

ON A GENERALIZED JACOBI TRANSFORM

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In this paper, we study a generalized Jacobi transform and obtain images of certain functions under this transform. Furthermore, we define a Jacobi random variable and derive its moments, distribution function, and characteristic function.

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1. Introduction

Kalla et al. [4] have studied the following integral,

$$I_{\nu, \alpha, \beta}^{a, b} = \int_{-1}^1 (1-x)^a (1+x)^b P_{\nu}^{(\alpha, \beta)}(x) dx \quad (1)$$

with $Re(a) > -1$, $Re(b) > -1$ and $P_{\nu}^{(\alpha, \beta)}$ is the Jacobi function, where

$$P_{\nu}^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_{\nu}}{\Gamma(\nu+1)} \cdot {}_2F_1 \left(\begin{matrix} -\nu, \nu+\lambda \\ \alpha+1 \end{matrix}; \frac{1-x}{2} \right) \quad (2)$$

and $\lambda = \alpha + \beta + 1$. The authors considered its partial derivatives with respect to a and b . Some more general results were obtained in [6]. These results were extended by Sarabia [10] using the following integral:

$$I_{\nu, \alpha, \beta}^{a, b, c, p} = \int_{-1}^1 (1-x)^a (1+x)^b P_{\nu}^{(\alpha, \beta, c, p)}(x) dx, \quad (3)$$

where $P_{\nu}^{(\alpha, \beta)}$ is a generalized Jacobi function defined as

$$P_{\nu}^{(\alpha, \beta, c, p)}(x) = \frac{(\alpha + 1)_{\nu}}{\Gamma(\nu + 1)} \cdot {}_3F_2\left(\begin{matrix} -\nu, \nu + \lambda, c \\ \alpha + 1, p \end{matrix}; \frac{1-x}{2}\right), \quad (4)$$

where

$$P \in C - Z^{-} U \{0\}; \alpha, \nu \in C - Z^{-}; \beta \in C; \operatorname{Re}(p - \beta - c) > 0. \quad (4.a)$$

Hence, $P_{\nu}^{(\alpha, \beta, c, p)}$ is continuous on $[-1, 1]$. For $p = c$, (4) reduces to (2).

In this paper, we define a generalized Jacobi transform and obtain images of certain functions under this transform. Furthermore, we define a random variable and derive some statistical properties such as: moments, a distribution function, and characteristic function.

2. Generalized Jacobi Transform

Let f be a real-valued function defined on $[-1, 1]$, with $\operatorname{Re}(a) > -1$, $\operatorname{Re}(b) > -1$, and conditions (4a) being held. Then the generalized Jacobi transform (GJT) of $f(x)$ is defined as

$$J_{\alpha, \beta}^{a, b, c, p}[f(x), \nu] = \int_{-1}^1 (1-x)^a (1+x)^b P_{\nu}^{(\alpha, \beta, c, p)}(x) f(x) dx. \quad (5)$$

For continuous or sectionally continuous f on $[a, b]$, integral (5) exists. For $c = p$, (5) reduces to the well known Jacobi transform [3].

Now, we obtain images of some functions under the generalized Jacobi transform.

(i) For $f(x) = 1$, we have [9, 10]:

$$J_{\alpha, \beta}^{a, b, c, p}[1; \nu] = 2^{a+b+1} \frac{(\alpha + 1)_{\nu} B(a+1, b+1)}{\Gamma(\nu + 1)} \cdot {}_4F_3\left(\begin{matrix} -\nu, \nu + \lambda, c, a + 1 \\ \alpha + 1, p, a + b + 2 \end{matrix}; 1\right), \quad (6)$$

(ii) $f(x) = \ln(1-x)$. Since $f(x)$ is not piecewise continuous on $[-1, 1]$, we have to demonstrate directly that (5) exists.

Indeed: $F(x, y) = (1-x)^y (1+x)^b P_{\nu}^{(\alpha, \beta, c, p)}(x)$ and $D_2 F(x, y) = \ln(1-x)(1-x)^y (1+x)^b P_{\nu}^{(\alpha, \beta, c, p)}(x)$ are continuous on $D = [A, B] \times \bar{B}_R(a)$, where

$$[A, B] \subset (-1, 1) \text{ and } \bar{B}_R(a) \subset S_{\delta} = \{a \in C: \operatorname{Re}(a) \geq \delta > -1\}.$$

In addition, for $\operatorname{Re}(y) > -1$ and $\bar{B}_R(a) \subset S_{\delta}$, $J_{\alpha, \beta}^{y, a, c, p}[1, \nu]$ exists.

