true in I) recursively enumerable? No general solution of these problems is known. We shall formulate only some partial results.

Definition. A quantifier Q has the *property* (E) if for each denumerable set I the function T determining Q_I satisfies the condition: the sets $\underset{n}{E}[T(n,\aleph_0) = \wedge]$, $\underset{n}{E}[T(n,\aleph_0) = \vee]$ are both denumerable.

THEOREM 2. *If among $Q^1,\ldots,Q^s$ occur the general and the existential quantifiers and at least one quantifier Q with the property* (E), *then the completeness problem for these quantifiers has a negative solution.*

Proof. We denote by A_0 the conjunction of axioms of the elementary theory of non-densely ordered rings (cf. Tarski [9], p. 69). In A_0 occur 4 predicate-variables $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4$ and the formulas $\mathrm{F}_1(\mathrm{x}), \mathrm{F}_2(\mathrm{x},\mathrm{y},\mathrm{z})$, $\mathrm{F}_3(\mathrm{x},\mathrm{y},\mathrm{z}), \mathrm{F}_4(\mathrm{x},\mathrm{y})$ are to be read thus: x is the zero of the ring, x is the sum of y and z, x is the product of y and z, x is less than y.

We abbreviate $(\exists\mathrm{u})[\mathrm{F}_1(\mathrm{u}) \wedge \mathrm{F}_4(\mathrm{u},\mathrm{x})]$ as $\mathrm{P}(\mathrm{x})$ (to be read as: x is positive) and denote by B_0 the formula

$$(\forall\mathrm{x})\Big[\mathrm{P}(\mathrm{x}) \supset (\exists\mathrm{y})\Big(\mathrm{F}_4(\mathrm{x},\mathrm{y}) \wedge \big[(\mathbf{Q}\mathrm{z})\big(\mathrm{P}(\mathrm{z}) \wedge \mathrm{F}_4(\mathrm{z},\mathrm{x})\big) \not\equiv (\mathbf{Q}\mathrm{z})\big(\mathrm{P}(\mathrm{z}) \wedge \mathrm{F}_4(\mathrm{z},\mathrm{y})\big)\big] \wedge$$
$$\wedge (\forall\mathrm{t})\big\{\mathrm{F}_4(\mathrm{x},\mathrm{t}) \wedge \mathrm{F}_4(\mathrm{t},\mathrm{y}) \supset \big[(\mathbf{Q}\mathrm{z})\big(\mathrm{P}(\mathrm{z}) \wedge \mathrm{F}_4(\mathrm{z},\mathrm{x})\big) \equiv (\mathbf{Q}\mathrm{z})\big(\mathrm{P}(\mathrm{z}) \wedge \mathrm{F}_4(\mathrm{z},\mathrm{t})\big)\big]\big\}\Big)\Big]\,.$$

The intuitive meaning of B_0 is this: for every positive x there is a smallest y such that the value assigned by the quantifier Q to the set $\underset{\mathrm{z}}{E}[0 < \mathrm{z} < \mathrm{x}]$ is different from the value assigned by Q to the set $\underset{\mathrm{z}}{E}[0 < \mathrm{z} < \mathrm{y}]$.

Finally we denote by A the conjunction of A_0 and B_0.

Let I be a set and M an I-valuation such that $[F_1]_M, \ldots, [F_4]_M$ satisfy A_0 in I. It is evident that the supports of $[F_1]_M, \ldots, [F_4]_M$ are $\{1\}$, $\{1,2,3\}$, $\{1,2,3\}$, $\{1,2\}$. From the properties of A_0 it is easy to derive the following facts:

(a) *There is exactly one element θ in I such that $[F_1]_M(\theta, \ldots) = \bigvee$* [8].

(b) *For every x, y in I there is exactly one element $z = x \oplus y$ in I such that $[F_2]_M(z, x, y, \ldots) = \bigvee$* [9] *and exactly one element $t = x \odot y$ such that $[F_3]_M(t, x, y, \ldots) = \bigvee$.*

(c) *The binary relation $\prec$ defined by means of the equivalence $(x \prec y) \equiv \big([F_4]_M(x, y, \ldots) = \bigvee\big)$ orders the set I.*

(d) *The set I is a non-densely ordered ring with respect to the operations $\oplus$, $\odot$, and the ordering $\prec$; θ is the zero of this ring.*

We shall now prove

(e) *The ring I as described in* (d) *is isomorphic to the ring of integers; in this isomorphism θ, $\oplus$, $\odot$, $\prec$ are mapped onto 0, addition, multiplication, and the "less-than" relation.*

A non-densely ordered ring is isomorphic to the ring of integers if and only if for each positive x there are finitely many elements between 0 and x. Hence it is sufficient to show that for each x in I such that $\theta \prec x$ the set $Z_x = \underset{y \in I}{E}[\theta \prec y \prec x]$ is finite. Let us assume that this is false and choose x so that the cardinal number $\mathfrak{m}$ of Z_x be infinite and as small as possible. Since $[F_1]_M, \ldots, [F_4]_M$ satisfy B_0 in I there is an element y in I such that $x \prec y$ and $\boldsymbol{Q}_I(Z_x) \neq \boldsymbol{Q}_I(Z_y)$ (we have identified here subsets of I with the propositional functions of one variable). This implies that one of the equations $\overline{\overline{Z}}_x = \overline{\overline{Z}}_y$, $\overline{\overline{I - Z_x}} = \overline{\overline{I - Z_y}}$ must be false. Now the complements of both Z_x and Z_y contain the set $\underset{z \in I}{E}[z \prec \theta]$, whose cardinal number is $\overline{\overline{I}}$; hence $\overline{\overline{I - Z_x}} = \overline{\overline{I - Z_y}}$ and therefore

$$\overline{\overline{Z}}_x < \overline{\overline{Z}}_y \leqslant \overline{\overline{I}}.$$

Denoting by T the function which determines $\boldsymbol{Q}_I$ we obtain further $T(\overline{\overline{Z}}_x, \overline{\overline{I}}) \neq T(\overline{\overline{Z}}_y, \overline{\overline{I}})$.

Now let t be the antecedent of y in I. It is evident that $\overline{\overline{Z}}_t = \overline{\overline{Z}}_y$ and $x \prec t \prec y$. The assumption that $[F_1]_M, \ldots, [F_4]_M$ satisfy B_0 in I yields therefore $\boldsymbol{Q}_I(Z_x) = \boldsymbol{Q}_I(Z_t)$, *i. e.*, $T(\overline{\overline{Z}}_x, \overline{\overline{I}}) = T(\overline{\overline{Z}}_y, \overline{\overline{I}})$. We have thus arrived at a contradiction, which proves (e).

[8] $(\theta, \ldots)$ denotes here a sequence of elements of I with the first term θ.

[9] $(x, y, z, \ldots)$ denotes a sequence of elements of I beginning with x, y, z.

As the last auxiliary statement we prove

(f) *Let I_0 be the set of integers and let M_0 be an I_0-valuation such that $[F_1]_{M_0}, \ldots, [F_4]_{M_0}$ satisfy the equivalences*

$$\{[F_1]_{M_0}(x_1, x_2, \ldots) = \bigvee\} \equiv (x_1 = 0),$$
$$\{[F_2]_{M_0}(x_1, x_2, \ldots) = \bigvee\} \equiv (x_1 = x_2 + x_3),$$
$$\{[F_3]_{M_0}(x_1, x_2, \ldots) = \bigvee\} \equiv (x_1 = x_2 x_3),$$
$$\{[F_4]_{M_0}(x_1, x_2, \ldots) = \bigvee\} \equiv (x_1 < x_2).$$

Then $[F_1]_{M_0}, \ldots, [F_4]_{M_0}$ satisfy A *in I_0.*

Indeed, $val_{M_0 I_0}(A_0) = \bigvee$ since I_0 is a non-densely ordered ring. In order to prove that $val_{M_0 I_0}(B_0) = \bigvee$ we choose an integer $x > 0$ and put $\boldsymbol{Q}_{I_0}\big(\underset{z \in I_0}{E}[0 < z < x]\big) = \alpha$. Since $\boldsymbol{Q}_{I_0}$ has the property (E), there must be in I_0 a smallest y such that $x < y$ and $\boldsymbol{Q}_{I_0}\big(\underset{z \in I_0}{E}[0 < z < y]\big) \neq \alpha$. Hence, if $x < t < y$, then $\boldsymbol{Q}_{I_0}\big(\underset{z \in I_0}{E}[0 < z < t]\big) = \alpha$. This proves that $val_{M_0 I_0}(B_0) = \bigvee$.

In order to prove theorem 2 we consider the set $\mathfrak{X}$ of formulas X in which there occur no quantifiers other than $\boldsymbol{\exists}, \boldsymbol{\forall}$, and no free variables other than $F_1, \ldots, F_4$. If $A \supset X$ is true, then (by (f)) $val_{M_0 I_0}(X) = \bigvee$. If, conversely, $val_{M_0 I_0}(X) = \bigvee$, then, by (e), $A \supset X$ is true since the unique model of A is a model of X. Hence, if the set of all true formulas of (S) were recursively enumerable, then so would be the set of all X in $\mathfrak{X}$ satisfying $val_{M_0 I_0}(X) = \bigvee$. It is known, however, that this set is not recursively enumerable: it is not even arithmetically definable. This accomplishes the proof of theorem 2.

