ON A HEATH-JARROW-MORTON APPROACH FOR STOCK OPTIONS

JAN KALLSEN PAUL KRÜHNER

ABSTRACT. This paper aims at transferring the philosophy behind Heath-Jarrow-Morton to the modelling of call options with all strikes and maturities. Contrary to a related contribution by Carmona and Nadtochiy [7], the key parametrisation of our approach involves time-inhomogeneous Lévy processes instead of local volatility models. We provide necessary and sufficient conditions for absence of arbitrage. Moreover we discuss the construction of arbitrage-free models. Specifically, we prove their existence and uniqueness given basic building blocks.

Keywords: Heath-Jarrow-Morton, option price surfaces, Lévy processes, affine models

1. INTRODUCTION

The traditional approach to modelling stock options takes the underlying as a starting point. If the dynamics of the stock are specified under a risk neutral measure for the whole market (i.e. all discounted asset price processes are martingales), then options prices are obtained as conditional expectations of their payoff. In reality standard options as calls and puts are liquidly traded. If one wants to obtain vanilla option prices which are consistent with observed market values, special care has to be taken. A common and also theoretically reasonable way is calibration, i.e. to choose the parameters for the stock dynamics such that the model approximates market values sufficiently well. After a while, models typically have to be recalibrated, i.e. different parameters have to be chosen in order for model prices to be still consistent with observed values. However, frequent recalibration is unsatisfactory from a theoretical point of view because model parameters are meant to be deterministic and constant. Its necessity indicates that the chosen class fails to describe the market consistently.

A possible way out is to model the whole surface of call options as a state variable, i.e. as a family of primary assets in their own right. This alternative perspective is motivated from the Heath-Jarrow-Morton (HJM, see [13]) approach in interest rate theory. Rather than considering bonds as derivatives on the short rate, HJM treat the whole family of zero bonds or equivalently the forward rate curve as state variable in the first place. In the context of HJM-type approaches for stock options, [24] and [22] consider the case of a single strike, whereas [7] allows for all strikes and maturities. Further important references in this context include [14, 23, 25]. The HJM approach has been adapted to other asset classes, e.g. credit models [4, 22] and variance swaps [5], cf. [6] for an overview and further references.

Similar to [7] we aim at modelling the whole call option price surface using the HJM methodology. However, our approach differs in the choice of the parametrisation or *codebook*, which constitutes a crucial step in HJM-type setups. By relying on time-inhomogeneous Lévy processes rather than Dupire's local volatility models, we can avoid some intrinsic difficulties of the framework in [7]. Very recently and independently of the present study, Carmona and Nadtochiy [8] have also put forward a HJM-type approach for the option price surface which is based on time-inhomogeneous Lévy processes. The similarities and differences of their and our approach are discussed in Section 5.

The paper is arranged as follows. We start in Section 2 with an informal discussion of the HJM philosophy, as a motivation to its application to stock options. Section 3 provides necessary and sufficient conditions for an option surface model to be arbitrage-free or, more precisely, risk-neutral. Subsequently, we turn to the construction of option surface models from basic building blocks. In particular, we provide a concrete example which turns out to be related to the stochastic volatility model in [2]. Mathematical tools and some technical proofs are relegated to the appendix.

Notation. Re and Im denote the real resp. imaginary part of a complex vector in $\mathbb{C}^d$. We write $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0\}$. For $a, b \in \mathbb{R}$ we denote the closed interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$, which is empty if $a > b$. We use the notations ∂_u and D for partial and total derivatives, respectively. We often write $\beta \bullet X_t = \int_0^t \beta_s dX_s$ for stochastic integrals. $L(X)$ denotes the set of X-integrable predictable processes for a semimartingale X. If we talk about an $m + n$-dimensional semimartingale (X, Y), we mean that X is an $\mathbb{R}^m$-valued semimartingale and Y is an $\mathbb{R}^n$-valued semimartingale. For $u, v \in \mathbb{C}^d$ we denote the scalar product of u and v by $uv := \sum_{k=1}^d u_k v_k$. Finally, I denotes the identity process, i.e. $I_t = t$. The abbreviation *PII* stands for processes with independent increments in the sense of [15]. Further unexplained notation is used as in [15].

2. Heath-Jarrow-Morton and Lévy models

This section provides an informal discussion of the HJM philosophy and its application to stock options.

2.1. **The Heath-Jarrow-Morton philosophy.** According to the fundamental theorem of asset pricing, there exists at least one equivalent probability measure that turns discounted prices of all traded securities into martingales or, more precisely, into σ-martingales. For simplicity we take the point of view of risk-neutral modelling in this paper, i.e. we specify the dynamics of all assets in the market directly under such an equivalent martingale measure (EMM). Moreover, we assume the existence of a constant bank account. Put differently, all prices are expressed in discounted terms.

Before we turn to our concrete setup, we want to highlight key features of the HJM approach in general. For more background and examples we refer the reader to the brilliant exposition [6], which greatly inspired our work. We proceed by stating seven informal axioms or steps.