Also, the constants $K_1 > 0$ and $K_2 > 0$ exist, such that:

$$\begin{aligned} & |\ln(1-x)(1-x)^y (1+x)^b P_{\nu}^{(\alpha, \beta, c, p)}(x)| \\ & \leq M(x) = \begin{cases} -K_1 \ln(1-x)(1-x)^{-\delta} & x \in (0, 1] \\ K_2 (1+x)^{\operatorname{Re}(b)} & x \in [0, 1]. \end{cases} \end{aligned}$$

Then, by applying the M-Criterion of Weierstrass, we obtain:

$$\int_{-1}^1 \ln(1-x)(1-x)^y (1+x)^b P_{\nu}^{(\alpha, \beta, c, p)}(x) dx,$$

which converges uniformly on $\bar{B}_R(a)$ and

$$\begin{aligned} & D_2 \int_{-1}^1 (1-x)^y(1+x)^b P_\nu^{(\alpha, \beta, c, p)}(x) dx \\ &= \int_{-1}^1 \ln(1-x)(1-x)^y(1+x)^b P_\nu^{(\alpha, \beta, c, p)}(x) dx \end{aligned}$$

on $\bar{B}_R(a)$. Then, for $a \in C$, so that $Re(a) > -1$, according to [13],

$$\begin{aligned} & D_2 \int_{-1}^1 (1-x)^a(1+x)^b P_\nu^{(\alpha, \beta, c, p)}(x) dx \\ &= \int_{-1}^1 \ln(1-x)(1-x)^a(1+x)^b P_\nu^{(\alpha, \beta, c, p)}(x) dx. \end{aligned} \tag{7}$$

By demonstrating that $J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu]$ exists and also

$$J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu] = \frac{\partial}{\partial \alpha} J_{\alpha, \beta}^{a, b, c, p}[1, \nu], \tag{8}$$

using (6) and (8), we get

$$\begin{aligned} & J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu] \\ &= \frac{\partial}{\partial \alpha} \left[\frac{(\alpha+1)_\nu \Gamma(b+1) 2^{b+1}}{\Gamma(\nu+1)} \cdot 2^a \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu+\lambda)_n (c)_n \Gamma(a+1+n)}{(\alpha+1)_n (p)_n \Gamma(a+b+2+n) n!} \right], \end{aligned}$$

and, hence

$$\begin{aligned} & J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu] = \ln 2 \cdot J_{\alpha, \beta}^{a, b, c, p}[1; \nu] \\ &+ \frac{(\alpha+1)_\nu B(a+1, b+1) 2^{a+b+1}}{\Gamma(\nu+1)} \cdot \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu+\lambda)_n (c)_n (a+1)_n}{(\alpha+1)_n (p)_n n! (a+b+2)_n} \\ &\cdot [\psi(a+1+n) - \psi(a+b+2+n)]. \end{aligned} \tag{9}$$

We can see that (9) has as a particular case (2.2) from [4] when $c = p$. Likewise and relatively speaking, it is a more simple representation than [10, eq. (6)].

(iii) $f(x) = \ln(1+x)$. Reasoning in a similar way as we did in the previous case, we have that if $Re(a), Re(b) > -1$ and the conditions considered in (4.a) hold, then the existence of $J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu]$ can be easily proved. Also, by criterions of uniform convergence, we obtain:

$$J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu] = \frac{\partial}{\partial b} [J_{\alpha, \beta}^{a, b, c, p}[1; \nu]]$$

and

$$\begin{aligned} J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu] &= [\phi(b+1) + \ln 2] J_{\alpha, \beta}^{a, b, c, p}[1; \nu] \\ &\quad - \frac{(\alpha+1)_\nu B(a+1, b+1) 2^{a+b+1}}{\Gamma(\nu+1)} \\ &\quad \cdot \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu+\lambda)_n (c)_n (a+1)_n}{(\alpha+1)_n (p)_n n! (a+b+2)_n} [\varphi(a+b+2) + S_n], \end{aligned} \quad (10)$$

where $S_n = \sum_{k=0}^{n-1} \frac{1}{(a+b+2+k)}$ for $n \geq 1$, $S_0 = 0$.