The condition given in theorem 2 is not necessary for the solution of the completeness problem to be negative. In fact neither of the quantifiers $\boldsymbol{\exists}, \boldsymbol{\forall}, \boldsymbol{S}, \boldsymbol{S}^0$ satisfies the condition (E) and yet we have

THEOREM 3. *If among $\boldsymbol{Q}^1, \ldots, \boldsymbol{Q}^s$ occur the quantifiers $\boldsymbol{\exists}, \boldsymbol{\forall}, \boldsymbol{S}$ or the quantifiers $\boldsymbol{\exists}, \boldsymbol{\forall}, \boldsymbol{S}^0$ then the completeness problem for these quantifiers as well as the completeness problem for these quantifiers limited to a denumerable set have both a negative solution.*

Proof of theorem 3 is similar to that of theorem 2. The only difference is that we have to take as A the conjunction of A_0 and of one of the following formulas:

$$(\boldsymbol{\forall} x)\{P(x) \supset (\boldsymbol{S} z)[P(z) \wedge F_4(z, x)]\},$$
$$(\boldsymbol{\forall} x)\{P(x) \supset (\boldsymbol{S}^0 z)[P(z) \wedge F_4(z, x)]\}.$$

The completeness problem for quantifiers limited to a denumerable set can easily be solved in all its generality:

THEOREM 4. *Let the quantifiers* $\mathbf{\exists}, \mathbf{\forall}$ *occur among* $Q^1, \ldots, Q^s$ *and let* I *be a denumerable set. A necessary and sufficient condition for the completeness problem for quantifiers* $Q_I^1, \ldots, Q_I^s$ *to have a positive solution is that the quantifiers* $Q_I^1, \ldots, Q_I^s$ *be numerical.*

Proof. The sufficiency is obvious. Let us now assume that one of the quantifiers $Q_I^1, \ldots, Q_I^s$ is not numerical. If there is a quantifier definable in I in terms of $Q_I^1, \ldots, Q_I^s$ and satisfying the condition (E), then the result follows from theorem 2. If there is no such quantifier, then the final result of section 2(e) proves that one of the quantifiers S_I, S_I^0 is definable in I in terms of $Q_I^1, \ldots, Q_I^s$, and hence the result follows from theorem 3.

The general case of the completeness problem as well as the completeness problem for quantifiers limited to non-denumerable sets remain open. We see no way of solving this problem even for the quantifiers $\mathbf{\exists}, \mathbf{\forall}, \mathbf{P}$ (see section 2(d)). The following result shows that the method used in previous theorems is not applicable to that case:

THEOREM 5. *If* Q *is a quantifier definable in terms of the quantifiers* $\mathbf{\exists}, \mathbf{\forall}, \mathbf{P}$, *then* $Q_I \neq S_I$ *and* $Q_I \neq S_I^0$ *for each denumerable set* I.

Proof. If S_I or S_I^0 were definable in terms of $\mathbf{\exists}_I$, $\mathbf{\forall}_I$, and P_I, then they would be definable in terms of $\mathbf{\exists}_I$ and $\mathbf{\forall}_I$ alone since P_I is identical with a constant quantifier which assigns the value $\vee$ to each propositional function. Theorems 3, 4, and the classical completeness-theorem show, however, that none of the quantifiers S_I, S_I^0 is definable in terms of $\mathbf{\exists}_I$ and $\mathbf{\forall}_I$.

5. The Skolem-Löwenheim theorem. Let Q be an unlimited quantifier.

Definition. We shall say that Q does *not distinguish infinite powers* if for any two infinite sets I_1, I_2 the functions T_1, T_2 which determine Q_{I_1} and Q_{I_2} satisfy the equations:

$$T_1(n, \overline{\overline{I}}_1) = T_2(n, \overline{\overline{I}}_2), \quad n = 0, 1, 2, \ldots,$$
$$T_1(\overline{\overline{I}}_1, n) = T_2(\overline{\overline{I}}_2, n), \quad n = 0, 1, 2, \ldots,$$
$$T_1(\mathfrak{m}_1, \mathfrak{n}_1) = T_2(\mathfrak{m}_2, \mathfrak{n}_2) \quad \text{for} \quad \mathfrak{m}_i + \mathfrak{n}_i = \overline{\overline{I}}_i,\ \mathfrak{m}_i, \mathfrak{n}_i \geq \aleph_0,\ i = 1, 2.$$

THEOREM 6. *If none of the quantifiers* $Q^1, \ldots, Q^s$ *distinguishes infinite powers, then each closed formula satisfiable in an infinite set is satisfiable in a denumerable set.*

Proof. Let Z be a closed formula satisfiable in an infinite set I and let $\overline{M}$ be an I-valuation such that $val_{\overline{M}I}(\mathrm{Z}) = \vee$. We enlarge the calculus (S) by adding to it an infinite number of individual constants σ_x, each element x in I determining exactly one constant σ_x. Formulas not containing the new constants will be called *proper*.

Semantical notions which we have introduced in section 2 can be extended so that they become applicable to improper formulas. The only change which is needed is the stipulation that $[\sigma_x]_M = x$ for each I-valuation M.

The following lemma can easily be proved by induction:

LEMMA (a). *If* A *is a proper formula with the free variables* $\mathrm{F}_1, \ldots, \mathrm{F}_k$, $\mathrm{x}_1, \ldots, \mathrm{x}_l$, *then* $val_{MI}(\mathrm{A}) = \vee$ *if and only if* $[\mathrm{F}_1]_M, \ldots, [\mathrm{F}_k]_M$ *satisfy in* I *the improper formula* A′ *resulting from* A *by a substitution of* $\sigma_{[\mathrm{x}_j]_M}$ *for* x_j $(j = 1, 2, \ldots, l)$.

We choose an arbitrary denumerable subset I_1 of I and denote by $\mathfrak{U}$ the set of proper formulas beginning with one of the symbols $\mathbf{Q}^1, \ldots, \mathbf{Q}^s$.

Let us assume that $k \geq 1$ and that a denumerable subset I_k of I has been defined. We are going to define a set I_{k+1}. To this end we arrange in a sequence

$$\mathrm{V}_1, \mathrm{V}_2, \ldots \tag{1}$$

all closed (proper and improper) formulas resulting from formulas X in $\mathfrak{U}$ by a substitution of symbols σ_x (x in I_k) for the free variables of X.

Each V_j determines a set I_{kj} in the following way: Assume that V_j is the formula $(\mathbf{Q}^t\mathrm{x})\mathrm{W}$. We denote by $\mathrm{W}(\sigma_a)$ the formula resulting from W by the substitution of σ_a for x and consider the sets

$$J_1 = \underset{a \in I}{E}\,[val_{\overline{M}I}(\mathrm{W}(\sigma_a)) = \vee], \quad J_2 = \underset{a \in I}{E}\,[val_{\overline{M}I}(\mathrm{W}(\sigma_a)) = \wedge]. \tag{2}$$

Let $\mathfrak{m}_1, \mathfrak{m}_2$ be the cardinal numbers of these sets.

If $\mathfrak{m}_1, \mathfrak{m}_2$ are both infinite, then we take as I_{kj} a denumerable subset of I having infinitely many elements in common with both J_1 and J_2. If $\mathfrak{m}_1$ is finite and $\mathfrak{m}_2$ infinite, then we take as I_{kj} a denumerable subset of I having infinitely many elements in common with J_2 and containing all the elements of J_1. If $\mathfrak{m}_2$ is finite and $\mathfrak{m}_1$ infinite, then we take as I_{kj} a denumerable subset of I having infinitely many elements in common with J_1 and containing all the elements of J_2.

We now put $I_{k+1} = \bigcup_{j=1}^{\infty} I_{kj}$. The sets I_k $(k = 1, 2, \ldots)$ are thus defined by induction.

We now put $I_0 = \bigcup_{k=1}^{\infty} I_k$ and obtain a denumerable subset of I. We shall show that Z is satisfiable in I_0.

For each I-valuation M we define an I_0-valuation M_0. M_0 differs from M by assigning to x_i an arbitrary element of I_0 whenever $[x_i]_M$ is not in I_0 and by assigning to F_j the propositional function $[F_j]_M$ restricted to I_0. If $x_1, x_2, \ldots$ are in I_0 we have therefore

(3) $$[F_j]_{M_0}(x_1, x_2, \ldots) = [F_j]_M(x_1, x_2, \ldots).$$

LEMMA (b). *Let M be an I-valuation satisfying the conditions*

(4) $$[x_i]_M \in I_0 \quad \text{for} \quad i = 1, 2, \ldots,$$

(5) $$[F_j]_M = [F_j]_{\overline{M}} \quad \text{for} \quad j = 1, 2, \ldots$$

If V is a proper formula and V′ results from V by a substitution of symbols σ_a (a in I_0) for some or for all free variables of V, then $val_{MI}(V') = val_{M_0 I_0}(V')$.

