

ON A GENERALIZATION OF THE PELL SEQUENCE

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Abstract. The Pell sequence $(P_n)_{n=0}^{\infty}$ is the second order linear recurrence defined by $P_n = 2P_{n-1} + P_{n-2}$ with initial conditions $P_0 = 0$ and $P_1 = 1$. In this paper, we investigate a generalization of the Pell sequence called the k -generalized Pell sequence which is generated by a recurrence relation of a higher order. We present recurrence relations, the generalized Binet formula and different arithmetic properties for the above family of sequences. Some interesting identities involving the Fibonacci and generalized Pell numbers are also deduced.

Keywords: generalized Fibonacci number; generalized Pell number; recurrence sequence

MSC 2020: 11B37, 11B39

1. INTRODUCTION

There are many integer sequences which are used in almost every field of modern sciences. For instance, the Fibonacci sequence $(F_n)_{n=0}^{\infty}$ is one of the most famous and curious numerical sequences in mathematics and has been widely studied in the literature. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation.

Dresden and Du in [4] consider a generalization of the Fibonacci sequence consisting of a recurrence relation of higher order. Specifically, they consider, for an integer $k \geq 2$, the k -generalized Fibonacci sequence which is like the Fibonacci sequence but

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starting with $0, 0, \dots, 0, 1$ (a total of k terms) and each term afterwards is the sum of the k preceding terms. In [4], a Binet-style formula that can be used to produce the k -generalized Fibonacci numbers and interesting arithmetic properties of these numbers is given. The k -generalized Fibonacci numbers and their properties have been studied by various authors (for more details see [2], [12], [13], [14], [16]). Other generalizations of the Fibonacci sequence have also been studied (see, for example, [3], [8], [15]).

Also, there is the Pell sequence, which is as important as the Fibonacci sequence. The Pell sequence $(P_n)_{n=0}^\infty$ is defined by the recurrence $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$. Further details about the Pell sequence can be found, for instance, in [1], [5], [9], [11].

In this paper we study, for an integer $k \geq 2$, a generalization of the Pell sequence which is generated by a recurrence relation of higher order; i.e., we consider the k -generalized Pell sequence or, for simplicity, the k -Pell sequence $P^{(k)} = (P_n^{(k)})_{n=-(k-2)}^\infty$ given by the recurrence

$$(1.1) \quad P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \dots + P_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \dots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$.

We shall refer to $P_n^{(k)}$ as the n th k -Pell number. We note that this generalization is in fact a family of sequences where each new choice of k produces a distinct sequence. For example, the usual Pell sequence $(P_n)_{n=0}^\infty$ is obtained for $k = 2$. Below we present the values of these numbers for the first few values of k and $n \geq 1$.

k	Name	First nonzero terms
2	Pell	1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, ...
3	3-Pell	1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, ...
4	4-Pell	1, 2, 5, 13, 34, 88, 228, 591, 1532, 3971, 10293, 26680, 69156, ...
5	5-Pell	1, 2, 5, 13, 34, 89, 232, 605, 1578, 4116, 10736, 28003, 73041, ...

Table 1. First nonzero k -Pell numbers

Generalized Pell numbers and their properties have been studied by some authors (see [8], [7], [10]). In [7], Kiliç presented some relations involving the usual Fibonacci and k -Pell numbers showing that the k -Pell numbers can be expressed as the summation of the usual Fibonacci numbers. The authors of [10] gave a new generalization of the Pell numbers in matrix representation and showed that the sums of the generalized Pell numbers could be derived directly using this representation.

The first interesting fact about the k -Pell sequence, showed by Kiliç in [7], is that the first terms in $P^{(k)}$ are Fibonacci numbers. In fact, Kiliç proved (in our notation)

that

$$(1.2) \quad P_n^{(k)} = F_{2n-1} \quad \text{for all } 1 \leq n \leq k+1,$$

while the next term is $P_{k+2}^{(k)} = F_{2k+3} - 1$. In addition, it was also proved in [7] that if $k+2 \leq n \leq 2k+2$, then

$$(1.3) \quad P_n^{(k)} = F_{2n-1} - \sum_{j=1}^{n-k-1} F_{2j-1} F_{2(n-k-j)}.$$

In this paper, we investigate the k -generalized Pell sequences and present recurrence relations, the generalized Binet formula, and different arithmetic properties for $P^{(k)}$. Some interesting identities involving the Fibonacci and generalized Pell numbers are also deduced and some well-known properties of $P^{(2)}$ are generalized to the sequence $P^{(k)}$. We also exhibit a good approximation to the n th k -Pell number and show the exponential growth of $P^{(k)}$.