Hence,

$$\begin{aligned} J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu] &= [\phi(b+1) + \ln 2 - \phi(a+b+2)] J_{\alpha, \beta}^{a, b, c, p}[1; \nu] \\ &\quad - \frac{(\alpha+1)_\nu 2^{a+b+1} B(a+1, b+1)}{\Gamma(\nu+1)} \\ &\quad \cdot \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu+\lambda)_n (c)_n (a+1)_n}{(\alpha+1)_n (p)_n n! (a+b+2)_n} S_n. \end{aligned} \quad (11)$$

Clearly, the above results lead to

$$\begin{aligned} J_{\alpha, \beta}^{a, b, c, p}[\ln(1-x); \nu] &= [\Psi(b+1) + 2\ln 2 - \Psi(a+b+2)] J_{\alpha, \beta}^{a, b, c, p}[1; \nu] \\ &\quad + \frac{(\alpha+1)_\nu B(a+1, b+1) 2^{a+b+1}}{\Gamma(\nu+1)} \\ &\quad \cdot \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu+\lambda)_n (c)_n (a+1)_n}{(\alpha+1)_n (p)_n n! (a+b+2)_n} \\ &\quad [\Psi(a+b+2) - \Psi(a+b+2+n) - S_n] \end{aligned} \quad (12)$$

and

$$\begin{aligned} J_{\alpha, \beta}^{a, b, c, p}[\ln\left(\frac{1-x}{1+x}\right); \nu] &= [\phi(a+b+2) - \phi(b+1)] J_{\alpha, \beta}^{a, b, c, p}[1; \nu] \\ &\quad + \frac{(\alpha+1)_\nu B(a+1, b+1) 2^{a+b+1}}{\Gamma(\nu+1)} \\ &\quad \cdot \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu+\lambda)_n (c)_n (a+1)_n}{(\alpha+1)_n (p)_n n! (a+b+2)_n} \end{aligned}$$

$$[S_n + \Psi(a + 1 + n) - \Psi(a + b + 2 + n)]. \tag{13}$$

(iv) $f(x) = (1 - x)^A(1 + x)^B$. Let $Re(a) > \max\{-1, -Re(A) - 1\}$, $Re(b) > \max\{-1, -Re(B) - 1\}$, that is to say $Re(a), Re(b), Re(a + A), Re(b + B) > -1$; also let

$$Re(p - \beta - c) > 0; \alpha, \nu \in C - Z^-; \beta \in C. \tag{14}$$

Hence, equation (6) becomes:

$$J_{\alpha, \beta}^{a, b, c, p}[(1 - x)^A(1 + x)^B; \nu] = J_{\alpha, \beta}^{a + A, b + B, c, p}[1; \nu]. \tag{15}$$

Hence, we arrive at

$$J_{\alpha, \beta}^{a, b, c, p}[(1 - x)^A(1 + x)^B; \nu] = \frac{(\alpha + 1)_\nu B(a + A + 1, b + B + 1) 2^{a + A + b + B + 1}}{\Gamma(\nu + 1)} \cdot {}_4F_3 \left(\begin{matrix} -\nu, \nu + \lambda, c, a + A + 1 \\ \alpha + 1, p, a + A + b + B + 2 \end{matrix}; 1 \right). \tag{15.a}$$

(v) $f(x) = P_{\mu}^{(\gamma, \delta, d, q)}(x)$. Let

$$\eta = \gamma + \delta + 1; q \in C - Z^- \cup \{0\}; \gamma, \mu \in C - Z; Re(q - \delta - d) > 0, \tag{16}$$

and let conditions of (4a) hold true. Then $f(x)$ is continuous on $[-1, 1]$ and, therefore, its GJT exists. Consequently,

$$J_{\alpha, \beta}^{a, b, c, p}[P_{\mu}^{(\gamma, \delta, d, q)}(x); \nu] = \frac{(\gamma + 1)_\mu}{\Gamma(\mu + 1)} \int_{-1}^1 (1 - x)^a(1 + x)^b P_{\nu}^{(\alpha, \beta, c, p)}(x) P_{\mu}^{(\gamma, \delta, d, q)}(x) dx.$$

Since the integral on interval $[r, R] \subset (-1, 1)$ can be interchanged with the series (by uniform convergence) and since the transform exists, we can interchange the integral on $[-1, 1]$ with the series by applying [1, (14.31)].

Hence,

$$J_{\alpha, \beta}^{a, b, c, p}[P_{\mu}^{(\gamma, \delta, d, q)}(x); \nu] = \frac{(\alpha + 1)_\nu (\gamma + 1)_\mu B(a + 1, b + 1)}{\Gamma(\nu + 1) \Gamma(\mu + 1)} \cdot \sum_{n=0}^{\infty} \frac{(-\mu)_n (\mu + \eta)_n (d)_n (a + 1)_n 2^{a + b + n + 1}}{(\gamma + 1)_n (q)_n (a + b + 2)_n n! 2^n} \cdot {}_4F_3 \left(\begin{matrix} -\nu, \nu + \lambda, c, a + n + 1 \\ \alpha + 1, p, a + b + n + 2 \end{matrix}; 1 \right),$$

and

$$J_{\alpha, \beta}^{a, b, c, p}[P_{\mu}^{(\gamma, \delta, d, q)}(x); \nu] = \frac{(\alpha + 1)_\nu (\gamma + 1)_\mu B(a + 1, b + 1) 2^{a + b + 1}}{\Gamma(\nu + 1) \Gamma(\mu + 1)}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-\mu)_n (\mu + \eta)_n (d)_n (a + 1)_n (-\nu)_k (\nu + \eta)_k (c)_k (a + n + 1)_k}{(\gamma + 1)_n (q)_n (a + b + 2)_n (\alpha + 1)_k (p)_k (a + b + n + 2)_k n! k!}. \quad (17)$$