If V is an atomic formula $F_j(x_{i_1}, \ldots, x_{i_k})$ or $x_i = x_j$, then our assertion follows immediately from (3) and (4). If the lemma holds for formulas V_1 and V_2, then it is clear that it holds for the formula $V_1 | V_2$. It remains thus to show that if the lemma holds for a formula V, it does so for the formula $(\mathbf{Q}^h x_i)V$.

Let $(\mathbf{Q}^h x_i)V'$ result from $(\mathbf{Q}^h x_i)V$ by a substitution of symbols σ_a (a in I_0) for some or all free variables of $(\mathbf{Q}^h x_i)V$. Let $y_1, \ldots, y_k$ be the free individual variables of $(\mathbf{Q}^h x_i)V'$. We substitute $\sigma_{[y_j]_M}$ (or, what is the same, $\sigma_{[y_j]_{M_0}}$) for y_j in $(\mathbf{Q}^h x_i)V'$ $(j = 1, 2, \ldots, k)$ and obtain a closed formula $(\mathbf{Q}^h x_i)W$. We can assume that this formula occurs in the sequence (1) and is identical with V_j.

According to lemma (a), p. 21,

$$val_{MI}[(\mathbf{Q}^h x_i)V'] = val_{MI}[(\mathbf{Q}^h x_i)W],$$
$$val_{M_0 I_0}[(\mathbf{Q}^h x_i)V'] = val_{M_0 I_0}[(\mathbf{Q}^h x_i)W].$$

Thus it is sufficient to show that

(6) $$val_{MI}[(\mathbf{Q}^h x_i)W] = val_{M_0 I_0}[(\mathbf{Q}^h x_i)W].$$

We shall first calculate the left-hand side of (6). According to the definitions given in section 2 we have to define a propositional function F on I with the support $\{i\}$ such that

$$F(y_1, y_2, \ldots) = val_{M(i, y_i), I}(W)$$

and then take the value $\mathbf{Q}_I^h(F)$. According to lemma (a) $val_{M(i,y_i),I}(W)$ is $\vee$ or $\wedge$ according as $val_{MI}(W(\sigma_{y_i}))$ is $\vee$ or $\wedge$. Hence if we denote by T_I^h the function which determines the quantifier Q_I^h and put

$$\mathfrak{m}_1 = \overline{\overline{\underset{a \in I}{E}[val_{MI}(W(\sigma_a)) = \vee]}}, \quad \mathfrak{m}_2 = \overline{\overline{\underset{a \in I}{E}[val_{MI}(W(\sigma_a)) = \wedge]}},$$

we obtain

(7) $$val_{MI}[(\mathbf{Q}^h x_i)W] = T_I^h(\mathfrak{m}_1, \mathfrak{m}_2).$$

Similar considerations show that

(8) $$val_{M_0 I_0}[(\mathbf{Q}^h x_i)W] = T_{I_0}^h(\mathfrak{m}_1^0, \mathfrak{m}_2^0)$$

where $T_{I_0}^h$ is the function which determines the quantifier $Q_{I_0}^h$ and

$$\mathfrak{m}_1^0 = \overline{\overline{\underset{a \in I_0}{E}[val_{M_0 I_0}(W(\sigma_a)) = \vee]}}, \quad \mathfrak{m}_2^0 = \overline{\overline{\underset{a \in I_0}{E}[val_{M_0 I_0}(W(\sigma_a)) = \wedge]}}.$$

We observe now that if a is in I_0, then $W(\sigma_a)$ results from V by a substitution of symbols σ_x (x in I_0) for all free variables of V and that we can therefore apply the inductive assumption to the formula $W(\sigma_a)$. This gives $val_{M_0 I_0}(W(\sigma_a)) = val_{MI}(W(\sigma_a))$ for $a \in I_0$. Furthermore $W(\sigma_a)$ is a closed formula and thus $val_{MI}(W(\sigma_a))$ depends only on $[F_j]_M$, which in view of (5) proves that

$$val_{MI}(W(\sigma_a)) = val_{\overline{M}I}(W(\sigma_a)).$$

This equation holds for arbitrary a in I, not only for a in I_0.

Taking these observations together we obtain the equations

$$\mathfrak{m}_1^0 = \overline{\overline{\underset{a \in I_0}{E}[val_{\overline{M}I}(W(\sigma_a)) = \vee]}}, \quad \mathfrak{m}_2^0 = \overline{\overline{\underset{a \in I_0}{E}[val_{\overline{M}I}(W(\sigma_a)) = \wedge]}},$$
$$\mathfrak{m}_1 = \overline{\overline{\underset{a \in I}{E}[val_{\overline{M}I}(W(\sigma_a)) = \vee]}}, \quad \mathfrak{m}_2 = \overline{\overline{\underset{a \in I}{E}[val_{\overline{M}I}(W(\sigma_a)) = \wedge]}}.$$

We now use the definition of sets I_{kj} given above. Remembering that $(\mathbf{Q}^h x_i)W$ is the jth term of the sequence (1) we see that $\mathfrak{m}_1, \mathfrak{m}_2$ are the cardinal numbers of sets (2). If $\mathfrak{m}_1, \mathfrak{m}_2$ are both infinite, then I_{kj} has infinitely many elements in common with both J_1, J_2 and hence $\mathfrak{m}_1^0 = \mathfrak{m}_2^0 = \aleph_0$. If $\mathfrak{m}_1 = n$ is finite and $\mathfrak{m}_2$ infinite, then $J_1 \subset I_0$ and hence $\mathfrak{m}_1^0 = n$, and $\mathfrak{m}_2^0 = \aleph_0$. Similarly if $\mathfrak{m}_2 = n$ is finite and $\mathfrak{m}_1$ infinite, then $\mathfrak{m}_2^0 = n$ and $\mathfrak{m}_1^0 = \aleph_0$. This proves that

$$T_I^h(\mathfrak{m}_1, \mathfrak{m}_2) = T_{I_0}^h(\mathfrak{m}_1^0, \mathfrak{m}_2^0)$$

since the quantifier Q^h does not distinguish infinite powers.

Comparing the last equation with (7) and (8) we obtain (6), which proves lemma (b).

We can now conclude the proof of theorem 6. Since Z is a closed formula, the value of $val_{\overline{M}I}(Z)$ does not depend on $[x_i]_{\overline{M}}$. Hence we can assume that $[x_i]_{\overline{M}}$ is in I_0 for all i. Using lemma (b) for $M = \overline{M}$ and $V = Z$, we obtain $val_{\overline{M}I}(Z) = val_{\overline{M}_0 I_0}(Z)$ and hence $val_{\overline{M}_0 I_0}(Z) = \vee$ since $val_{\overline{M}I}(Z) = \vee$ by the definition of $\overline{M}$. This proves that Z is satisfiable in I_0.

We shall now prove that conditions given in theorem 6 are not only sufficient but also necessary for the validity of the Skolem-Löwenheim theorem.

THEOREM 7. *If $\mathbf{\exists}$ and $\mathbf{\forall}$ occur among the quantifiers $\boldsymbol{Q}^1,\dots,\boldsymbol{Q}^s$ and if at least one of these quantifiers distinguishes infinite powers, then there are closed formulas satisfiable in non-denumerable sets but not satisfiable in denumerable sets.*

Proof. Let us assume that the quantifier $\boldsymbol{Q}^1=\boldsymbol{Q}$ distinguishes infinite powers, *i. e.*, that there are infinite sets I_1, I_2 such that one of the following cases holds:

(a) there is an n such that $T_1(n,\overline{\overline{I_1}})\neq T_2(n,\overline{\overline{I_2}})$,

(b) there is an n such that $T_1(\overline{\overline{I_1}},n)\neq T_2(\overline{\overline{I_2}},n)$,

(c) there are infinite cardinals $\mathfrak{m}_i,\mathfrak{n}_i$ $(i=1,2)$ such that $\mathfrak{m}_i+\mathfrak{n}_i=\overline{\overline{I_i}}$ $(i=1,2)$ and $T_1(\mathfrak{m}_1,\mathfrak{n}_1)\neq T_2(\mathfrak{m}_2,\mathfrak{n}_2)$.

Here T_i is the function which determines the quantifiers $\boldsymbol{Q}_{I_i}$, $i=1,2$.

Case (a). Let $\mathfrak{m}_2$ be the least infinite cardinal such that if $\overline{\overline{I_2}}=\mathfrak{m}_2$, then there is an infinite set $I_1\subset I_2$ satisfying the conditions given in (a). Evidently $\mathfrak{m}_2>\aleph_0$.

We can assume that $T_2(n,\mathfrak{m}_2)=\bigvee$ and $T_1(n,\mathfrak{m}_1)=\bigwedge$ for all cardinal numbers $\mathfrak{m}_1$ satisfying the inequality $\aleph_0\leqslant\mathfrak{m}_1<\mathfrak{m}_2$.