2. PRELIMINARY RESULTS

First of all, we denote the characteristic polynomial of the k -Pell sequence $P^{(k)}$ by

$$\Phi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \dots - x - 1.$$

In 2013, Wu and Zang (see [17]) showed that if a_1, a_2, \dots, a_m are positive integers satisfying $a_1 \geq a_2 \geq \dots \geq a_m$, then for the polynomial

$$p(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \dots - a_{m-1} x - a_m,$$

we have:

- (i) The polynomial $p(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$;
- (ii) The other $m - 1$ zeros of $p(x)$ lie within the unit circle in the complex plane.

From the above we deduce that $\Phi_k(x)$ has exactly one positive real zero, and that it is located between 2 and 3. Throughout this paper, $\gamma := \gamma(k)$ denotes that single zero which is a Pisot number of degree k , since the other zeros of the characteristic polynomial $\Phi_k(x)$ are strictly inside the unit circle. To simplify notation, we shall omit the dependence on k of γ whenever no confusion may arise. This important property of γ leads us to call it the *dominant root* of $P^{(k)}$. Since γ is a Pisot number of minimal polynomial $\Phi_k(x)$, it follows that this polynomial is irreducible over $\mathbb{Q}[x]$.

We now consider, for each integer $k \geq 2$, the function $h_k(x)$ defined by

$$(2.1) \quad h_k(x) = (x-1)\Phi_k(x) = x^{k+1} - 3x^k + x^{k-1} + 1.$$

Since $P^{(k)}$ is a linear recurrence of order k with characteristic polynomial $\Phi_k(x)$ and $\Phi_k(x)$ divides $h_k(x)$, we deduce that $P^{(k)}$ is also a linear recurrence of order $k+1$ with characteristic polynomial $h_k(x)$. Hence, we obtain our first result of the paper, which is a “shift formula”, that will be used in the sequel.

Theorem 2.1. *Let $k \geq 2$ be integer. Then*

$$P_n^{(k)} = 3P_{n-1}^{(k)} - P_{n-2}^{(k)} - P_{n-k-1}^{(k)} \quad \text{for all } n \geq 3.$$

One can use Theorem 2.1 and mathematical induction to give alternative proofs of expressions (1.2) and (1.3) (details are left to the reader). Another application of Theorem 2.1 will enable us to derive an extended version of (1.3) in the following form:

Theorem 2.2. *Let $k \geq 2$ be an integer. Then*

$$P_n^{(k)} = F_{2n-1} - \sum_{j=1}^{n-k-1} F_{2j} P_{n-k-j}^{(k)} \quad \text{for all } n \geq k+2.$$

Note that Theorem 2.2 immediately shows that the n th k -Pell number does not exceed the Fibonacci number with index $2n-1$, i.e.,

$$P_n^{(k)} < F_{2n-1} \quad \text{holds for all } k \geq 2 \text{ and } n \geq k+2.$$

Proof. We shall prove Theorem 2.2 by induction on n . According to (1.3), we have

$$P_{k+2}^{(k)} = F_{2k+3} - 1 = F_{2k+3} - F_2 P_1^{(k)},$$

and

$$P_{k+3}^{(k)} = F_{2k+5} - 5 = F_{2k+5} - F_2 P_2^{(k)} - F_4 P_1^{(k)}.$$

Then the result holds for $n = k+2$ and $n = k+3$. Let $s \geq k+4$ be an integer and suppose that

$$P_l^{(k)} = F_{2l-1} - \sum_{j=1}^{l-k-1} F_{2j} P_{l-k-j}^{(k)} \quad \text{holds for all } k+4 \leq l \leq s.$$

We have to prove that

$$P_{s+1}^{(k)} = F_{2s+1} - \sum_{j=1}^{s-k} F_{2j} P_{s+1-k-j}^{(k)}.$$

Indeed, by Theorem 2.1 and the induction hypothesis we get

$$P_{s+1}^{(k)} = 3F_{2s-1} - 3 \sum_{j=1}^{s-k-1} F_{2j} P_{s-k-j}^{(k)} - F_{2s-3} + \sum_{j=1}^{s-k-2} F_{2j} P_{s-1-k-j}^{(k)} - P_{s-k}^{(k)},$$

which implies

$$\begin{aligned} P_{s+1}^{(k)} &= 3F_{2s-1} - F_{2s-3} - 3 \sum_{j=2}^{s-k-1} F_{2j} P_{s-k-j}^{(k)} + \sum_{j=1}^{s-k-2} F_{2j} P_{s-1-k-j}^{(k)} \\ &\quad - 3F_2 P_{s-k-1}^{(k)} - P_{s-k}^{(k)}. \end{aligned}$$