Recollecting that $(h + n)_K = \frac{(h)_{n+K}}{(h)_n}$ and setting

$$H = \frac{(\alpha + 1)_\nu (\gamma + 1)_\mu B(a + 1, b + 1) 2^{a+b+1}}{\Gamma(\nu + 1) \gamma(\mu + 1)} \quad (18)$$

we get,

$$\begin{aligned} & J_{\alpha, \beta}^{a, b, c, p} [P_{\mu}^{(\gamma, \delta, d, q)}(x); \nu] \\ &= H \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-\mu)_n (\mu + \eta)_n (d)_n (a + 1)_{n+k} (-\nu)_k (\nu + \lambda)_k (c)_k}{(\gamma + 1)_n (q)_n (a + b + 2)_{n+k} (\alpha + 1)_k (p)_k n! k!}. \end{aligned} \quad (19)$$

Using the Kampé de Fériet's hypergeometric function (see [8, p. 160]) and since the transform exists, we have:

$$J_{\alpha, \beta}^{a, b, c, p} [P_{\mu}^{(\gamma, \delta, d, q)}(x); \nu] = \frac{(\alpha + 1)_\nu (\gamma + 1)_\mu B(a + 1, b + 1) 2^{a+b+1}}{\Gamma(\nu + 1) \Gamma(\mu + 1)} \cdot F \left(\begin{array}{c|c|c} 1 & a + 1 & \\ 3 & -\mu, -\nu; \mu + \eta, \nu + \lambda; d, c & \\ 1 & a + b + 2 & \\ 2 & \gamma + 1, \alpha + 1; q, p & \\ \hline & & 1, 1 \end{array} \right). \quad (20)$$

3. Some Inequalities

In this section, we will obtain some inequalities to the GJT, based on the Luke's Inequality (see [7, p. 254], formula (6)):

$$(1 - \theta z)^{-\sigma} < {}_{p+1}F_p \left(\begin{array}{c} \sigma; \alpha_1, \dots, \alpha_p \\ \rho_1, \dots, \rho_p \end{array}; z \right) < 1 - \theta + \theta(1 - z)^{-\sigma}, \quad (21)$$

where $\theta = \prod_{j=1}^p \left(\frac{\alpha_j}{\rho_j} \right)$, with $0 < z < 1$; $\sigma > 0$; $\rho_j \geq \alpha_j > 0$ ($j = 1, \dots, p$).

We consider the following conditions:

$$\begin{aligned} 0 > \nu > -\min\{1, 1 + \alpha\}; 0 < \lambda + \nu < \alpha + 1; \\ 0 < c \leq p; p > \beta + c; a + 1, b > 0. \end{aligned} \quad (22)$$

Then $0 < \theta = \frac{(\lambda + \nu)c}{(\alpha + 1)p} < 1$, and since $0 \leq \frac{1-x}{2} \leq 1$, we obtain

$$1 \leq \left[1 - \theta \left(\frac{1-x}{2} \right) \right]^\nu \leq {}_3F_2 \left(\begin{matrix} -\nu, \lambda + \nu, c \\ \alpha + 1, p \end{matrix}; \frac{1-x}{2} \right) \leq 1 - \theta + \theta \left(\frac{1+x}{2} \right)^\nu \tag{23}$$

and

$$\begin{aligned} & \frac{(\alpha + 1)_\nu}{\Gamma(\nu + 1)} (1-x)^a (1+x)^b \cdot {}_3F_2 \left(\begin{matrix} -\nu, \lambda + \nu, c \\ \alpha + 1, p \end{matrix}; \frac{1-x}{2} \right) \\ & \leq \frac{(\alpha + 1)_\nu}{\Gamma(\nu + 1)} (1-x)^a (1+x)^b \left[1 - \theta + \theta \left(\frac{1+x}{2} \right)^\nu \right]. \end{aligned} \tag{24}$$