Let A be the formula

$$\sim(\exists x)G(x,x)\wedge(\forall x,y)[\sim G(x,y)\vee\sim G(y,x)]\wedge$$
$$\wedge(\forall x,y,z)[G(x,y)\wedge G(y,z)\supset G(x,z)]\wedge(\forall x,y)[G(x,y)\vee x=y\vee G(y,x)\wedge$$
$$\wedge(\exists x_0)\{(\exists x_1,\dots,x_n)(\mathbf{K}_{i<j}(x_i\neq x_j)\wedge(\forall z)[G(z,x_0)\equiv(z=x_1\vee\dots\vee z=x_n)])\wedge$$
$$\wedge(\exists y)(G(x_0,y)\wedge(\forall s)\{G(s,y)\supset(\exists t)[G(s,t)\wedge G(t,y)]\})\wedge(\boldsymbol{Q}u)G(u,x_0)\}.$$

In this formula the symbol $\mathbf{K}_{i<j}$ $(x_i\neq x_j)$ denotes the conjunction of all formulas $x_i\neq x_j$ where $i,j=1,2,\dots,n$ and $i<j$.

The meaning of the formula A is this: the relation G orders the universe of discourse and there is an element x_0 such that 1° there are exactly n elements preceding x_0, 2° the set of elements following x_0 contains a limit point (and hence is infinite); the quantifier $\boldsymbol{Q}$ assigns the value $\bigvee$ to the set of elements preceding x_0.

Formula A is satisfiable in I_2. Indeed, if $\prec$ is a relation which determines a well-ordering of I_2 and G a propositional function on I_2 such that $G(y_1,y_2,\dots)\equiv y_1\prec y_2$, then G satisfies A in I_2. If A were satisfiable in a denumerable set I, then I would be an ordered set having a segment S with exactly n elements and with the corresponding rest R infinite (since R would contain at least one limit-point). Moreover, the

segment S would satisfy the equation $\boldsymbol{Q}_I(S)=\bigvee$. This, however, is impossible, since $\overline{\overline{S}}=n$, $\overline{\overline{I-S}}=\aleph_0$ and $T_I(n,\aleph_0)=\bigwedge$.

Case (b) can be treated similarly as Case (a).

Case (c). Let I_2 be a set of as small power as possible such that there is a set $I_1\subset I_2$ and infinite cardinals $\mathfrak{m}_i,\mathfrak{n}_i$ satisfying the conditions $\mathfrak{m}_i+\mathfrak{n}_i=I_i$ $(i=1,2)$ and $T_1(\mathfrak{m}_1,\mathfrak{n}_1)\neq T_2(\mathfrak{m}_2,\mathfrak{n}_2)$. Hence if $\overline{\overline{I'}}<\overline{\overline{I_2}}$, $I_1'\subset I'$ and $\mathfrak{m}_1',\mathfrak{n}_1',\mathfrak{m}',\mathfrak{n}'$ are infinite cardinals such that

$$\mathfrak{m}_1'+\mathfrak{n}_1'=\overline{\overline{I_1'}},\qquad \mathfrak{m}'+\mathfrak{n}'=\overline{\overline{I'}},$$

then $T_{I_1'}(\mathfrak{m}_1',\mathfrak{n}_1')=T_{I'}(\mathfrak{m}',\mathfrak{n}')$. In particular, if $\overline{\overline{I}}=\aleph_0$, then $T_I(\aleph_0,\aleph_0)\neq T_{I_2}(\mathfrak{m}_2,\mathfrak{n}_2)$. We can assume that $T_I(\aleph_0,\aleph_0)=\bigwedge$ and $T_{I_2}(\mathfrak{m}_2,\mathfrak{n}_2)=\bigvee$.

Now let A be t.. formula

$$\sim(\exists x)G(x,x)\wedge(\forall x,y)[\sim G(x,y)\vee\sim G(y,x)]\wedge$$
$$\wedge(\forall x,y,z)[G(x,y)\wedge G(y,z)\supset G(x,z)]\wedge$$
$$\wedge(\forall x,y)[G(x,y)\vee x=y\vee G(y,x)]\wedge$$
$$\wedge(\exists x_0)\{(\boldsymbol{Q}u)G(u,x_0)\wedge(\exists y_1)[G(y_1,x_0)\wedge(\forall s)\{G(s,y_1)\supset(\exists t)[G(s,t)\wedge G(t,y_1)]\}]\wedge$$
$$\wedge(\exists y_2)[G(x_0,y_2)\wedge(\forall s)\{G(s,y_2)\supset(\exists t)[G(s,t)\wedge G(t,y_2)]\}]\}.$$

The intuitive meaning of this formula is: the relation G orders the universe of discourse and there is an element x_0 such that 1° the quantifier $\boldsymbol{Q}$ assigns the value $\bigvee$ to the set of elements preceding x_0; 2° both the set of elements preceding x_0 and the set of elements following x_0 possess limit-points (and thus are infinite).

If $\prec$ is a relation which determines a well ordering of I_2 such that I_2 has a segment of power $\mathfrak{m}_2$ and the corresponding rest of power $\mathfrak{n}_2$, then the propositional function G defined thus:

$$G(y_1,y_2,\dots)\equiv y_1\prec y_2$$

satisfies A in I_2.

If A were satisfiable in a denumerable set I, then I would be an ordered set having a segment S and the corresponding rest such that $\boldsymbol{Q}_I(S)=\bigvee$ and $\overline{\overline{S}}=\overline{\overline{R}}=\aleph_0$. This, however, would contradict the formula $T_I(\aleph_0,\aleph_0)=\bigwedge$.

Theorem 7 is thus proved.

The Skolem-Löwenheim's theorem has been generalized by Tarski (see Skolem [6], p. 161) in the following way: if a formula (of the classical functional calculus) is satisfiable in an infinite set, then it is satisfiable in every infinite set. This theorem cannot be extended to the case of arbitrary quantifiers. We have in fact the following

THEOREM 8. *If $\boldsymbol{\exists}$ and $\boldsymbol{\forall}$ occur among $\boldsymbol{Q}^1,\ldots,\boldsymbol{Q}^s$ and if each formula of* (S) *satisfiable in an infinite set is satisfiable in every infinite set, then for each set J the quantifiers $\boldsymbol{Q}_J^1,\ldots,\boldsymbol{Q}_J^s$ are all definable in terms of $\boldsymbol{\exists}$ and $\boldsymbol{\forall}$.*

Proof. It follows from theorem 7 that the quantifiers $\boldsymbol{Q}^1,\ldots,\boldsymbol{Q}^s$ do not distinguish infinite powers. Since there exist formulas A involving no symbols for quantifiers other than $\boldsymbol{\exists},\boldsymbol{\forall},\boldsymbol{Q}$ and such that if $\boldsymbol{Q}$ possesses the property (E), then A is satisfiable in denumerable sets only (cf. the proof of theorem 2), we can assume that quantifiers definable in terms of $\boldsymbol{Q}^1,\ldots,\boldsymbol{Q}^s$ do not have the property (E).

In section 2(e) we have shown that if $\overline{\overline{I}}=\aleph_0$ and $\boldsymbol{Q}_I$ is a non-numerical quantifier such that neither $\boldsymbol{Q}_I$ nor its dual $\boldsymbol{Q}_I^*$ has the property (E), then one of the quantifiers $\boldsymbol{S}_I,\boldsymbol{S}_I^0,\boldsymbol{S}_I^*$ is expressible as a Boolean polynomial in $\boldsymbol{Q}_I$, $\overset{(n)}{\sum_I}$, and $\overset{(m)}{\prod_I}$ $(n,m=1,2,\ldots)$. It will be sufficient to consider only the case when $\boldsymbol{S}_I$ is thus expressible. Let us therefore assume that

$$\boldsymbol{S}_I=\Phi\Big(\boldsymbol{Q}_I,\overset{(n_1)}{\sum_I},\ldots,\overset{(n_k)}{\sum_I},\overset{(m_1)}{\prod_I},\ldots,\overset{(m_l)}{\prod_I}\Big) \tag{9}$$

where Φ is a Boolean polynomial. For an arbitrary $X\subset I$ we have thus

$$\boldsymbol{S}_I(X)=\Phi\Big(\boldsymbol{Q}_I(X),\overset{(n_1)}{\sum_I}(X),\ldots,\overset{(n_k)}{\sum_I}(X),\overset{(m_1)}{\prod_I}(X),\ldots,\overset{(m_l)}{\prod_I}(X)\Big). \tag{10}$$

Let $T_J^S,T_J^Q,T_J^{n_1},\ldots,T_J^{n_k},\widetilde{T}_J^{m_1},\ldots,\widetilde{T}_J^{m_l}$ be functions determining the quantifiers $\boldsymbol{S}_J,\boldsymbol{Q}_J,\overset{(n_1)}{\sum_J},\ldots,\overset{(n_k)}{\sum_J},\overset{(m_1)}{\prod_J},\ldots,\overset{(m_l)}{\prod_J}$ (J — an arbitrary set). Substituting in (10) for X a set with exactly n elements we obtain

$$\Phi\big(T_I^Q(n,\aleph_0),T_I^{n_1}(n,\aleph_0),\ldots,T_I^{n_k}(n,\aleph_0),\widetilde{T}_I^{m_1}(n,\aleph_0),\ldots,\widetilde{T}_I^{m_l}(n,\aleph_0)\big)=\vee. \tag{11}$$