From the above we have, after some elementary algebra, that

$$P_{s+1}^{(k)} = F_{2s+1} - \sum_{j=1}^{s-k-2} (3F_{2j+2} - F_{2j}) P_{s-k-(j+1)}^{(k)} - 3P_{s-k-1}^{(k)} - P_{s-k}^{(k)},$$

and therefore

$$P_{s+1}^{(k)} = F_{2s+1} - \sum_{j=3}^{s-k} F_{2j} P_{s+1-k-j}^{(k)} - F_4 P_{s-k-1}^{(k)} - F_2 P_{s-k}^{(k)}.$$

Consequently

$$P_{s+1}^{(k)} = F_{2s+1} - \sum_{j=1}^{s-k} F_{2j} P_{s+1-k-j}^{(k)}.$$

This proves Theorem 2.2. □

3. MAIN RESULTS

This section is devoted to stating and proving the main results of the paper. These results are concerned with the generalized Binet formula for $P^{(k)}$ and its exponential growth, in which we demonstrate that the k -Pell numbers grow at an exponential rate equal to the dominant root γ , extending a result known for the usual Pell numbers. We also show that a good approximation to the n th k -Pell number is just the term of the Binet-style formula involving the dominant root.

We summarize the main results in the following theorem.

Theorem 3.1. *Let $k \geq 2$ be an integer. Then*

(a) *For all $n \geq 2 - k$, we have*

$$P_n^{(k)} = \sum_{i=1}^k g_k(\gamma_i) \gamma_i^n \quad \text{and} \quad |P_n^{(k)} - g_k(\gamma) \gamma^n| < \frac{1}{2},$$

where $\gamma := \gamma_1, \gamma_2, \dots, \gamma_k$ are the roots of characteristic polynomial $\Phi_k(x)$ and

$$(3.1) \quad g_k(z) := \frac{z - 1}{(k + 1)z^2 - 3kz + k - 1}.$$

(b) *For all $n \geq 1$, we have*

$$(3.2) \quad \gamma^{n-2} \leq P_n^{(k)} \leq \gamma^{n-1}.$$

In order to prove Theorem 3.1, we establish some lemmas which give us interesting properties of the dominant root of $P^{(k)}$, and which we believe are of independent interest.

3.1. Generalized Binet formula. In [6], Kalman proved that if $(a_n)_{n \geq 0}$ is a linear recurrence sequence of order $k \geq 2$ with initial conditions $(a_0, a_1, \dots, a_{k-1}) = (0, 0, \dots, 0, 1)$ and recurrence

$$a_{n+k} = c_{k-1}a_{n+k-1} + \dots + c_1a_{n+1} + c_0a_n \quad \text{for all } n \geq 0,$$

where c_0, c_1, \dots, c_{k-1} are constants, then

$$a_n = \sum_{i=1}^k \frac{\alpha_i^n}{P'(\alpha_i)},$$

where $P(t) = t^k - c_{k-1}t^{k-1} - \dots - c_1t - c_0$ is the characteristic polynomial of $(a_n)_{n \geq 0}$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ are the roots of $P(t)$.

If we put $a_n = P_{n-(k-2)}^{(k)}$ for all $n \geq 0$, then we have that $P(t) = \Phi_k(t)$ and

$$h'_k(\gamma_i) = (\gamma_i - 1)\Phi'_k(\gamma_i) \quad \text{for all } 1 \leq i \leq k,$$

where $h_k(x)$ is given by (2.1). So, by using Kalman's result above, we obtain

$$P_n^{(k)} = \sum_{i=1}^k \frac{\gamma_i - 1}{(k + 1)\gamma_i^2 - 3k\gamma_i + k - 1} \gamma_i^n.$$

This proves the first part of (a) in Theorem 3.1.