(1) In HJM-type setups there typically exists a canonical underlying asset or reference process, namely the money market account in interest rate theory or the stock in the present paper. The object of interest, namely bonds in interest rate theory or vanilla options in stock markets, can be interpreted as derivatives on the canonical process. HJM-type approaches typically focus on a whole manifold of such — at least in theory — liquidly traded derivatives, e.g. the one-dimensional manifold of bonds with all maturities or the two-dimensional manifold of call options with all strikes and maturities. As first and probably most important HJM axiom we claim that this manifold of liquid derivatives is to be treated as the set of primary assets. It — rather than the canonical reference asset — constitutes the object whose dynamics should be modelled in the first place.

(2) The first axiom immediately leads to the second one: do *not* model the canonical reference asset in detail under the market's risk-neutral measure. Indeed, otherwise all derivative prices would be entirely determined, leaving no room for a specification of their dynamics.

(3) Direct modelling of the above manifold typically leads to awkward constraints. Zero bond price processes must terminate in 1, vanilla options in their respective payoff. Rather than prices themselves one should therefore consider a convenient parametrisation (or *codebook* in the language of [6]), e.g. instantaneous forward rates in interest rate theory. Specifying the dynamics of this codebook leads immediately to a model for the manifold of primary assets. If the codebook is properly chosen, then the above awkward constraints are satisfied automatically.

(4) It is generally understood that choosing a convenient parametrisation constitutes a crucial step for a successful HJM-type approach. This is particularly obvious in the context of call options. Their prices are linked by a number of non-trivial static arbitrage constraints, which must hold independently of any particular model, cf. [11]. These static constraints have to be respected by any codebook dynamics. Specifying the latter properly may therefore be a difficult task unless the codebook is chosen such that the constraints naturally hold. We now suggest a way to choose such a codebock.

The starting point is a family of simple risk-neutral models for the canonical underlying whose parameter space has the same dimension as the manifold of the liquid derivatives. Provided sufficient regularity holds, the presently observed manifold of derivative prices is explained by one and only one of these models.

Example 1. In interest rate theory consider bank accounts of the form

$$S_t^0 = \exp\left(\int_0^t r(s)ds\right)$$

with deterministic short rate $r(T), T \in \mathbb{R}_+$. A differentiable curve of bond prices $B(t,T), T \in \mathbb{R}_+$ at a given time $t \in \mathbb{R}_+$ is consistent with one and only one of these models, namely for

$$r(T) := -\partial_T \log(B(t,T)). \tag{2.1}$$

Example 2. Consider Dupire's local volatility models

$$dS_t = S_t \sigma(S_t, t) dW_t$$

for a discounted stock, where W denotes standard Brownian motion and $\sigma : \mathbb{R}_+^2 \to \mathbb{R}$ a deterministic function. Up to regularity, any surface of discounted call option prices $C_t(T,K)$ with varying maturity T and strike K and fixed current date $t \in \mathbb{R}_+$ is obtained by one and only one local volatility function σ, namely by

$$\sigma(T,K) := \frac{2\partial_T C_t(T,K)}{K^2 \partial_{KK} C_t(T,K)} \tag{2.2}$$

If market data follows the simple model, the parameter manifold, e.g. $(r(T))_{T\in\mathbb{R}_+}$ in Example 1, is deterministic and does not depend on the time when the derivative prices are observed. Generally, however, market data does not follow such a simple model as in the two examples. Hence, evaluation of the right-hand side of (2.1) and (2.2) leads to a parameter manifold which changes randomly over time.

Example 1. The *instantaneous forward rate* curve

$$f(t,T) := -\partial_T \log(B(t,T)), \quad T \in \mathbb{R}_+$$

for fixed $t \in \mathbb{R}_+$ can be interpreted as the family of deterministic short rates that is consistent with the presently observed bond price curve $B(t,T), T \in \mathbb{R}_+$.

Example 2. The *implied local volatility*

$$\sigma_t(T,K) := \frac{2\partial_T C_t(T,K)}{K^2\partial_{KK}C_t(T,K)}, \quad T\in\mathbb{R}_+, K>0$$

for fixed $t\in\mathbb{R}_+$ can be interpreted as the unique local volatility function that is consistent with the presently observed discounted call prices $C_t(T,K)$, $T\in\mathbb{R}_+$, $K>0$.

The idea now is to take this present parameter manifold as a parametrisation or codebook for the manifold of derivatives.

(5) In a next step, we set this parameter manifold "in motion." We consider the codebook, e.g. the instantaneous forward curve $f(t,T)$ or the implied local volatility $\sigma_t(T,K)$, as an infinite-dimensional stochastic process. It is typically modelled by a stochastic differential equation, e.g.

$$df(t,T) = \alpha(t,T)dt + \beta(t,T)dW_t,$$

where W denotes standard Brownian motion. As long as the solution to this equation moves within the parameter space for the family of small models, one automatically obtains derivative prices that satisfy any static arbitrage constraints. Indeed, since the current bond prices resp. call prices coincide with the prices from an arbitrage-free model, they cannot violate any such constraints, however complicated they might be. This automatic absence of static arbitrage motivates the codebook choice in Step 4.

(6) Absence of static arbitrage does not imply absence of arbitrage altogether. Under the risk neutral modelling paradigm, all discounted assets must be martingales. In interest rate theory this leads to the well known HJM drift condition. More generally it means that the drift part of the codebook dynamics of Step 5 is determined by its diffusive component.

(7) Finally we come back to Step 2. The dynamics of the canonical reference asset process is typically implied by the current state of the codebook. E.g. in interest rate theory the short rate is determined by the so called consistency condition

$$r(t) = f(t,t).$$

Similar conditions determine the current stock volatility in [24, 25, 7].