If we take for X a set whose complement contains exactly n elements and a set which is infinite together with its complement, we obtain similarly

$$\Phi\big(T_I^Q(\aleph_0,n),T_I^{n_1}(\aleph_0,n),\ldots,T_I^{n_k}(\aleph_0,n),\widetilde{T}_I^{m_1}(\aleph_0,n),\ldots,\widetilde{T}_I^{m_l}(\aleph_0,n)\big)=\wedge, \tag{12}$$

$$\Phi\big(T_I^Q(\aleph_0,\aleph_0),T_I^{n_1}(\aleph_0,\aleph_0),\ldots,T_I^{n_k}(\aleph_0,\aleph_0),\widetilde{T}_I^{m_1}(\aleph_0,\aleph_0),\ldots,\widetilde{T}_I^{m_l}(\aleph_0,\aleph_0)\big)=\wedge. \tag{13}$$

Assuming that $\boldsymbol{Q}$ does not distinguish infinite powers, we have for an arbitrary infinite set J and for arbitrary infinite cardinals $\mathfrak{m},\mathfrak{n}$ with $\mathfrak{m}+\mathfrak{n}=\overline{\overline{J}}$

$$T_I^Q(n,\aleph_0)=T_J^Q(n,\overline{\overline{J}}),\qquad T_I^Q(\aleph_0,n)=T_J^Q(\overline{\overline{J}},n),\qquad T_I^Q(\aleph_0,\aleph_0)=T_J^Q(\mathfrak{m},\mathfrak{n}). \tag{14}$$

Similar equations hold for functions T_J^n and $\widetilde{T}_I^m$ $(n,m=1,2,\ldots)$ since the quantifiers $\overset{(n)}{\sum}$ and $\overset{(m)}{\prod}$ do not distinguish infinite powers. Formulas (11)-(13) yield now

$$\Phi\big(T_J^Q(n,\overline{\overline{J}}),T_J^{n_1}(n,\overline{\overline{J}}),\ldots,T_J^{n_k}(n,\overline{\overline{J}}),\widetilde{T}_J^{m_1}(n,\overline{\overline{J}}),\ldots,\widetilde{T}_J^{m_l}(n,\overline{\overline{J}})\big)=\vee=T_J^S(n,\overline{\overline{J}}),$$

$$\Phi\big(T_J^Q(\overline{\overline{J}},n),T_J^{n_1}(\overline{\overline{J}},n),\ldots,T_J^{n_k}(\overline{\overline{J}},n),\widetilde{T}_J^{m_1}(\overline{\overline{J}},m),\ldots,\widetilde{T}_J^{m_l}(\overline{\overline{J}},n)\big)=\wedge=T_J^S(\overline{\overline{J}},n),$$

$$\Phi\big(T_J^Q(\mathfrak{m},\mathfrak{n}),T_J^{n_1}(\mathfrak{m},\mathfrak{n}),\ldots,T_J^{n_k}(\mathfrak{m},\mathfrak{n}),\widetilde{T}_J^{m_1}(\mathfrak{m},\mathfrak{n}),\ldots,\widetilde{T}_J^{m_l}(\mathfrak{m},\mathfrak{n})\big)=\wedge=T_J^S(\mathfrak{m},\mathfrak{n}),$$

which proves that formula (9) holds for every infinite set I.

Let us apply this result to $\boldsymbol{Q}=\boldsymbol{Q}^j$ $(j=1,2,\ldots,s)$. If $\boldsymbol{Q}_I^j$ (where $\overline{\overline{I}}=\aleph_0$) were a non-numerical quantifier, then according to the result obtained above formula (9) (or a similar formula with $\boldsymbol{S}$ replaced by $\boldsymbol{S}^0$ or $\boldsymbol{S}^*$) would hold for every infinite set I. Since in the proof of theorem 3 we have exhibited formulas involving exclusively the quantifiers $\boldsymbol{S},\boldsymbol{\exists},\boldsymbol{\forall}$ (or $\boldsymbol{S}^0,\boldsymbol{\exists},\boldsymbol{\forall}$) satisfiable in denumerable sets only, we see that if $\boldsymbol{Q}_I^j$ (with $\overline{\overline{I}}=\aleph_0$) were non-numerical, then there would be formulas satisfiable in some infinite sets but not satisfiable in all of them. Hence $\boldsymbol{Q}_I^j$ must be numerical for $\overline{\overline{I}}=\aleph_0$: $\boldsymbol{Q}_I^j=\Psi\Big(\overset{(n_1)}{\sum_I},\ldots,\overset{(n_k)}{\sum_I},\overset{(m_1)}{\prod_I},\ldots,\overset{(m_l)}{\prod_I}\Big)$ where Ψ is a Boolean polynomial. Since neither $\boldsymbol{Q}^j$ nor $\overset{(n)}{\sum}$ nor $\overset{(m)}{\prod}$ distinguishes infinite powers, formulas (14) hold for these quantifiers, whence by the same method as above we obtain the formula $\boldsymbol{Q}_J^j=\Psi\Big(\overset{(n_1)}{\sum_J},\ldots,\overset{(n_k)}{\sum_J},\overset{(m_1)}{\prod_J},\ldots,\overset{(m_l)}{\prod_J}\Big)$ for every infinite set J.

Our theorem is thus proved for infinite J. For finite J the assertion of the theorem is evident since for $\overline{\overline{J}}<\aleph_0$ each quantifier $\boldsymbol{Q}_J$ limited to J is definable in terms of $\boldsymbol{\exists}$ and $\boldsymbol{\forall}$.

Remark. The above proof gives in fact a little stronger result than that stated in the theorem: we have shown that under the assumptions of theorem 8 each $\boldsymbol{Q}_J^j$ with an infinite J is expressible as a Boolean polynomial in the quantifiers $\overset{(n)}{\sum_J},\overset{(m)}{\prod_J}$ $(n,m=1,2,\ldots)$ and the form of this polynomial is independent of J. It is not true, however, that under the assumptions of theorem 8 the unlimited quantifiers $\boldsymbol{Q}^1,\ldots,\boldsymbol{Q}^s$ are necessarily definable in terms of $\boldsymbol{\exists},\boldsymbol{\forall}$. For let $\boldsymbol{Q}$ be the unlimited quantifier such that $\boldsymbol{Q}_I=\sum_I$ for infinite I and $\boldsymbol{Q}_I=\prod_I$ for finite I. If $\boldsymbol{Q}^1=\boldsymbol{\exists}$, $\boldsymbol{Q}^2=\boldsymbol{\forall}$, $\boldsymbol{Q}^3=\boldsymbol{Q}^4=\ldots=\boldsymbol{Q}^s=\boldsymbol{Q}$, then the assumptions of theorem 8 are satisfied but $\boldsymbol{Q}^3,\ldots,\boldsymbol{Q}^s$ are not definable in terms of $\boldsymbol{\exists}$ and $\boldsymbol{\forall}$.

6. The monadic calculus. Let (S^M) be the subsystem of (S) whose formulas contain only monadic functional variables (*i. e.*, functional variables with one argument). (S_n^M) will denote the subsystem of (S^M) whose formulas contain only the functional variables $F_1, \ldots, F_n$.

THEOREM 9. *There are quantifiers* $\boldsymbol{Q}^1, \ldots, \boldsymbol{Q}^s$ *such that the set of those formulas of* (S^M) *which are satisfiable in a denumerable set is not recursive.*

Proof. For each set A of integers we denote by $H(A)$ the set of integers p with the following property: there are $n_1, n_2, \ldots, n_p$ in A such that $n_1+n_2+\ldots+n_p \in A$. We shall show the existence of an A such that the set $H(A)$ is not recursive. Since there are only denumerably many recursive sets, it is sufficient to prove that there are more than $\aleph_0$ sets of the form $H(A)$ and this results immediately from the

LEMMA [10]). *If* A_0 *is the set of integers* $1, 2, 2^3, 2^9, \ldots, 2^{3^n}, \ldots$ *then* $H(A_1) \neq H(A_2)$ *for arbitrary subsets* A_1, A_2 *of* A_0 *such that* $A_1 \neq A_2$ *and* $1 \in A_1$, $1 \in A_2$.

Indeed, assuming that $p = 2^{3^n} \in A_1 - A_2$ we have $2^{3^n} \in H(A_1)$ for $2^{3^n} = 1+1+\ldots+1$ and $1 \in A_1$. From $p \in H(A_2)$ it would follow that in A_2 there are integers $n_0, n_1, \ldots, n_p$ $(n_1 \leqslant n_2 \leqslant \ldots \leqslant n_p)$ such that $n_0 = n_1+n_2+ +\ldots+n_p$. Not all n_j are 1 since otherwise we should have $n_0 = p$ and p would be an element of A_2. Hence $n_p > 1$ and we can assume $n_p = 2^{3^s}$ with $s \geqslant 0$. Since $p > 1$ we have also $n_0 = 2^{3^t}$ with $t > 0$. Evidently $n_0 > n_p$ whence $3^t > 3^s$ and $t-1 \geqslant s$. Since $n_0 \leqslant p n_p$, we obtain $p \geqslant n_0/n_p = 2^{3^t - 3^s} \geqslant 2^{3^t - 3^{t-1}} = 2^{3^{t-1} \cdot 2}$. On the other hand $p < n_0 = 2^{3^t}$ and, since $p = 2^{3^n}$, we obtain the inequalities $2^{3^{t-1} \cdot 2} \leqslant 2^{3^n} < 2^{3^t}$, which entail a contradiction.