3.2. Properties of the dominant root. First of all, if we consider the function $g_k(x)$ defined in (3.1) as a function of a real variable, then it is not difficult to see that $g_k(x)$ has a vertical asymptote in

$$c_k := \frac{3k + \sqrt{5k^2 + 4}}{2(k + 1)},$$

and is positive and continuous in (c_k, ∞) . Further,

$$g'_k(x) = -\frac{k(x^2 - 2x + 2) + (x - 1)^2}{(k(x^2 - 3x + 1) + x^2 - 1)^2}$$

is negative in (c_k, ∞) , so $g_k(x)$ is decreasing in (c_k, ∞) . Put

$$a_k = \frac{3k + \sqrt{5k^2 + 4}}{2(k + 1)} + \frac{1}{k} \quad \text{for all } k \geq 2.$$

We next show that the sequence $(a_k)_{k \geq 2}$ is increasing. To do this, let f be the real function defined by

$$f(x) = \frac{3x + \sqrt{5x^2 + 4}}{2(x + 1)} + \frac{1}{x}.$$

It is then a simple matter to show that

$$f'(x) = \frac{5x - 4}{2(x + 1)^2 \sqrt{5x^2 + 4}} + \frac{3}{2(x + 1)^2} - \frac{1}{x^2} = 0$$

implies that $4(x + 1)^2(5x^4 - 10x^3 + 3x^2 - 8x - 4) = 0$. Thus, f has a critical point at $x_0 = 2.14813\dots$ and is increasing in $[x_0, \infty)$. This, of course, tells us that $(a_k)_{k \geq 2}$ is an increasing sequence. In addition, note that

$$\lim_{k \rightarrow \infty} c_k = \lim_{k \rightarrow \infty} \frac{3k + \sqrt{5k^2 + 4}}{2(k + 1)} = \lim_{k \rightarrow \infty} \frac{3 + \sqrt{5 + 4k^{-2}}}{2 + 2k^{-1}} = \varphi^2,$$

where $\varphi = \frac{1}{2}(1 + \sqrt{5})$. Thus, $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} (c_k + k^{-1}) = \varphi^2$, and $c_k \leq \varphi^2 - k^{-1}$ for all $k \geq 2$. Finally, taking into account that $k < \varphi^{k-2}$ for all $k \geq 6$, it is easy to see that $\varphi^2 - k^{-1} < \varphi^2(1 - \varphi^{-k})$ for all $k \geq 6$.

We summarize what we have proved so far in the following lemma.

Lemma 3.1. *Keep the above notation and let $k \geq 2$ be an integer. Then*

- (a) *The function $g_k(x)$ is positive, decreasing and continuous in the interval (c_k, ∞) and $g_k(x)$ has a vertical asymptote in c_k .*
- (b) *If $k \geq 2$, then $c_k \leq \varphi^2 - k^{-1}$. In addition, if $k \geq 6$, then the inequalities*

$$c_k \leq \varphi^2 - k^{-1} < \varphi^2(1 - \varphi^{-k})$$

hold.

Recall that each choice of k produces a distinct k -generalized Pell sequence which in turn has an associated dominant root $\gamma(k)$. For the convenience of the reader, let us denote by $(\gamma(k))_{k \geq 2}$ the sequence of the dominant roots of the k -Pell family of sequences.

We have the following lemma in which we prove that this dominant root is strictly increasing as k increases. We also prove, in the second part of the lemma, that this dominant root approaches φ^2 as k approaches infinity, and that it is larger than $\varphi^2(1 - \varphi^{-k})$. The rest of the statements of the lemma are some technical results which will be used later.

Lemma 3.2. *Let $k, l \geq 2$ be integers. Then*

- (a) *If $k > l$, then $\gamma(k) > \gamma(l)$.*
- (b) *$\varphi^2(1 - \varphi^{-k}) < \gamma(k) < \varphi^2$.*
- (c) *If $k \geq 6$, then*

$$c_k \leq \varphi^2 - k^{-1} < \varphi^2(1 - \varphi^{-k}) < \gamma(k) < \varphi^2.$$

- (d) *$g_k(\varphi^2) = 1/(\varphi + 2)$.*
- (e) *$0.276 < g_k(\gamma(k)) < 0.5$.*