2.2. **Time-inhomogeneous Lévy models.** According to the above interpretation, the approach of [7] to option surface modelling relies on the family of Dupire's local volatility models. We suggest another family of simple models for the stock, also relying on a two-dimensional parameter manifold. To this end, suppose that the discounted stock is a martingale of the form $S=e^X$, where the *return process* X denotes a process with independent increments (or time-inhomogeneous Lévy process, henceforth PII) on some filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}_+},P)$ and P denotes a risk neutral measure. More specifically, the characteristic function of X is assumed to be absolutely continuous in time, i.e.

$$E(e^{iuX_t}) = \exp\left(iuX_0 + \int_0^t \Psi(s,u)ds\right) \tag{2.3}$$

with some function $\Psi:\mathbb{R}_+\times\mathbb{R}\to\mathbb{C}$.

We assume call options of all strikes and maturities to be liquidly traded. Specifically, we write $C_t(T,K)$ for the discounted price at time t of a call which expires at T with discounted strike K. A slight extension of [3, Proposition 1] shows that option prices can be expressed in terms of Ψ. To this end, we define *modified option prices*

$$\mathscr{O}_t(T,x) := e^{-(x+X_t)}C_t(T,e^{x+X_t}) - (e^{-x}-1)^+.$$

Since call option prices are obtained from $C_t(T,K) = E((S_T - K)^+|\mathscr{F}_t)$, by call-put parity, and by $E(S_T|\mathscr{F}_t) = S_t$, we have

$$\mathscr{O}_t(T,x) = \begin{cases} E((e^{(X_T - X_t)-x} - 1)^+|\mathscr{F}_t) & \text{if } x \geq 0, \\ E((1 - e^{(X_T - X_t)-x})^+|\mathscr{F}_t) & \text{if } x < 0. \end{cases}$$

Proposition B.4 yields

$$\mathscr{O}_t(T,x) = \mathscr{F}^{-1}\left\{u \mapsto \frac{1 - E(e^{iu(X_T - X_t)}|\mathscr{F}_t)}{u^2 + iu}\right\}(x), \tag{2.4}$$

$$\mathscr{F}\{x \mapsto \mathscr{O}_t(T,x)\}(u) = \frac{1 - E(e^{iu(X_T - X_t)}|\mathscr{F}_t)}{u^2 + iu} \tag{2.5}$$

where $\mathscr{F}^{-1}$ and $\mathscr{F}$ denote the improper inverse Fourier transform and the improper Fourier transform, respectively, in the sense of (B.1, B.2) in Section B.1 in the appendix. Since

$$C_t(T,K) = (S_t - K)^+ + K\mathscr{O}_t\left(T, \log\frac{K}{S_t}\right) \tag{2.6}$$

and

$$E(e^{iu(X_T - X_t)}|\mathscr{F}_t) = \exp\left(\int_t^T \Psi(s,u)ds\right), \tag{2.7}$$

we can compute option prices according to the following diagram:

$$\Psi \to \exp\left(\int_t^T \Psi(s,\cdot)ds\right) \to \mathscr{O}_t(T,\cdot) \to C_t(T,\cdot).$$

For the last step we also need the current stock price S_t. Under sufficient smoothness we can invert all transformations. Indeed, we have

$$\Psi(T,u) = \partial_T \log\Big(1 - (u^2 + iu)\mathscr{F}\{x \mapsto \mathscr{O}_t(T,x)\}(u)\Big). \tag{2.8}$$

Hence we obtain option prices from Ψ and vice versa as long as we know the current stock price.

2.3. **Setting Lévy in motion.**

Generally we do not assume that the *return process*

$$X := \log(S) \tag{2.9}$$

follows a time-inhomogeneous Lévy process. Hence the right-hand side of Equation (2.8) will typically change randomly over time. In line with Step 4 above, we define modified option prices

$$\mathscr{O}_t(T,x) := e^{-(x+X_t)}C_t(T, e^{x+X_t}) - (e^{-x} - 1)^+ \tag{2.10}$$

as before and

$$\Psi_t(T,u) := \partial_T \log\Big(1 - (u^2 + iu)\mathscr{F}\{x \mapsto \mathscr{O}_t(T,x)\}(u)\Big). \tag{2.11}$$

This constitutes our *codebook process* for the surface of discounted option prices. As in Section 2.2 the asset price processes S and $C(T,K)$ can be recovered from X and $\Psi(T,u)$

via

$$
\begin{aligned}
S &= \exp(X), \\
\mathscr{O}_t(T,x) &= \mathscr{F}^{-1}\left\{u \mapsto \frac{1-\exp(\int_t^T \Psi_t(s,u)ds)}{u^2+iu}\right\}(x), \\
C_t(T,K) &= (S_t-K)^+ + K\mathscr{O}_t\left(T, \log\frac{K}{S_t}\right).
\end{aligned}
$$

In the remainder of this paper we assume that the infinite-dimensional codebook process satisfies an equation of the form

$$
d\Psi_t(T,u) = \alpha_t(T,u)dt + \beta_t(T,u)dM_t, \tag{2.12}
$$

driven by some finite-dimensional semimartingale M.

3. Model setup and risk neutrality

As before we fix a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ with trivial initial σ-field $\mathscr{F}_0$. In this section we single out conditions such that a given pair (X,Ψ) corresponds via (2.9 – 2.11) to a risk neutral model for the stock and its call options.