Let A be a set such that $H(A)$ is not recursive and let $\boldsymbol{Q}_I$ be a quantifier limited to a denumerable set I such that $\boldsymbol{Q}_I(X) = \bigvee$ if and only if $\overline{\overline{X}} \in A$. We take as $\boldsymbol{Q}^1$ and $\boldsymbol{Q}^2$ the quantifiers $\boldsymbol{V}$ and $\boldsymbol{\exists}$ and as $\boldsymbol{Q}^3, \ldots, \boldsymbol{Q}^s$ such quantifiers that $\boldsymbol{Q}_I^3 = \ldots = \boldsymbol{Q}_I^s = \boldsymbol{Q}_I$.

We consider now an infinite sequence of monadic functional variables $F_1, F_2, \ldots$ We shall write $F_j^0(x)$ for $F_j(x)$ and $F_j^1(x)$ for $\sim F_j(x)$. The 2^n formulas $F_1^{i_1}(x) \wedge F_2^{i_2}(x) \wedge \ldots \wedge F_n^{i_n}(x)$ we call *constituents of order* n. The lexicographical ordering of the n-tuples $(i_1, \ldots, i_n)$ determines the similar ordering of the constituents. The first n constituents of order n will be denoted by $S_1(x), \ldots, S_n(x)$.

Let W_n be the formula of (S^M)

$$[(\boldsymbol{Q}^3 x) S_1(x)] \wedge [(\boldsymbol{Q}^3 x) S_2(x)] \wedge \ldots \wedge \wedge [(\boldsymbol{Q}^3 x) S_n(x)] \wedge \big[(\boldsymbol{Q}^3 x)\big(S_1(x) \vee S_2(x) \vee \ldots \vee S_n(x)\big)\big].$$

[10]) The proof of this lemma has been kindly communicated to me by J. Mycielski. It remains an open question whether there are recursive sets A such that $H(A)$ is not recursive.

It is evident that this formula is satisfiable in a denumerable set I if and only if n is in $H(A)$; this proves that the set of formulas of (S^M) satisfiable in I is not recursive.

Theorem 9 is thus proved. It follows of course from this theorem that there are quantifiers $\boldsymbol{Q}^1, \ldots, \boldsymbol{Q}^s$ such that the set of formulas of (S^M) true in a denumerable set is not recursive.

The characterization of quantifiers for which the set of true formulas of (S^M) is recursive remains an open problem. In the next theorem we give examples of non-trivial quantifiers satisfying this condition:

THEOREM 10. *Let* $\mathfrak{m}_1 < \mathfrak{m}_2 < \ldots < \mathfrak{m}_s$ *be s cardinals such that* $\mathfrak{m}_2, \ldots, \mathfrak{m}_s$ *are infinite and* $\mathfrak{m}_1$ *either is* 1 *or is infinite. Let* $\boldsymbol{Q}^1, \boldsymbol{Q}^2, \ldots, \boldsymbol{Q}^s$ *be quantifiers such that for each* I

$$\{\boldsymbol{Q}_I^j(F) = \bigvee\} \equiv (\overline{\overline{F}} \geqslant \mathfrak{m}_j), \qquad j = 1, 2, \ldots, s.$$

Then the set of true formulas of (S^M) *is recursive* [11]).

Remark. If $\mathfrak{m}_1 = 1$, then $\boldsymbol{Q}^1 = \boldsymbol{\exists}$; if $\mathfrak{m}_2 = \aleph_0$, then $\boldsymbol{Q}^2 = \sim \boldsymbol{S}$; if $\mathfrak{m}_3 = \aleph_1$, then $\boldsymbol{Q}^3 = \sim \boldsymbol{P}$.

The proof of theorem 10 will be based on some lemmas:

(a) *Quantifiers* $\boldsymbol{Q}^j$ *satisfy the equations*

$$\boldsymbol{Q}_I^j(F \vee G) = \boldsymbol{Q}_I^j(F) \vee \boldsymbol{Q}_I^j(G) \qquad (\textit{additivity}), \qquad \boldsymbol{Q}_I^j(\bigwedge) = \bigwedge.$$

(b) *If* $Z_1, \ldots, Z_n$ *are formulas of* (S) *containing the free variable* x_i *and if* $W_1, \ldots, W_n$ *are formulas of* (S) *not containing* x_i, *then the formula*

$$(\boldsymbol{Q}^j x_i)[(Z_1 \wedge W_1) \vee \ldots \vee (Z_n \wedge W_n)] \equiv [W_1 \wedge (\boldsymbol{Q}^j x_i) Z_1] \vee \ldots \vee [W_n \wedge (\boldsymbol{Q}^j x_i) Z_n]$$

is true.

We abbreviate the left-hand and the right-hand side of this equivalence as L and R respectively and denote by I an arbitrary set and by M an I-valuation. Finally we denote by F the propositional function on I with the support $\{i\}$ such that

$$F(y_1, y_2, \ldots) = val_{M(i,y_i),I}[(Z_1 \wedge W_1) \vee \ldots \vee (Z_n \wedge W_n)]$$

and by F_h and G_h $(h = 1, 2, \ldots, n)$ propositional functions on I with supports $\{i\}$ such that

$$F_h(y_1, y_2, \ldots) = val_{M(i,y_i),I}(Z_h),$$
$$G_h(y_1, y_2, \ldots) = val_{M(i,y_i),I}(W_h).$$

[11]) This theorem presents a generalization of the classical theorem of Löwenheim. For our proof see Hilbert-Ackermann [2], p. 101.

Since W_h does not contain x_l, G_h has a constant value $w_h = val_{MI}(W_h)$. Now we have

$$F(y_1, y_2, \ldots) = [w_1 \wedge val_{M(l,y_l),I}(Z_1)] \vee \ldots \vee [w_n \wedge val_{M(l,y_l),I}(Z_n)]$$
$$= [w_1 \wedge F_1(y_1, y_2, \ldots)] \vee \ldots \vee [w_n \wedge F_n(y_1, y_2, \ldots)]$$

and hence, by (a),

$$val_{MI}(L) = Q_I^j(F) = Q_I^j(w_1 \wedge F_1) \vee \ldots \vee Q_I^j(w_n \wedge F_n).$$

Similar calculations yield

$$val_{MI}(R) = [w_1 \wedge Q_I^j(F_1)] \vee \ldots \vee [w_n \wedge Q_I^j(F_n)].$$

It remains thus to verify that $w_h \wedge Q_I^j(F_h) = Q_I^j(F_h \wedge w_h)$. But this is evident if $w_h = \bigvee$ and follows from (a) if $w_h = \bigwedge$. Lemma (b) is thus proved.

Let $F_1, \ldots, F_n$ be n monadic functional variables. We put

$$A^j_{i_1 i_2 \ldots i_n} = (Q^j x)[F_1^{i_1}(x) \wedge \ldots \wedge F_n^{i_n}(x)]$$

where, as before, $F_i^0(x) = F_i(x)$ and $F_i^1(x) = \sim F_i(x)$ and denote by $\Re(x_1, \ldots, x_m)$ the least class of formulas that contains the formulas $F_h(x_l)$ and $A^j_{i_1 \ldots i_n}$ ($h = 1, 2, \ldots, n$, $j = 1, 2, \ldots, s$, $l = 1, 2, \ldots, m$, $i_t = 0$ or 1 for $t = 1, 2, \ldots, n$) and that satisfies the condition: if Z_1, Z_2 are in $\Re(x_1, \ldots, x_m)$, then so is $Z_1|Z_2$. We do not exclude the case $m = 0$. In this case we denote the class simply by $\Re$; of course $\Re$ contains only closed formulas built from formulas $A^j_{i_1 i_2 \ldots i_n}$ by means of the stroke.

(c) *For each formula Z of* (S^M) *containing* $x_1, \ldots, x_m$ *as its unique free individual variables and* $F_1, \ldots, F_n$ *as its unique functional variables there is a formula U in* $\Re(x_1, \ldots, x_m)$ *such that the formula* $Z \equiv U$ *is true. Formula U can be found explicitly if Z is explicitly given.*

If Z contains no symbols $Q^1, \ldots, Q^s$, then the assertion of (c) is evident. Let us assume the validity of (c) for formulas containing at most $p-1$ of these symbols and let Z contain p of them. It is clear that the lemma will be proved in general if we show that it holds for the case when Z has the form $(Q^j x)Z_1$ with Z_1 in $\Re(x_1, \ldots, x_m, x)$. From the definition of this class it follows that Z_1 can be represented in the form of a logical sum of formulas $F_1^{i_1}(x) \wedge \ldots \wedge F_n^{i_n}(x) \wedge W$ with W in $\Re(x_1, \ldots, x_m)$. Using (b) we obtain therefore an U in $\Re(x_1, \ldots, x_m)$ such that the formula $Z \equiv U$ is true.