Before proving this, we note, as an immediate consequence of the preceding lemma, that

$$\lim_{k \rightarrow \infty} \gamma(k) = \varphi^2.$$

Proof. To prove (a), we proceed by contradiction by assuming that $\gamma(k) \leq \gamma(l)$; hence $1/\gamma(l)^i \leq 1/\gamma(k)^i$ holds for all $i \geq 1$. Taking into account that $\Phi_l(\gamma(l)) = 0$, one has that

$$\gamma(l)^l = 2\gamma(l)^{l-1} + \gamma(l)^{l-2} + \dots + \gamma(l) + 1,$$

and, of course, the same conclusion remains valid for $\gamma(k)$. From this, we get that

$$\begin{aligned} 1 &= \frac{2}{\gamma(l)} + \frac{1}{\gamma(l)^2} + \frac{1}{\gamma(l)^3} + \dots + \frac{1}{\gamma(l)^l} \\ &< \frac{2}{\gamma(k)} + \frac{1}{\gamma(k)^2} + \frac{1}{\gamma(k)^3} + \dots + \frac{1}{\gamma(k)^k} = 1, \end{aligned}$$

which is a contradiction.

We next prove (b). First, we turn back to expression (2.1), which we rewrite here as follows

$$(3.3) \quad h_k(x) = (x - 1)\Phi_k(x) = x^{k-1}(x^2 - 3x + 1) + 1.$$

Notice here that φ^2 is a root of $x^2 - 3x + 1$. It then follows from (3.3) that $h_k(\varphi^2) = 1$ and therefore $\Phi_k(\varphi^2) = 1/\varphi > 0$. In the above we have used the fact that $\varphi^2 - 1 = \varphi$. Since $\Phi_k(2) = 1 - 2^{k-1} < 0$, and recalling that $\Phi_k(x)$ has just one positive real zero, we find that $2 < \gamma < \varphi^2$.

On the other hand, by using once more the fact that φ^2 is a root of $x^2 - 3x + 1$ and evaluating expression (3.3) at γ , we get the relations

$$\varphi^4 - 3\varphi^2 + 1 = 0 \quad \text{and} \quad \gamma^2 - 3\gamma + 1 = \frac{-1}{\gamma^{k-1}}.$$

Subtracting the two expressions above and rearranging some terms, one obtains

$$(\varphi^2 - \gamma)(\varphi^2 + \gamma - 3) = \frac{1}{\gamma^{k-1}}.$$

From this, and using the facts that $\varphi^2 + \gamma - 3 > 1/\varphi$ and $\varphi < \gamma$, which are easily seen, we get that $\varphi^2 - \gamma < \varphi^2/\varphi^k$ and so $\varphi^2(1 - \varphi^{-k}) < \gamma$. This finishes the proof of (b).

The proof of (c) is a direct combination of the second part of this lemma and Lemma 3.1 (b). To prove (c), we observe that

$$g_k(\varphi^2) = \frac{\varphi^2 - 1}{(k+1)\varphi^4 - 3k(\varphi+1) + k - 1} = \frac{\varphi}{3\varphi+1} = \frac{1}{\varphi+2},$$

where we used the facts that $\varphi^2 = \varphi + 1$, $\varphi^4 = 3\varphi + 2$ and $\varphi(\varphi + 2) = 3\varphi + 1$.

We now prove (d). Since $g_k(x)$ is decreasing in the interval (c_k, ∞) and the inequality $c_k \leq \varphi^2 - k^{-1} < \gamma(k) < \varphi^2$ holds for all $k \geq 6$, we have

$$\frac{1}{\varphi+2} = g_k(\varphi^2) < g_k(\gamma(k)) < g_k(\varphi^2 - k^{-1}).$$

But

$$\begin{aligned} g_k(\varphi^2 - k^{-1}) &= \frac{\varphi^2 - k^{-1} - 1}{(k+1)(\varphi^2 - k^{-1})^2 - 3k(\varphi^2 - k^{-1}) + k - 1} \\ &= \frac{\varphi - k^{-1}}{\varphi - 2\varphi k^{-1} - k^{-1} + k^{-2} + 2} < \frac{1}{2}, \end{aligned}$$

where the last inequality holds for all $k \geq 6$. Hence, $0.276 < g_k(\gamma(k)) < 0.5$ holds for all $k \geq 6$. Finally, computationally we get that

k	2	3	4	5
$g_k(\gamma(k))$	0.35...	0.30...	0.29...	0.28...

which shows that $0.276 < g_k(\gamma(k)) < 0.5$ also holds for all $k \geq 2$. This finishes the proof of the lemma. □

3.3. Sequence of errors. For a fixed integer $k \geq 2$ and $n \geq 2 - k$, define $E_n^{(k)}$ to be the error of the approximation of the n th k -Pell number with the dominant term of the Binet-style formula of $P^{(k)}$ given in Theorem 3.1 (a), i.e.,

$$(3.4) \quad E_n^{(k)} = P_n^{(k)} - g_k(\gamma)\gamma^n,$$

for γ the dominant root of $\Phi_k(x)$ and $g_k(x)$ defined as in (3.1).