Integrating (24) between -1 and 1 , and due to $0 \leq \theta \left(\frac{1-x}{2} \right) \leq \theta < 1$ we have the following, by applying criterions of uniform convergence and simplifying for $\frac{(\alpha + 1)_\nu B(a + 1, b + 1) 2^{a+b+1}}{\Gamma(\nu + 1)}$, under the same conditions as that of (22),

$$\begin{aligned} 1 & \leq {}_2F_1 \left(\begin{matrix} -\nu, a + 1 \\ a + b + 2 \end{matrix}; \theta \right) \leq {}_4F_3 \left(\begin{matrix} -\nu, \nu + \lambda, c, a + 1 \\ \alpha + 1, p, a + b + 2 \end{matrix}; 1 \right) \\ & \leq 1 - \theta + \theta \frac{\Gamma(a + b + 2)\Gamma(b + \nu + 1)}{\Gamma(b + 1)\Gamma(a + b + \nu + 2)}. \end{aligned} \tag{25}$$

Throughout this section, using (19) we will get a bound to the GJT of $f(x)$, which is continuous on $[-1, 1]$.

Thus, let $M = \max\{|f(x)| : x \in [-1, 1]\}$, then

$$\begin{aligned} |J_{\alpha, \beta}^{a, b, c, p}[f(x); \nu]| & \leq M J_{\alpha, \beta}^{a, b, c, p}[1; \nu] L \cdot {}_4F_3 \left(\begin{matrix} -\nu, \nu + \lambda, c, a + 1 \\ \alpha + 1, p, a + b + 2 \end{matrix}; 1 \right) \\ & \leq L \left[1 - \theta - \theta \frac{\Gamma(a + b + 2)\Gamma(b + \nu + 1)}{\Gamma(b + 1)\Gamma(a + b + \nu + 2)} \right] \end{aligned} \tag{26}$$

where,

$$L = \frac{M(\alpha + 1)_\nu B(a + 1, b + 1) 2^{a+b+1}}{\Gamma(\nu + 1)}; \lambda = \alpha + \beta + 1; \theta = \frac{(\lambda + \nu)c}{(\alpha + 1)p},$$

along with conditions (22).

For example, let $f(x) = \frac{1}{(1+x^2)}$, $m = \frac{1}{2}$ (minimum) and $M = 1$ (maximum), with $a = 0, b = 1; -1 < \nu < -\frac{1}{2}; \alpha = \beta = 0; c = 1, p = 2$ and $0 < \theta < \frac{1}{4}$, then

$$1 \leq J_{0,0}^{0,1,1,2} \left[\frac{1}{1+x^2}; \nu \right] \leq \frac{-\nu^2 + \nu + 4}{\nu + 2}. \tag{27}$$

4. Generalized Jacobi Random Variable

Statistical distributions have been used in a variety of applications, including the field of reliability. Recently, Kalla et al. [5] have studied a unified form of gamma-type distributions. Here we define a Jacobi random variable and derive some statistical properties.

4.1 Density Function

We define the generalized Jacobi random variable with parameters $(\alpha, \beta, c, p, \nu; a, b)$ as a random variable, whose density function is given by

$$g_{a,b}^{(\alpha, \beta, c, p, \nu)}(x) = \begin{cases} K(1-x)^a(1+x)^b P_{\nu}^{(\alpha, \beta, c, p)}(x) & \text{for } x \in [-1, 1] \\ 0 & \text{otherwise,} \end{cases} \quad (28)$$

where

$$a, b > -1; \alpha, \beta, c > 0; p > \beta + c; -1 < \nu < 0. \quad (29)$$

Furthermore, $K = \frac{\Gamma(\nu+1)2^{-(a+b+1)}}{(\alpha+1)_{\nu} B(a+1, b+1)R}$ and

$$R = {}_4F_3 \left(\begin{matrix} -\nu, \nu + \lambda, c, a + 1 \\ \alpha + 1, p, a + b + 2 \end{matrix} ; 1 \right). \quad (30)$$

We can see that (27) becomes a density function of the family of β -random variables whose parameters are $(a+1, b+1)$, displaced on $x = 1 - 2u$, when $\nu = 0$.

Indeed

$$h(u) = g_{a,b}^{(\alpha, \beta, c, p, 0)}(1-2u) = \frac{1}{B(a+1, b+1)} u^a (1-u)^b. \quad (31)$$

Hence, we can consider the generalized Jacobi random variable as a generalization of the β -random variable.

4.2 Moments of Order $m > 0$

Clearly, $E(X^0) = 1$. For this reason, we let $m > 0$ and condition (29) hold.