We now have to find the criterion of truth for formulas in $\Re$. This criterion will be expressed by means of some auxiliary notions.

Let Φ be a Boolean polynomial in the variables $a^j_{i_1 \ldots i_n}$ ($j = 1, 2, \ldots, s$, $i_t = 0, 1$ for $t = 1, 2, \ldots, n$). A function γ assigning the values $\bigvee$, $\bigwedge$ to these variables is called an allowable valuation if

$$\gamma(a^{j+1}_{i_1 \ldots i_n}) \leqslant \gamma(a^j_{i_1 \ldots i_n})$$

for $j = 1, 2, \ldots, s-1$, $i_t = 0, 1$ for $t = 1, 2, \ldots, n$. The number of allowable valuations is obviously finite. We shall say that Φ has the property (T) if each allowable valuation gives it the value $\bigvee$.

(d) *A formula* Z *in* $\Re$ *is true if and only if it results by a substitution of* $A^j_{i_1 \ldots i_n}$ *for* $a^j_{i_1 \ldots i_n}$ *from a Boolean polynomial with property* (T).

In order to show this lemma we first assume that Φ has the property (T) and that the formula Z resulting from Φ by the substitution described in the lemma is not true in a set I. If M is an I-valuation such that $val_{MI}(Z) = \bigwedge$, then the function

$$\gamma(a^j_{i_1 \ldots i_n}) = val_{MI}(A^j_{i_1 \ldots i_n})$$

is an allowable valuation. Indeed, if $val_{MI}(A^{j+1}_{i_1 \ldots i_n}) = \bigvee$, then the set of elements x in I such that $[F_1]_M, \ldots, [F_n]_M, x$ satisfy the formula $F_1^{i_1}(x) \wedge \ldots \wedge F_n^{i_n}(x)$ in I has the cardinal number $\geqslant \mathfrak{m}_{j+1}$; hence this cardinal number is $\geqslant \mathfrak{m}_j$ and hence $val_{MI}(A^j_{i_1 \ldots i_n}) = \bigvee$. The allowable valuation γ gives to Φ the value $val_{MI}(Z)$ against the assumption that Φ has the property (T).

Let us now assume that Z results by the substitution described in the lemma from a polynomial Φ without the property (T). We are going to define a set I and an I-valuation such that $val_{MI}(Z) = \bigwedge$.

To this end we consider 2^n disjoint sets $X'_{i_1 \ldots i_n}$ each of power $\mathfrak{m}_s$. If $j = j_{i_1 \ldots i_n}$ is the greatest integer $\leqslant s$ such that $\gamma(a^j_{i_1 \ldots i_n}) = \bigvee$, then we remove from $X'_{i_1 \ldots i_n}$ as many elements as to leave a set $X_{i_1 \ldots i}$ of power $\mathfrak{m}_j$. Now we take as I the union of all sets $X_{i_1 \ldots i_n}$ and define an I-valuation M by taking as $[x_j]_M$ an arbitrary element of I ($j = 1, 2, \ldots$) and as $[F_k]_M$ the union of those $X_{i_1 \ldots i_n}$ for which $i_k = 0$ ($k = 1, 2, \ldots, n$). It is easy to see that

$$\{val_{M(h,y),I}[F_1^{i_1}(x_h) \wedge \ldots \wedge F_n^{i_n}(x_h)] = \bigvee\} \equiv (y \in X_{i_1 \ldots i_n})$$

and hence

$$\{val_{MI}(A^j_{i_1 \ldots i_n}) = \bigvee\} \equiv \{\overline{\overline{X}}_{i_1 \ldots i_n} \geqslant \mathfrak{m}_j\} \equiv \{\mathfrak{m}_{j_{i_1 \ldots i_n}} \geqslant \mathfrak{m}_j\}$$
$$\equiv \{j_{i_1 \ldots i_n} \geqslant j\} \equiv \gamma(a^j_{i_1 \ldots i_n}) = \bigvee.$$

Lemma (d) is thus proved.

Theorem 10 results immediately from lemmas (c) and (d) since the set of polynomials with the property (T) is recursive.

We conclude with a discussion of the systems (S_n^M) with $n=1,2,\dots$ In contrast to theorem 9 we have the following

THEOREM 11. *For each $n\geqslant 1$ the set of true formulas of (S_n^M) and the set of formulas of (S_n^M) which are true in any given set I are recursive.*

Proof. Let the $m=2^n$ constituents $F_1^{i_1}(x)\wedge\dots\wedge F_n^{i_n}(x)$ be denoted by $S_1(x),\dots,S_m(x)$. Each Boolean polynomial $W(x)$ in $F_1(x),\dots,F_n(x)$ which does not vanish identically has a "canonical representation"

$$S_{k_1}(x)\vee S_{k_2}(x)\vee\dots\vee S_{k_h}(x)$$

which is unique up to the order of summands. We put

$$A_W^j=(\mathbf{Q}^jx)[S_{k_1}(x)\vee\dots\vee S_{k_h}(x)]$$

and

$$A_0^j=(\mathbf{Q}^jx)[F_1(x)\wedge\sim F_1(x)]\,.$$

(a) *If A and B are formulas not containing the free variable x and M(x) and N(x) are formulas in which x occurs free, then the equivalences*

$$(\mathbf{Q}^jx)[(A\wedge M(x))\vee(B\wedge N(x))]\equiv\{(\sim A\wedge\sim B\wedge A_0^j)\vee(\sim A\wedge B\wedge(\mathbf{Q}^jx)N(x))\vee$$
$$\vee(A\wedge\sim B\wedge(\mathbf{Q}^jx)M(x))\vee(A\wedge B\wedge(\mathbf{Q}^jx)[M(x)\vee N(x)])\}$$

are true $(j=1,2,\dots,s)$.

Proof of this lemma is evident.

Let us now consider a formula Z of the form

$$M_0(x)\vee[C_1\wedge N_1(x)]\vee\dots\vee[C_p\wedge N_p(x)]$$

where $C_1,\dots,C_p$ do not contain x. For each set $i_1,\dots,i_p$ of indices $(=0,1)$ we denote by $\sum_{i_1,\dots,i_p}(x)$ the sum $M_0(x)\vee N_{i_{k_1}}(x)\vee\dots\vee N_{i_{k_t}}(x)$ where $k_1,\dots,k_t$ are all integers $k\leqslant p$ for which $i_k=0$. Denoting by $\bigvee_{i_1,\dots,i_p}$ the Boolean sum over the sets of 2^p indices we have

(b) *The equivalence*

$$(\mathbf{Q}^jx)Z\equiv\bigvee_{i_1,\dots,i_p}[C_1^{i_1}\wedge\dots\wedge C_p^{i_p}\wedge(\mathbf{Q}^jx)\sum_{i_1,\dots,i_p}(x)]$$

is true.

We show this by induction on p. If $p=1$, then we take in (a) $A=(\mathbf{Q}^jx)F_1(x)\vee\sim(\mathbf{Q}^jx)F_1(x)=A_0$, $B=C_1$, $M(x)=M_0(x)$, $N(x)=N_1(x)$. If the lemma holds for the number $p-1$, then we take in (a) $A=A_0$, $B=C_p$, $M(x)=M_0(x)\vee[C_1\wedge N_1(x)]\vee\dots\vee[C_{p-1}\wedge N_{p-1}(x)]$, $N(x)=N_p(x)$ and obtain the (true) equivalence

$$(\mathbf{Q}^jx)Z\equiv C_p^1\wedge(\mathbf{Q}^jx)\{M_0(x)\vee[C_1\wedge N_1(x)]\vee\dots\vee[C_{p-1}\wedge N_{p-1}(x)]\}\vee$$
$$\vee C_p^0\wedge(\mathbf{Q}^jx)\{[M_0(x)\vee N_p(x)]\vee[C_1\wedge N_1(x)]\vee\dots\vee[C_{p-1}\wedge N_{p-1}(x)]\}\,.$$

Lemma (b) results now immediately if we use twice the inductive assumption.

We introduce now the class $\mathfrak{R}(x_1,\dots,x_m)$ in much the same way as in the proof of theorem 10 (cf. definitions preceding lemma (c) on p. 30). The only difference is that we require from the present class $\mathfrak{R}(x_1,\dots,x_m)$ that it should contain the s formulas A_0^j and the $s(2^{2^n}-1)$ formulas A_W^j instead of the former 2^ns formulas $A_{i_1\dots i_n}^j$. Lemma (c) of the proof of theorem 10 holds in the present case and will be referred to as lemma (c_n). In order to prove this lemma it is sufficient to show its validity for Z having the form $(\mathbf{Q}^jx)Z_1$ where Z_1 is either the formula $F_1(x)\wedge\sim F_1(x)$ or the formula $[C_1\wedge S_{k_1}(x)]\vee\dots\vee[C_p\wedge S_{k_p}(x)]$ with C_h independent of x. In the former case it is sufficient to take $U=A_0^j$ and in the latter (c_n) results immediately from (b).