Given a polynomial f , the set of all possible linear recurrence sequences of real numbers having the characteristic equation $f(x) = 0$ is a vector space over the real numbers. Since $P^{(k)}$ and $(\gamma^n)_n$ satisfy the characteristic equation $\Phi_k(x) = 0$, it follows from (3.4) that the sequence $(E_n^{(k)})_n$ satisfies the same recurrence relation as the k -Pell sequence. We record this as follows.

Lemma 3.3. *Let $k \geq 2$ be an integer. Then*

$$E_n^{(k)} = 2E_{n-1}^{(k)} + E_{n-2}^{(k)} + \dots + E_{n-k}^{(k)} \quad \text{for all } n \geq 2.$$

Furthermore, if $n \geq 3$, then

$$E_n^{(k)} = 3E_{n-1}^{(k)} - E_{n-2}^{(k)} - E_{n-k-1}^{(k)}.$$

The last result of this subsection is the following.

Lemma 3.4. *For a fixed integer $k \geq 2$ we have*

$$\lim_{n \rightarrow \infty} E_n^{(k)} = 0.$$

Proof. Using the fact that $\lim_{n \rightarrow \infty} |\gamma_j|^n = 0$ for all $2 \leq j \leq k$ and taking into account that

$$|E_n^{(k)}| \leq \sum_{j=2}^k |g_k(\gamma_j)| |\gamma_j|^n,$$

we deduce that

$$\lim_{n \rightarrow \infty} |E_n^{(k)}| = 0.$$

This proves the lemma. □

To conclude this subsection, we prove the second part of Theorem 3.1 (a). Indeed, with the notation above, we have to prove that

$$|E_n^{(k)}| < \frac{1}{2} \quad \text{for all } k \geq 2 \text{ and } n \geq 2 - k.$$

In order to do this, we prove the following three claims:

Claim 3.1. $|E_n^{(k)}| < \frac{1}{2}$ for all $2 - k \leq n \leq 0$.

Claim 3.2. $|E_1^{(k)}| < \frac{1}{2}$.

Claim 3.3. $|E_n^{(k)}| < \frac{1}{2}$ for all $n \geq 2$.

Proof of Claim 3.1. Because of the initial conditions of $P^{(k)}$, we have that $P_n^{(k)} = 0$ for all $2 - k \leq n \leq 0$, so $E_n^{(k)} = -g_k(\gamma)\gamma^n$ for all $2 - k \leq n \leq 0$. For the case $n = 0$, we have, by Lemma 3.2 (e), that $|E_0^{(k)}| = g_k(\gamma) < \frac{1}{2}$. If $2 - k \leq n \leq -1$, then $\gamma^n < \gamma^{-1} < 1$ and therefore $g_k(\gamma)\gamma^n < g_k(\gamma) < \frac{1}{2}$ for all $k \geq 2$. This proves Claim 3.1. \square

Proof of Claim 3.2. First, note that $E_1^{(k)} = P_1^{(k)} - g_k(\gamma)\gamma = 1 - g_k(\gamma)\gamma$. By Lemma 3.2 (e) we have that $0.276\gamma < g_k(\gamma)\gamma < \frac{1}{2}\gamma$. However, $0.66 < 0.276\gamma(2) \leq 0.276\gamma$ and $\frac{1}{2}\gamma < \frac{1}{2}\varphi^2 < 1.31$, and so $0.66 < g_k(\gamma)\gamma < 1.31$. Thus, $-0.31 < 1 - g_k(\gamma)\gamma < 0.34$, which implies that $|E_1^{(k)}| = |1 - g_k(\gamma)| < \frac{1}{2}$ for all $k \geq 2$. So, the proof of Claim 3.2 is complete. \square