3.1. **Option surface models.** We denote by Π the set of characteristic exponents of Lévy processes L such that $E(e^{L_1}) = 1$. More precisely, Π contains all functions $\psi : \mathbb{R} \to \mathbb{C}$ of the form

$$
\psi(u) = -\frac{u^2+iu}{2}c + \int (e^{iux} - 1 - iu(e^x-1))K(dx),
$$

where $c \in \mathbb{R}_+$ and K denotes a Lévy measure on $\mathbb{R}$ satisfying $\int_{\{x>1\}} e^x K(dx) < \infty$.

Definition 3.1. A quintuple $(X, \Psi_0, \alpha, \beta, M)$ is an *option surface model* if

- (X, M) is a $1+d$-dimensional semimartingale that allows for local characteristics in the sense of Section A.1,
- $\Psi_0 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ with $\int_0^T |\Psi_0(r,u)|dr < \infty$ for any $T \in \mathbb{R}_+$, $u \in \mathbb{R}$,
- $\alpha(T,u), \beta(T,u)$ are $\mathbb{R}$- resp. $\mathbb{R}^d$-valued predictable processes for any $T \in \mathbb{R}_+$, $u \in \mathbb{R}$,
- $(\omega, t, T, u) \mapsto \alpha_t(T,u)(\omega), \beta_t(T,u)(\omega)$ are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}$-measurable, where $\mathscr{P}$ denotes the predictable σ-field on $\Omega \times \mathbb{R}_+$,
- $\int_0^t \int_0^T |\alpha_s(r,u)|drds < \infty$ for any $t, T \in \mathbb{R}_+$, $u \in \mathbb{R}$,
- $\int_0^T (\beta_t(r,u))^2 dr < \infty$ for any $t, T \in \mathbb{R}_+$, $u \in \mathbb{R}$,
- $\left((\int_0^T (\beta_t(r,u))^2 dr)^{\frac{1}{2}}\right)_{t\in\mathbb{R}_+} \in L(M)$ for any fixed $T \in \mathbb{R}_+$, $u \in \mathbb{R}$,
- the corresponding *codebook process*

$$
\Psi_t(T,u) := \Psi_0(T,u) + \int_0^{t\wedge T} \alpha_s(T,u)ds + \int_0^{t\wedge T} \beta_s(T,u)dM_s \tag{3.1}
$$

  has the following properties:
  (1) $(\omega, t, T, u) \mapsto \Psi_t(T,u)(\omega)$ is $\mathscr{O} \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}$-measurable, where $\mathscr{O}$ denotes the optional σ-field on $\Omega \times \mathbb{R}_+$,
  (2) $u \mapsto \int_t^T \Psi_s(r,u)dr(\omega)$ is in Π for any $T \in \mathbb{R}_+$, $t \in [0,T]$, $s \in [0,t]$, $\omega \in \Omega$.

In line with Section 2.3, the discounted stock and call price processes associated with an option surface model are defined by

$$\begin{aligned} S_t &:= \exp(X_t), & (3.2)\\ \mathscr{O}_t(T,x) &:= \mathscr{F}^{-1}\left\{u \mapsto \frac{1-\exp\left(\int_t^T \Psi_t(r,u)dr\right)}{u^2+iu}\right\}(x), & (3.3)\\ C_t(T,K) &:= (S_t-K)^+ + K\mathscr{O}_t\left(T, \log\frac{K}{S_t}\right) & (3.4) \end{aligned}$$

for any $T \in \mathbb{R}_+$, $t \in [0,T]$, $x \in \mathbb{R}$, $K \in \mathbb{R}_+$, where $\mathcal{F}^{-1}$ denotes the improper inverse Fourier transform in the sense of Section B.1. We denote the local exponents of (X,M), X by $\psi^{(X,M)}$, ψ^X and their domains by $\mathscr{U}^{(X,M)}$, $\mathscr{U}^X$, cf. Definitions A.4 and A.6.

Remark 3.2. The existence of these processes is implied by the assumptions above. Indeed, by Fubini's theorem for ordinary and stochastic integrals [20, Theorem IV.65], we have

$$\int_0^T |\Psi_t(r,u)|dr < \infty.$$

Since $u \mapsto \int_t^T \Psi_t(r,u)dr(\omega) \in \Pi$, there is an infinitely divisible distribution Q on $(\mathbb{R},\mathscr{B})$ such that

$$\int e^{iux}Q(dx) = \exp\left(\int_t^T \Psi_t(r,u)dr(\omega)\right),$$

where $\omega \in \Omega$ is fixed. The random variable $Y : \mathbb{R} \to \mathbb{R}, t \mapsto t$ has the property that e^Y is Q-integrable with expectation 1 because the characteristic function of Y is in Π. Thus Proposition B.4 yields the existence of the inverse Fourier transform in Equation (3.3). Moreover, it implies $C_t(T,K)(\omega) = E_Q((S_t(\omega)e^Y - K)^+)$ and thus we have $0 \le C_t(T,K)(\omega) \le S_t(\omega)$ and $0 \le P_t(T,K)(\omega) \le K$ for any $K \in \mathbb{R}_+$, $T \in \mathbb{R}_+$, $t \in [0,T]$, where $P_t(T,K) := C_t(T,K) + K - S_t$ for any $K \in \mathbb{R}_+$, $T \in \mathbb{R}_+$, $t \in [0,T]$.