Thus, from [1, (14.31)], we have:

$$E(X^m) = K \cdot \sum_{s=0}^{\infty} \binom{m}{s} (-1)^s J_{\alpha, \beta}^{a, b, c, p} [(1-x)^s; \nu].$$

Then from (15a), for $A = s$, $B = 0$, we get:

$$E(X^m) = K \sum_{s=0}^{\infty} \frac{\binom{m}{s} (-1)^s (\alpha + 1)_\nu B(a + s + 1, b + 1) 2^{a+s+b+1}}{\Gamma(\nu + 1)} \cdot {}_4F_3 \left(\begin{matrix} -\nu, \nu + \lambda, c, a + s + 1 \\ \alpha + 1, p, a + s + b + 2 \end{matrix}; 1 \right), \tag{32}$$

which can also be expressed as,

$$E(X^m) = \frac{1}{R} \cdot F \left(\begin{matrix} 1 & a + 1 \\ 3 & -m, -\nu; 1, \nu + \lambda; 1, c \\ 1 & a + b + 2 \\ 2 & 1, \alpha + 1; 1, p \end{matrix} \middle| 2, 1 \right). \tag{33}$$

It is also possible to calculate the modified moment of the form, $E((I - X)^A(I + X)^B)$ with $A > -(1 + a); B > -(1 + b)$.

Indeed, from (15a), we have (with conditions (29)),

$$E((I - X)^A(I + X)^B) = \frac{2^{A+B} B(a + A + 1, b + B + 1)}{R \cdot B(a + 1, b + 1)} \cdot {}_4F_3 \left(\begin{matrix} -\nu, \nu + \lambda, c, a + A + 1 \\ \alpha + 1, p, a + A + b + B + 2 \end{matrix}; 1 \right). \tag{34}$$

4.3 Distribution Function

The distribution function of the generalized Jacobi random variable is defined as:

$$\begin{aligned} G_{a,b}^{(\alpha, \beta, c, p, \nu)}(x) &= \int_{-\infty}^x g_{a,b}^{(\alpha, \beta, c, p, \nu)}(x) dx \\ &= K \int_{-1}^x (1 - t)^a (1 + t)^b P_\nu^{(\alpha, \beta, c, p)}(t) dt, \end{aligned}$$

for $|x| \leq 1$. Here and in the sequel, we abbreviate this distribution as $G(x)$.

Thus, by virtue of [1, (14.31)], we obtain:

$$G(x) = \frac{K(\alpha + 1)_\nu}{\Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu + \lambda)_n (c)_n}{(\alpha + 1)_n (p)_n n! 2^n} \int_{-1}^x (1 - t)^{a+n} (1 + t)^b dt. \tag{35}$$

Moreover, since

$$\int_{-1}^x (1 - t)^{a+n} (1 + t)^b dt$$

$$= 2^{a+b+n+1} [B(a+n+1, b+1) - B_{\frac{1-x}{2}}(a+n+1, b+1)],$$

where $B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$ is the incomplete beta function, $G(x) = G_1(x) - G_2(x)$, where:

$$G_1(x) = \frac{K(\alpha+1)_\nu 2^{a+b+1} B(a+1, b+1)}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu+\lambda)_n (c)_n (a+1)_n}{(\alpha+1)_n (p)_n (a+b+2)_n n!} \quad (36)$$

$$= \frac{1}{R} \cdot {}_4F_3 \left(\begin{matrix} -\nu, \nu+\lambda, c, a+1 \\ \alpha+1, p, a+b+2 \end{matrix}; 1 \right), \quad (37)$$

since $p - \beta - c + b + 1 > 0$.

On the other hand, due to [2, p. 87], we have that

$$B_x(p, q) = \frac{x^p}{p} \cdot {}_2F_1 \left(\begin{matrix} p, 1-q \\ p+1 \end{matrix}; x \right) \quad (p, q > 0; 0 < x < 1). \quad (38)$$

Therefore,

$$\begin{aligned} G_2(x) &= \frac{K(\alpha+1)_\nu}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-\nu)_n (\nu+\lambda)_n (c)_n 2^{a+b+n+1}}{(\alpha+1)_n (p)_n n! 2^n} B_{\frac{1-x}{2}}(a+n+1, b+1) \\ &= \frac{K(\alpha+1)_\nu 2^{a+b+1}}{\Gamma(\nu+1)} \left(\frac{1-x}{2} \right)^{a+1} \\ &\cdot \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a+1)_{n+k} (-\nu)_n (\nu+\lambda)_n (c)_n (-b)_k}{(a+2)_{n+k} (\alpha+1)_n (p)_n n! k!} \left(\frac{1-x}{2} \right)^{n+k}, \end{aligned} \quad (39)$$

and by using Kampé de Fériét function, (39) can be expressed as,

$$\begin{aligned} G_2(x) &= \frac{1}{B(a+1, b+1)R} \left(\frac{1-x}{2} \right)^{a+1} \\ &\cdot F \left(\begin{matrix} 1 & a+1 \\ 3 & -\nu, -b; 1, \nu+\lambda; c, 1 \\ 1 & a+2 \\ 2 & \alpha+1, 1; p, 1 \end{matrix} \middle| \frac{1-x}{2}, \frac{1-x}{2} \right). \end{aligned} \quad (39.a)$$