Proof of Claim 3.3. Suppose for the sake of contradiction that $|E_n^{(k)}| \geq \frac{1}{2}$ for some integer $n \geq 2$, and let n_0 be the smallest positive integer such that $|E_{n_0}^{(k)}| \geq \frac{1}{2}$. Since $|E_{n_0-1}^{(k)}| < \frac{1}{2}$ and $|E_{n_0-k}^{(k)}| < \frac{1}{2}$ we get $|E_{n_0-1}^{(k)} + E_{n_0-k}^{(k)}| < 1$. According to Lemma 3.3, $E_{n_0+1}^{(k)} = 3E_{n_0}^{(k)} - (E_{n_0-1}^{(k)} + E_{n_0-k}^{(k)})$ and so

$$|E_{n_0+1}^{(k)}| \geq 3|E_{n_0}^{(k)}| - |E_{n_0-1}^{(k)} + E_{n_0-k}^{(k)}|.$$

Hence

$$|E_{n_0+1}^{(k)}| - |E_{n_0}^{(k)}| \geq 2|E_{n_0}^{(k)}| - |E_{n_0-1}^{(k)} + E_{n_0-k}^{(k)}| > 0$$

giving

$$|E_{n_0+1}^{(k)}| > |E_{n_0}^{(k)}|.$$

Since $n_0 - k + 1 < n_0$, we infer that $|E_{n_0-k+1}^{(k)}| < \frac{1}{2} \leq |E_{n_0}^{(k)}| < |E_{n_0+1}^{(k)}|$ and therefore $|E_{n_0}^{(k)} + E_{n_0-k+1}^{(k)}| < 2|E_{n_0+1}^{(k)}|$. By using this and Lemma 3.3, we obtain

$$|E_{n_0+2}^{(k)}| \geq 3|E_{n_0+1}^{(k)}| - |E_{n_0}^{(k)} + E_{n_0-k+1}^{(k)}| > 3|E_{n_0+1}^{(k)}| - 2|E_{n_0+1}^{(k)}|.$$

Hence, $|E_{n_0+2}^{(k)}| > |E_{n_0+1}^{(k)}|$.

Now suppose that $|E_{n_0}^{(k)}| < |E_{n_0+1}^{(k)}| < \dots < |E_{n_0+i-1}^{(k)}|$ for some integer $i \geq 4$. We distinguish two cases according to whether $n_0 + i - k - 1 < n_0$ or $n_0 \leq n_0 + i - k - 1$.

First, if $n_0 + i - k - 1 < n_0$, then we get

$$|E_{n_0+i-k-1}^{(k)}| < \frac{1}{2} \leq |E_{n_0}^{(k)}| < |E_{n_0+1}^{(k)}| < \dots < |E_{n_0+i-1}^{(k)}|.$$

If $n_0 \leq n_0 + i - k - 1 < n_0 + i - 1$, then we obtain that $|E_{n_0+i-k-1}^{(k)}| < |E_{n_0+i-1}^{(k)}|$.

In any case, we have that the inequality

$$|E_{n_0+i-k-1}^{(k)}| < |E_{n_0+i-1}^{(k)}|$$

always holds. For this reason

$$|E_{n_0+i-2}^{(k)} + E_{n_0+i-k-1}^{(k)}| < 2|E_{n_0+i-1}^{(k)}|.$$

From Lemma 3.3 once more, we have that $E_{n_0+i}^{(k)} = 3E_{n_0+i-1}^{(k)} - (E_{n_0+i-2}^{(k)} + E_{n_0+i-k-1}^{(k)})$ and so

$$\begin{aligned} |E_{n_0+i}^{(k)}| &\geq 3|E_{n_0+i-1}^{(k)}| - |E_{n_0+i-2}^{(k)} + E_{n_0+i-k-1}^{(k)}| \\ &> 3|E_{n_0+i-1}^{(k)}| - 2|E_{n_0+i-1}^{(k)}| = |E_{n_0+i-1}^{(k)}|. \end{aligned}$$

Consequently,

$$|E_{n_0+i}^{(k)}| > |E_{n_0+i-1}^{(k)}| > \dots > |E_{n_0+1}^{(k)}| > |E_{n_0}^{(k)}|,$$

contradicting Lemma 3.4, which says that the error must eventually go to 0. This proves Claim 3.3 and therefore the proof of the second part of Theorem 3.1 (a). \square

3.4. Exponential growth. We begin by mentioning that for the Fibonacci sequence it is well-known that

$$(3.5) \quad \varphi^{n-2} \leq F_n \leq \varphi^{n-1} \quad \text{holds for all } n \geq 1.$$

Similarly, by induction and using the Binet formula for the Pell sequence (namely, the case $k = 2$), one can easily prove that

$$(3.6) \quad \gamma^{n-2} \leq P_n \leq \gamma^{n-1} \quad \text{holds for all } n \geq 1.$$

In the above, the value of γ is $\gamma(2) = 1 + \sqrt{2}$. This of course exhibits an exponential growth of the Fibonacci and Pell numbers. We finally prove (3.2), which shows that the above inequality (3.6) holds for the k -Pell sequence as well. This will be done using induction on n .