As noted above, we model asset prices under a risk neutral measure for the whole market. Put differently, we are interested in risk-neutral option surface models in the following sense.

Definition 3.3. An option surface model $(X, \Psi_0, \alpha, \beta, M)$ is called *risk neutral* if the corresponding stock S and all European call options $C(T,K)$, $T \in \mathbb{R}_+$, $K > 0$ are σ-martingales or, equivalently, local martingales (cf. [16, Proposition 3.1 and Corollary 3.1]). It is called *strongly risk neutral* if S and all $C(T,K)$ are martingales.

Below, risk-neutral option surface models are characterized in terms of the following properties.

Definition 3.4. An option surface model $(X, \Psi_0, \alpha, \beta, M)$ satisfies the *consistency condition* if

$$\psi_t^X(u) = \Psi_{t-}(t,u), \quad u \in \mathbb{R}.$$

outside some $dP \otimes dt$-null set. Moreover, it satisfies the *drift condition* if

$$\left(u, -i\int_t^T \beta_t(r,u)dr\right)_{t\in\mathbb{R}_+} \in \mathscr{U}^{(X,M)}$$

and

$$\int_t^T \alpha_t(r,u)dr = \psi_t^X(u) - \psi_t^{(X,M)}\left(u, -i\int_t^T \beta_t(r,u)dr\right) \tag{3.5}$$

outside some $dP \otimes dt$ null set for any $T \in \mathbb{R}_+$, $u \in \mathbb{R}$. Finally, the option surface model satisfies the *conditional expectation condition* if

$$\exp\left(\int_t^T \Psi_t(r,u)dr\right) = E(e^{iu(X_T - X_t)}|\mathscr{F}_t)$$

for any $T \in \mathbb{R}_+$, $t \in [0,T]$, $u \in \mathbb{R}$.

Remark 3.5. The drift condition can be rewritten as

$$\alpha_t(T,u) = -\partial_T\left(\psi_t^{(X,M)}\left(u, -i\int_t^T \beta_t(r,u)dr\right)\right)$$

for almost all $T \in \mathbb{R}_+$. It gets even simpler if X and M are assumed to be locally independent in the sense of Definition A.10:

$$\alpha_t(T,u) = -\partial_T\left(\psi_t^M\Big(-i\int_t^T \beta_t(r,u)dr\Big)\right).$$

If the derivative $\psi_t'(u) := \partial_u \psi_t^M(u)$ exists as well, the drift condition simplifies once more and turns into

$$\alpha_t(T,u) = i\psi_t'\Big(-i\int_t^T \beta_t(r,u)dr\Big)\beta_t(T,u).$$

Now consider the situation that M is a one-dimensional Brownian motion which is locally independent of the return process X. Then $\psi^M(u) = -\frac{u^2}{2}$ and the drift condition reads as

$$\alpha_t(T,u) = -\beta_t(T,u)\int_t^T \beta_t(r,u)dr.$$

Thus the drift condition for option surface models is similar to the HJM drift condition (cf. [13]).

3.2. **Necessary and sufficient conditions.** The goal of this section is to prove the following characterisation of risk-neutral option surface models.

Theorem 3.6. *For any option surface model $(X, \Psi_0, \alpha, \beta, M)$ the following statements are equivalent.*

(1) *It is strongly risk neutral.*
(2) *It is risk neutral.*
(3) *It satisfies the conditional expectation condition.*
(4) *It satisfies the consistency and drift conditions.*

The remainder of this section is devoted to the proof of Theorem 3.6. We proceed according to the following scheme

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1).$$

We use the notation

$$\begin{aligned}
\delta_t(T,u) &:= \int_t^T \alpha_t(r,u)dr - \psi_t^X(u),\\
\sigma_t(T,u) &:= \int_t^T \beta_t(r,u)dr,\\
\Gamma_t(T,u) &:= \int_0^T \Psi_0(r,u)dr + \int_0^t \delta_s(T,u)ds + \int_0^t \sigma_s(T,u)dM_s.
\end{aligned}$$

δ, σ correspond to the integrated drift and diffusion parts in the original HJM setup. The existence of the integrals above are implied by the condition for option surface models. Observe that $\Gamma(T, u)$ is a semimartingale.

Lemma 3.7. *For all $u \in \mathbb{R}, T \in \mathbb{R}_+, t \in [0, T]$ we have*

$$\Gamma_t(T, u) - \Gamma_t(t, u) = \int_t^T \Psi_t(r, u)dr.$$

Proof. Using the definition of Γ, δ, σ and applying Fubini's theorem as in [20, Theorem IV.65] yields

$$\begin{aligned}\Gamma_t(T, u) - \Gamma_t(t, u) &= \int_t^T \Psi_0(r, u)dr + \int_t^T \int_0^t \alpha_s(r, u)dsdr + \int_t^T \int_0^t \beta_s(r, u)dM_s dr \\ &= \int_t^T \Psi_t(r, u)dr.\end{aligned}$$