We introduce now the concept of a valuation allowable for a set I. This is a function γ which assigns the truth-values to the $s\cdot 2^{2^n}$ formulas A_0^j and A_W^j in such a way that there exists an assignment of cardinal numbers $\mathfrak{m}_k$ to constituents $S_k(x)$ $(k=1,2,\dots,m)$ satisfying the conditions:

1° $\mathfrak{m}_1+\mathfrak{m}_2+\dots+\mathfrak{m}_m=\overline{\overline{I}}$,

2° *if $W(x)$ is a Boolean polynomial in $F_1(x),\dots,F_n(x)$ and*

$$(15)\qquad S_{k_1}(x)\vee\dots\vee S_{k_p}(x),\qquad S_{l_1}(x)\vee\dots\vee S_{l_q}(x)$$

are the canonical representations of $W(x)$ and $\sim W(x)$, then $\gamma(A_W^j)=T^j(\sum_{i=1}^{p}\mathfrak{m}_{k_i},\sum_{i=1}^{q}\mathfrak{m}_{l_i})$ where T^j is the function which determines the quantifier $\mathbf{Q}_I^j$.

Of course one of the sums (15) disappears if $W(x)$ or $\sim W(x)$ vanishes identically; the corresponding sum of cardinals is then 0. Our definition covers the case when W is identically $\wedge$ if we agree that A_W^j is then to be interpreted as A_0^j. This we do tacitly in the rest of the proof.

(d) *If M is an I-valuation, then the function*

$$\gamma(A_W^j)=val(A_W^j)$$

is an allowable valuation for I.

Proof. We set

$$R_k = \underset{x \in I}{E}\left[val_{M(i,x),I}(S_k(x_i)) = \vee\right], \quad k = 1, 2, \ldots, m$$

and assign to $S_k(x)$ the cardinal number $\mathfrak{m}_k = \overline{\overline{R}}_k$. The condition 1° is obviously satisfied since the R_k are disjoint and I is their union. Now let (15) be the canonical representations of $W(x)$ and $\sim W(x)$. We have then

$$\underset{x \in I}{E}\left[val_{M(i,x),I}(W(x_i)) = \vee\right] = R_{k_1} \cup \ldots \cup R_{k_p},$$
$$\underset{x \in I}{E}\left[val_{M(i,x),I}(W(x_i)) = \wedge\right] = R_{l_1} \cup \ldots \cup R_{l_q}$$

and hence, by the definitions of section 2,

$$\gamma(A^j_W) = val_{MI}(A^j_W) = val_{MI}\left((Q^j x_i) W(x_i)\right)$$
$$= T^j(\overline{\overline{R_{k_1} \cup \ldots \cup R_{k_p}}}, \overline{\overline{R_{l_1} \cup \ldots \cup R_{l_q}}}) = T^j\left(\sum_{i=1}^{p} \mathfrak{m}_{k_i}, \sum_{i=1}^{q} \mathfrak{m}_{l_i}\right).$$

This proves that γ satisfies 2°.

(e) *For each valuation γ allowable for I there is an I-valuation M such that $\gamma(A^j_W) = val_{MI}(A^j_W)$ for each* W.

Proof. Let $\mathfrak{m}_k$ be the cardinal number correlated with the kth constituent $S_k(x)$ in accordance with conditions 1° and 2°. Let further $I = R_1 \cup \ldots \cup R_m$ be a partition of I into m disjoint sets such that $\overline{\overline{R}}_k = \mathfrak{m}_k$ $(k = 1, 2, \ldots, m)$. We take as $[x_i]_M$ an arbitrary element of I $(i = 1, 2, \ldots)$ and define $[F_i]_M$ as the union of sets R_k corresponding to constituents $S_k(x)$ contained in $F_i(x)$ $(i = 1, 2, \ldots, n)$. It is then easy to show that

$$\{val_{M(h,y),I}(S_k(x_h)) = \vee\} \equiv \{y \in R_k\}. \tag{16}$$

If $W(x)$ is a polynomial in $F_1(x), \ldots, F_n(x)$ and (15) are the canonical representations of $W(x)$ and of $\sim W(x)$, then it follows from (16) that

$$\{val_{M(h,y),I}(W(x_h)) = \vee\} \equiv \{y \in R_{k_1} \cup \ldots \cup R_{k_p}\},$$
$$\{val_{M(h,y),I}(W(x_h)) = \wedge\} \equiv \{y \in R_{l_1} \cup \ldots \cup R_{l_q}\},$$

and hence, by the definition of the function *val*,

$$val_{MI}(A^j_W) = T^j\left(\sum_{i=1}^{p} \mathfrak{m}_{k_i}, \sum_{i=1}^{q} \mathfrak{m}_{l_i}\right).$$

Since γ is an allowable valuation, the right-hand side of this equation is equal to $\gamma(A^j_W)$. Lemma (e) is thus proved.

We are now able to prove theorem 11.

Let I be a set. The set of formulas in $\mathfrak{R}$ which have the value $\vee$ for a single fixed valuation γ is of course recursive and so is the set of formulas which have the value $\vee$ for a finite set of such valuations. Since the set of valuations allowable for I is finite, it follows by lemmas (d) and (e) that the set of formulas in $\mathfrak{R}$ which are true in I is recursive. By lemma (c_n) for each closed formula Z of (S^M_n) we can find effectively a formula in $\mathfrak{R}$ which is true in I if and only if Z is true in I. Hence the set of formulas which are true in I is recursive, which proves the second half of the theorem.

Now denote by $\mathfrak{K}_I$ the set of valuations allowable for I. The number of such sets is of course finite (since so is the set of functions assigning truth values to formulas A^j_0 and A^j_W). Let $\mathfrak{K}$ be the union of all different $\mathfrak{K}_I$'s. Replacing in the previous proof the words "valuations allowable for I" by "valuations which belong to $\mathfrak{K}$", we obtain the proof of the first half of the theorem.

In spite of its generality (or perhaps just because of its generality) theorem 11 has no practical applications. We illustrate this by means of the following example: let Q^1, Q^2 be quantifiers such that for each infinite set I and for $X \subset I$

$$\{Q^1_I(X) = \vee\} \equiv \{\overline{\overline{X}} \text{ is a prime}\}, \quad \{Q^2_I(X) = \vee\} \equiv \{\overline{\overline{X}} \text{ has the form } 2^{2^n}+1\}.$$

The problem whether the formula

$$[(Q^1x)F(x) \wedge (Q^2x)F(x)] \supset [(\overset{(3)}{\Sigma} x)F(x) \vee (\overset{(5)}{\Sigma} x)F(x)$$
$$\vee (\overset{(17)}{\Sigma} x)F(x) \vee (\overset{(257)}{\Sigma} x)F(x) \vee (\overset{(65537)}{\Sigma} x)F(x)]\ ^{12)}$$

is true in an infinite set I is equivalent to the famous number-theoretical problem whether there are more than 5 Fermat primes. It is, however, impossible to solve this problem on the basis of theorem 11 in spite of the fact that theorem 11 asserts the existence of a finitary method for testing whether an individually given formula is or is not true in I. Such a test would indeed be possible if we knew effectively the recursion equations for the characteristic function of the set of formulas which are true in I. Unfortunately our proof of theorem 11 does not provide us with those equations. We have merely proved their existence (in a non-effective way) and cannot therefore draw any practical consequence from our result.

12) Symbols $\overset{(n)}{\Sigma}$ occurring in this formula denote quantifiers defined in section 2(b).

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On computable sequences

by

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A real number α $(0<\alpha<1)$ is said to be *computable* (cf. Robinson [9], Rice [8]) if there is a general recursive function φ such that

(i) $|\alpha-\varphi(n)/n|<1/n$ for $n=1,2,\ldots$

This definition is equivalent to each of the following ones [1]):

(ii) There is a general recursive function ψ such that

$$\alpha=\sum_{n=1}^{\infty}\psi(n)/10^n \quad \text{and} \quad \psi(n)<10 \quad \text{for} \quad n=1,2,\ldots$$

(iii) The relation R which p bears to q if and only if $p/q<\alpha$ is general recursive. (In other words the function ϑ such that $\vartheta(p,q)\leqslant 1$ and $\{\vartheta(p,q)=1\}\equiv\{p/q<\alpha\}$ is general recursive [2]).)

Several other equivalent formulations of (i) are known.

Let us now pass from numbers to sequences. If we replace in the definitions given above α by α_k and φ,ψ,ϑ by $\varphi_k,\psi_k,\vartheta_k$ where the index k runs over integers and if we further require that these functions be general recursive in all variables (including "k"), then we obtain three definitions of what may be called *computable sequences*. It will be proved below that no two of these definitions and of a couple of others, which we shall formulate later, are equivalent.

There is no doubt that of these various definitions the one which best expresses the existence of an algorithm permitting one to calculate uniformly the terms of a sequence with any desired degree of accuracy is that which corresponds to (i). The other definitions represent merely a mathematical curiosity. It seems to us, however, that the following circumstance deserves emphasis: if we replace in the definitions (i)-(iii)

[1]) The equivalence of these definitions has been first observed by Robinson [9]. Cf. further Rice [8] and Myhill [6].

[2]) These definitions have been formulated by Mazur [3]. The definition given by Rice [8] is equivalent to the first of these definitions.