Then from (36) and (39.a), the distribution function reduces to:

$$\begin{aligned} G_{a,b}^{(\alpha, \beta, c, p, \nu)}(x) &= \frac{1}{R} \left[{}_4F_3 \left(\begin{matrix} -\nu, \nu+\lambda, c, a+1 \\ \alpha+1, p, a+b+2 \end{matrix}; 1 \right) \right. \\ &\quad \left. - \frac{1}{B(a+1, b+1)} \left(\frac{1-x}{2} \right)^{a+1} \right] \end{aligned}$$

$$\cdot F \left(\begin{array}{c|c|c} 1 & a+1 & \\ 3 & -\nu, -b; \nu+\lambda, 1; c, 1 & \\ \hline 1 & a+2 & \\ 2 & \alpha+1, 1; p, 1 & \end{array} \middle| \begin{array}{c} \frac{1-x}{2}, \frac{1-x}{2} \end{array} \right). \tag{40}$$

4.4 Characteristic Function

For a random variable X with continuous density function $f(x)$, the characteristic function of X is defined by:

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx = \sqrt{2\pi} \mathcal{F}[f(x); t], \tag{41}$$

where \mathcal{F} is the Fourier's transform of $f(x)$.

For the generalized Jacobi random variable, under the conditions given in (29), we have

$$\begin{aligned}
 \varphi_X(t) &= \frac{K(\alpha+1)_\nu}{\Gamma(\nu+1)} \int_{-1}^1 e^{itx} (1-x)^a (1+x)^b \\
 &\cdot {}_3F_2 \left(\begin{array}{c} -\nu, \nu+\lambda, c \\ \alpha+1, p \end{array} ; \frac{1-x}{2} \right) dx. \tag{42}
 \end{aligned}$$

If we set $u = \frac{1-x}{2}$ in (42), it becomes

$$\begin{aligned}
 \varphi_X(t) &= \frac{K(\alpha+1)_\nu 2^{a+b+1}}{\Gamma(\nu+1)} \int_{-1}^1 e^{it(1-2u)} u^a (1-u)^b \\
 &\cdot {}_3F_2 \left(\begin{array}{c} -\nu, \nu+\lambda, c \\ \alpha+1, p \end{array} ; u \right) du. \tag{43}
 \end{aligned}$$

As $F(u) = {}_3F_2 \left(\begin{array}{c} -\nu, \nu+\lambda, c \\ \alpha+1, p \end{array} ; u \right)$ and it is continuous on $[0, 1]$, due to $p - \beta - c > 0$, $M_1, M_2 > 0$ exist so that:

$$| e^{-2itu} u^a (1-u)^b F(u) | \leq h(t) = \begin{cases} M_1 (1-u)^b & \text{in } [\frac{1}{2}, 1] \\ M_2 u^a & \text{in } [0, \frac{1}{2}] \end{cases}.$$

Furthermore, $h(t)$ is integrable on $[0, 1]$, since $a > -1$ and $b > 0 > -1$, and this ensures the convergence of (43).

Also, from the uniform convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n (2itu)^n}{n!} = e^{-2itu}$ on $[r, s]$, $\forall r, s$

with $0 < r \leq s \leq 1$, we have that [1, (14.31)] is applicable. Hence,

$$\varphi_X(t) = \frac{K(\alpha+1)_\nu 2^{a+b+1} e^{it}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n (2it)^n}{n!} \int_0^1 u^{a+n} (1-u)^b \cdot {}_3F_2 \left(\begin{matrix} -\nu, \nu+\lambda, c \\ \alpha+1, p \end{matrix}; u \right) du. \quad (44)$$

Using [7, p. 161 (2)], with $p = 3 = q + 1$, $a + 1 > 0$, $b > 0$, and $p - \beta - c > 0$, we have the following:

$$\varphi_X(t) = \frac{K(\alpha+1)_\nu 2^{a+b+1} B(a+1, b+1) e^{it}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(a+1)_n (-2it)^n}{(a+b+2)_n n!} \cdot {}_4F_3 \left(\begin{matrix} a+n+1, -\nu, \nu+\lambda, c \\ a+b+n+2, \alpha+1, p \end{matrix}; 1 \right). \quad (45)$$