□

Lemma 3.8. *Let $u \in \mathbb{R}, t \in \mathbb{R}_+$ and $Y_t := \Gamma_t(t, u)$. Then*

$$Y_t = \int_0^t \left(\Psi_{s-}(s, u) - \psi_s^X(u)\right) ds.$$

Proof. Observe that

$$\begin{aligned}Y_t &= \int_0^t \Psi_0(r, u)dr + \int_0^t \delta_s(t, u)ds + \int_0^t \sigma_s(t, u)dM_s \\ &= \int_0^t \left(\Psi_0(r, u) + \int_0^r \alpha_s(r, u)ds - \psi_r^X(u)\right)dr + \int_0^t \int_s^t \beta_s(r, u)drdM_s.\end{aligned}$$

By Fubini's theorem for stochastic integrals the last term equals $\int_0^t \int_0^r \beta_s(r, u)dM_s dr$. This yields the claim. □

Lemma 3.9. *If $(X, \Psi_0, \alpha, \beta, M)$ is risk neutral, then it satisfies the conditional expectation condition.*

Proof. Let $T \in \mathbb{R}_+$. We define

$$O_t(T, x) := \begin{cases} e^{-x} C_t(T, e^x) & \text{if } x \geq 0, \\ e^{-x} P_t(T, e^x) & \text{if } x < 0, \end{cases}$$

where $P_t(T, K) := C_t(T, K) + K - S_t$ for any $K \in \mathbb{R}_+$, $t \in [0, T], x \in \mathbb{R}$. Then we have

$$O_t(T, x) = \begin{cases} (e^{X_t - x} - 1)^+ + \mathscr{O}_t(T, x - X_t) & \text{if } x \geq 0, \\ (1 - e^{X_t - x})^+ + \mathscr{O}_t(T, x - X_t) & \text{if } x < 0. \end{cases}$$

We calculate the Fourier transform of $O_t(T, x)$ in two steps by considering the summands separately. The improper Fourier transform of the second summand $\mathscr{O}_t(T, x - X_t)$ exists and satisfies

$$\begin{aligned}\mathscr{F}\{x \mapsto \mathscr{O}_t(T, x - X_t)\}(u) &= \mathscr{F}\{x \mapsto \mathscr{O}_t(T, x)\}(u)e^{iuX_t} \\ &= \frac{1 - \exp\left(\int_t^T \Psi_t(r, u)dr\right)}{u^2 + iu} e^{iuX_t}\end{aligned}$$

for any $u \in \mathbb{R}\backslash\{0\}$ by Remark 3.2, Proposition B.4 and the translation property for the Fourier transform, which holds for the improper Fourier transform as well. The Fourier transform of the first summand $A_t(T,x) := O_t(T,x) - \mathscr{O}_t(T, x - X_t)$ exists and equals

$$\mathscr{F}\{x \mapsto A_t(T,x)\}(u) = \frac{1}{iu} - \frac{e^{X_t}}{iu-1} - \frac{e^{iuX_t}}{u^2+iu}$$

for any $u \in \mathbb{R}\backslash\{0\}$. Therefore the improper Fourier transform of $x \mapsto O_t(T,x)$ exists and is given by

$$\mathscr{F}\{x \mapsto O_t(T,x)\}(u) = \frac{1}{iu} - \frac{e^{X_t}}{iu-1} - \frac{\exp(iuX_t + \int_t^T \Psi_t(r,u)dr)}{u^2+iu}, \tag{3.6}$$

for any $u \in \mathbb{R}\backslash\{0\}$. By Lemmas 3.7 and 3.8 we have that the right-hand side of (3.6) is a semimartingale, in particular it has càdlàg paths. Remark 3.2 yields that $0 \leq P_t(T,K) \leq K$. Hence $(P_t(T,K))_{t\in[0,T]}$ is a martingale because it is a bounded local martingale. Let $(\tau_n)_{n\in\mathbb{N}}$ denote a common localising sequence for $(C_t(T,1))_{t\in[0,T]}$ and S, i.e. S^{τ_n}, $C^{\tau_n}(T,1)$ are uniformly integrable martingales for any $n \in \mathbb{N}$. Since $C_t^{\tau_n}(T,K) \leq C_t^{\tau_n}(T,1)$ for $K \in [1,\infty)$, we have that $(\tau_n)_{n\in\mathbb{N}}$ is a common localising sequence for all European calls with maturity T and strike $K \geq 1$. The definition of $O_t(T,x)$ yields that it is a local martingale for any $x \in \mathbb{R}$ and $(\tau_n)_{n\in\mathbb{N}}$ is a common localising sequence for $(O_t(T,x))_{t\in[0,T]}, x \in \mathbb{R}$.

Fix $\omega \in \Omega$. Since $u \mapsto \int_t^T \Psi_t(r,u)(\omega)dr$ is in Π for any $t \in [0,T]$, there is an infinitely divisible distribution Q on $(\mathbb{R},\mathscr{B})$ such that

$$\int e^{iux}Q(dx) = \exp\left(\int_t^T \Psi_t(r,u)(\omega)dr\right).$$

Then $\int e^y Q(dy) = 1$ and

$$O_t(T,x)(\omega) = \begin{cases} \int (S_t(\omega)e^{y-x} - 1)^+ Q(dy) & \text{if } x \geq 0, \\ \int (1 - S_t(\omega)e^{y-x})^+ Q(dy) & \text{if } x < 0, \end{cases}$$

cf. Remark 3.2. By Corollary B.3 we have $|\int_{-C}^{\infty} e^{iux}O_t(T,x)dx| \leq S_t(\omega) + \frac{1+2|u|}{u^2}$. Proposition B.5 yields that