To begin with, we show that inequality (3.2) holds for $n = 1, 2, \dots, k$. It is clear that the result is true for $n = 1$ because $\gamma > 1$, so we assume that $2 \leq n \leq k$. In this case, we deduce by (1.2) that $F_n^{(k)} = F_{2n-1}$, so we need to show that

$$(3.7) \quad \gamma^{n-2} \leq F_{2n-1} \leq \gamma^{n-1} \quad \text{for } 2 \leq n \leq k.$$

By Lemma 3.2 (b) and (3.5), we get

$$\gamma^{n-2} < \varphi^{2(n-2)} < \varphi^{2n-3} \leq F_{2n-1}$$

and therefore the left-hand side of the above inequality (3.7) holds. Then, it remains to prove that

$$(3.8) \quad F_{2n-1} \leq \gamma^{n-1} \quad \text{holds for } 2 \leq n \leq k.$$

Computationally, one checks that the inequality (3.8) holds true for $2 \leq k \leq 6$, so we may assume that $k \geq 7$. Now, by making use of the famous Binet formula for the Fibonacci numbers, we get

$$F_{2n-1} = \frac{\varphi^{2n-1} + \varphi^{-(2n-1)}}{\sqrt{5}} = \frac{\varphi^{2n-1}}{\sqrt{5}} \left(1 + \frac{1}{\varphi^{4n-2}} \right).$$

Since $\varphi^{2(n-1)}(1 - \varphi^{-k})^{n-1} < \gamma^{n-1}$ because $\varphi^2(1 - \varphi^{-k}) < \gamma$ by Lemma 3.2 (b), it suffices to prove that

$$\frac{\varphi^{2n-1}}{\sqrt{5}} \left(1 + \frac{1}{\varphi^{4n-2}} \right) \leq \varphi^{2(n-1)}(1 - \varphi^{-k})^{n-1},$$

which is equivalent to

$$(3.9) \quad 1 + \frac{1}{\varphi^{4n-2}} \leq \frac{\sqrt{5}}{\varphi} (1 - \varphi^{-k})^{n-1}.$$

Using the fact that the function $x \rightarrow (1 - \varphi^{-x})^{x-1}$ is increasing for $x \geq 7$ and taking into account that $2 \leq n \leq k$ and $k \geq 7$, we deduce that

$$1 + \frac{1}{\varphi^{4n-2}} \leq 1 + \frac{1}{\varphi^6} = 1.05572\dots,$$

whereas

$$\frac{\sqrt{5}}{\varphi} (1 - \varphi^{-k})^{n-1} \geq \frac{\sqrt{5}}{\varphi} (1 - \varphi^{-k})^{k-1} \geq \frac{\sqrt{5}}{\varphi} (1 - \varphi^{-7})^6 = 1.11987\dots$$

This proves inequality (3.9). Thus, we have proved that inequality (3.2) holds for the first k nonzero terms of $P^{(k)}$.

Finally, suppose that (3.2) holds for all terms $P_m^{(k)}$ with $m \leq n-1$ for some $n > k$. It then follows from the recurrence relation of $P^{(k)}$ that

$$2\gamma^{n-3} + \gamma^{n-4} + \dots + \gamma^{n-k-2} \leq P_n^{(k)} \leq 2\gamma^{n-2} + \gamma^{n-3} + \dots + \gamma^{n-k-1}.$$

So

$$\gamma^{n-k-2}(2\gamma^{k-1} + \gamma^{k-2} + \dots + 1) \leq P_n^{(k)} \leq \gamma^{n-k-1}(2\gamma^{k-1} + \gamma^{k-2} + \dots + 1),$$

which combined with the fact that $\gamma^k = 2\gamma^{k-1} + \gamma^{k-2} + \dots + 1$ gives the desired result. Thus, inequality (3.2) holds for all positive integers n . So, the proof of Theorem 3.1 is now complete.

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