Finally, $\varphi_X(t)$ can be expressed in terms of the Kampé de Fériét's function,

$$\varphi_X(t) = \frac{e^{it}}{R} \cdot F_{1:2;0}^{1:3;0} \left[\begin{matrix} a+1; -\nu, \nu+\lambda, c; -; \\ a+b+2; \alpha+1, p; -; \end{matrix} \quad 1, -2it \right]. \quad (46)$$

Observing that, with $t = -i\tau$ in (46), we have the moment generating function of X . Indeed,

$$M_X(\tau) = \frac{e^\tau}{R} \cdot F_{1:2;0}^{1:3;0} \left[\begin{matrix} a+1; -\nu, \nu+\lambda, c; -; \\ a+b+2; \alpha+1, p; -; \end{matrix} \quad 1, -2\tau \right]. \quad (47)$$

The above expression can be reached by e^{xt} expansion in

$$M_X(t) = \int_{-1}^1 e^{tx} K(1-x)^a (1+x)^b P_\nu^{(\alpha, \beta, c, p)}(x) dx.$$

Similarly, we have the Fourier transform of

$$(1-x)^a (1+x)^b {}_3F_2 \left(\begin{matrix} -\nu, \nu+\lambda, c \\ \alpha+1, p \end{matrix}; \frac{1-x}{2} \right)$$

from (46) and (41). Indeed:

$$\begin{aligned} & \mathfrak{F} \left[(1-x)^a (1+x)^b {}_3F_2 \left(\begin{matrix} -\nu, \nu + \lambda, c \\ \alpha + 1, p \end{matrix}; \frac{1-x}{2} \right); t \right] \\ &= \frac{(\alpha + 1)_\nu B(a + 1, b + 1) 2^{a+b+1} e^{it}}{\sqrt{2\pi} \Gamma(\nu + 1)} \cdot F_{1:2;0}^{1:3;0} \left[\begin{matrix} a + 1: -\nu, \nu + \lambda, c - ; \\ a + b + 2: \alpha + 1, p; - ; \end{matrix} \middle| 1, -2\tau \right]. \end{aligned} \tag{48}$$

4.5 The Distribution of $X + X'$, Where Both of Them are Independent Generalized Jacobi Random Variables

Let X be a generalized Jacobi random variable whose parameters are $(\alpha, \beta, c, p, \nu; a, b)$ and X' be a generalized Jacobi random variable whose parameters are $(\alpha', \beta', c', p', \nu'; a', b')$, under the additional assumption that both group of parameters satisfy (29) and that these random variables are independent.

Since $\varphi_{X + X'}(t) = \varphi_X(t)\varphi_{X'}(t)$, from (43) we have:

$$\begin{aligned} \varphi_{X + X'}(t) &= \frac{e^{2it}}{R \cdot R'} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} P(a, b, \nu, \lambda, c, \alpha, p, n, k) \\ & P(a', b', \nu', \lambda', c', \alpha', n', k') (-2it)^{n+n'}. \end{aligned} \tag{49}$$

Here the change of order is made possible due to the absolute convergence of the intermediate series of the quadruple series as well. Likewise:

$$P = P(a, b, \nu, \lambda, c, \alpha, p, n, k) = \frac{(a + 1)_{n+k} (-\nu)_k (\nu + \lambda)_k (c)_k}{(a + b + 2)_{n+k} (\alpha + 1)_k (p)_k n! k!}. \tag{50}$$

Then, from (41), we have that the density function of $X + X'$ in the form:

$$f_{X + X'}(x) = \frac{1}{\sqrt{2\pi}} \mathfrak{F}^{-1} [\varphi_{X + X'}(t); x] \tag{51}$$

Because of uniform convergence, we have from (51) the following result:

$$f_{X + X'}(x) = \frac{1}{\sqrt{2\pi} R R'} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} (-2i)^{n+n'} P P' \mathfrak{F}^{-1} [e^{2it} t^{n+n'}; x]. \tag{52}$$

Using [8, p. 517(19)], we have from (50) that

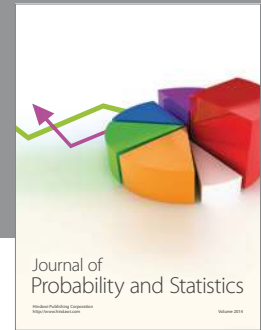
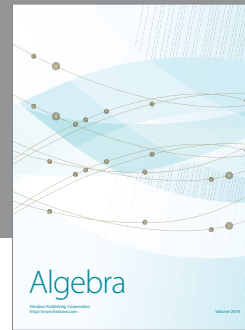
$$f_{X + X'}(x) = \frac{1}{R R'} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} (-2)^{n+n'} P P' \delta^{(n+n')}(2-x), \tag{53}$$

where $\delta(x)$ is the Dirac generalized function (see [3, p. 11-13] and [11, p. 484-504]).

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