$$(\mathscr{F}\{x \mapsto O_t(T,x)\}(u))_{t\in[0,T]}$$

and hence $(\Phi_t(u))_{t\in[0,T]}$ given by

$$\Phi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{C}, (\omega,t,u) \mapsto \Phi_t(u)(\omega) := \exp\left(iuX_t(\omega) + \int_t^T \Psi_t(r,u)(\omega)dr\right)$$

are local martingales for any $u \in \mathbb{R}\backslash\{0\}$. Since $u \mapsto \int_t^T \Psi_t(r,u)dr(\omega)$ is in Π for any $t \in [0,T]$, $\omega \in \Omega$, its real part is bounded by 0 from above cf. Lemma A.14. Hence $|\Phi_t(u)| \leq 1$ and thus $(\omega,t) \mapsto \Phi_t(u)(\omega)$ is a true martingale for any $u \in \mathbb{R}\backslash\{0\}$. By $\Phi_t(0) = 1$ it is a martingale for $u = 0$ as well. Since $\Phi_T(u) = \exp(iuX_T)$, the two martingales $(\Phi_t(u))_{t\in[0,T]}$ and $(E(\exp(iuX_T)|\mathscr{F}_t))_{t\in[0,T]}$ coincide for any $u \in \mathbb{R}$. Thus we have

$$\exp\left(\int_t^T \Psi_t(r,u)dr\right) = \exp(-iuX_t)\Phi_t(u) = E(e^{iu(X_T - X_t)}|\mathscr{F}_t)$$

for any $u \in \mathbb{R}, t \in [0,T]$. □

Lemma 3.10 (Drift condition in terms of δ and σ)**.** *If $(X, \Psi_0, \alpha, \beta, M)$ satisfies the conditional expectation condition, we have the drift condition*

$$\delta_t(T,u) = \Psi_{t-}(t,u) - \psi_t^X(u) - \psi_t^{(X,M)}(u, -i\sigma_t(T,u))$$

outside some $dP \otimes dt$-null set for $T \in \mathbb{R}_+$, $u \in \mathbb{R}$. In particular, $(u, -i\sigma(T,u)) \in \mathscr{U}^{(X,M)}$.

Proof. For $u \in \mathbb{R}$ and $T \in \mathbb{R}_+$ define the process $Z_t := iuX_t + \int_t^T \Psi_t(r,u)dr$. The conditional expectation condition yields that $\exp(Z_t) = E(e^{iuX_T}|\mathscr{F}_t)$ is a martingale. Hence $-i \in \mathscr{U}^Z$ and $\psi_t^Z(-i) = 0$ by Proposition A.16. With $Y_t := \Gamma_t(t,u)$ we obtain

$$\begin{aligned} 0 &= \psi_t^Z(-i) \\ &= \psi_t^{iuX+\Gamma(T,u)-Y}(-i) \\ &= \psi_t^{iuX+\Gamma(T,u)}(-i) - \big(\Psi_{t-}(t,u) - \psi_t^X(u)\big) \\ &= \psi_t^{(iuX,\Gamma(T,u))}(-i,-i) - \Psi_{t-}(t,u) + \psi_t^X(u) \\ &= \psi_t^{(X,M)}(u, -i\sigma_t(T,u)) + \delta_t(T,u) - \Psi_{t-}(t,u) + \psi_t^X(u), \end{aligned}$$

where the second equation follows from Lemma 3.7, the third from Lemmas 3.8 and A.19, the fourth from Lemma A.18 and the last from Lemmas A.17 and A.19. □

Corollary 3.11 (Consistency condition)**.** *If $(X, \Psi_0, \alpha, \beta, M)$ satisfies the conditional expectation condition, then it satisfies the consistency condition.*

Proof. Lemma 3.10 and the definition of δ yield

$$\Psi_{t-}(t,u) = \delta_t(t,u) + \psi_t^X(u) + \psi_t^{(X,M)}(u,0) = \psi_t^X(u).$$

□

Corollary 3.12 (Drift condition)**.** *If $(X, \Psi_0, \alpha, \beta, M)$ satisfies the conditional expectation condition, then it satisfies the drift condition.*

Proof. This follows from Lemma 3.10 and Corollary 3.11. □

Lemma 3.13. *If the option surface model satisfies the consistency condition, then*

$$\Gamma_t(T,u) = \int_t^T \Psi_t(r,u)dr$$

for any $T \in \mathbb{R}_+$, $t \in [0,T]$, $u \in \mathbb{R}$.

Proof. This is a direct consequence of Lemmas 3.7 and 3.8. □

Lemma 3.14. *If the option surface model satisfies the drift condition, then $(\Phi_t(T,u))_{t\in[0,T]}$ defined by*

$$\Phi_t(T,u) := \exp(iuX_t + \Gamma_t(T,u))$$

is a local martingale for any $u \in \mathbb{R}$, $T \in \mathbb{R}_+$.

Proof. Fix T, u and define $Z_t := iuX_t + \Gamma_t(T,u)$. By the drift condition and Lemmas A.17 – A.19 we have

$$\begin{aligned} 0 &= \psi_t^{(X,M)}(u, -i\sigma_t(T,u)) + \delta_t(T,u) \\ &= \psi_t^{(X,\sigma(T,u)\bullet M)}(u,-i) + \delta_t(T,u) \\ &= \psi_t^{(iuX,\Gamma(T,u))}(-i,-i) \\ &= \psi_t^{iuX+\Gamma(T,u)}(-i) \\ &= \psi_t^Z(-i). \end{aligned}$$

Hence $\exp(Y)$ is a local martingale by Proposition A.16. □

Lemma 3.15. *$(X, \Psi_0, \alpha, \beta, M)$ satisfies the drift and consistency conditions if and only if it satisfies the conditional expectation condition.*

Proof. $\Leftarrow$: This is a restatement of Corollaries 3.12 and 3.11.

$\Rightarrow$: Fix $u \in \mathbb{R}$, $T \in \mathbb{R}_+$. Lemma A.14 implies that the absolute value of

$$\Phi_t(T,u) := \exp\left(iuX_t + \int_t^T \Psi_t(r,u)dr\right)$$

is bounded by 1. By Lemmas 3.13 and 3.14, $\Phi(T,u)$ is a local martingale and hence a martingale. This yields

$$\Phi_t(T,u) = E(\Phi_T(T,u)|\mathscr{F}_t) = E(e^{iuX_T}|\mathscr{F}_t).$$

□

Lemma 3.16. *If the option surface model $(X, \Psi_0, \alpha, \beta, M)$ satisfies the conditional expectation condition, then $S = e^X$ is a martingale.*

Proof. For $T \in \mathbb{R}_+$, $t \in [0,T]$ define $Y := X_T - X_t$. Let $P^{Y|\mathscr{F}_t}$ denote the conditional distribution of Y given $\mathscr{F}_t$. Moreover, let

$$\Phi(u) := E(e^{iuY}|\mathscr{F}_t) = \exp\left(\int_t^T \Psi_t(r,u)dr\right).$$

Fix $\omega \in \Omega$. Since $u \mapsto \int_t^T \Psi_t(r,u)dr(\omega)$ is in Π, we have

$$E_{\widetilde{P}}(e^{iuL_1}) = \exp\left(\int_t^T \Psi_t(r,u)dr\right)(\omega)$$

for some Lévy process L on some filtered probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, (\widetilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+}, \widetilde{P})$. Consequently, we have $P^{Y|\mathscr{F}_t}(\omega) = \widetilde{P}$, which in turn implies

$$E(e^Y|\mathscr{F}_t)(\omega) = E_{\widetilde{P}}(e^{L_1}) = 1.$$

□

Lemma 3.17. *If the option surface model $(X, \Psi_0, \alpha, \beta, M)$ satisfies the conditional expectation condition, it is strongly risk neutral.*

Proof. Lemma 3.16 implies that e^X is a martingale and in particular that e^{X_t} is integrable for any $t \in \mathbb{R}_+$. Let $T \in \mathbb{R}_+$, $t \in [0,T]$. We define

$$\begin{aligned}
\widetilde{C}(K) &:= E((e^{X_T} - K)^+|\mathscr{F}_t),\\
\widetilde{\mathscr{O}}(x) &:= e^{-(x+X_t)}\widetilde{C}(e^{x+X_t}) - (e^{-x}-1)^+,\\
Y &:= X_T - X_t
\end{aligned}$$

for any $K \in \mathbb{R}_+$. Obviously we have

$$\widetilde{\mathscr{O}}(x) = \begin{cases} E((e^{Y-x}-1)^+|\mathscr{F}_t) & \text{if } x \geq 0,\\ E((1-e^{Y-x})^+|\mathscr{F}_t) & \text{if } x < 0 \end{cases}$$

and $E(e^Y|\mathscr{F}_t) = 1$. Hence Corollary B.4, the conditional expectation condition, and the definition of $\mathscr{O}$ yield

$$\begin{aligned}\mathscr{F}\{x \mapsto \widetilde{\mathscr{O}}(x)\}(u) &= \frac{1 - E(e^{iuY}|\mathscr{F}_t)}{u^2 + iu} \\ &= \frac{1 - \exp\left(\int_t^T \Psi_t(r,u)dr\right)}{u^2 + iu}\end{aligned}$$

as well as

$$\begin{aligned}\widetilde{\mathscr{O}}(x) &= \mathcal{F}^{-1}\left\{u \mapsto \frac{1 - \exp\left(\int_t^T \Psi_t(r,u)dr\right)}{u^2 + iu}\right\}(x) \\ &= \mathscr{O}_t(T, x).\end{aligned}$$

Thus we have

$$\widetilde{C}(K) = C_t(T, K)$$

for any $K \in \mathbb{R}_+$. Consequently, the option surface model $(X, \Psi_0, \alpha, \beta, M)$ is strongly risk neutral. □

Proof of Theorem 3.6. $(1) \Rightarrow (2)$ is obvious.
$(2) \Rightarrow (3)$ has been shown in Lemma 3.9.
$(3) \Leftrightarrow (4)$ is the conclusion of Lemma 3.15.
$(3) \Rightarrow (1)$ has been shown in Lemma 3.17. □

4. EXAMPLES AND EXISTENCE RESULTS

4.1. **Building blocks.** The goal in this section is to construct risk-neutral option surface models from basic building blocks. In order to model the forward rate

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \beta(s,T)dW_s \tag{4.1}$$

in the original HJM setup, one specifies the initial state $f(0,T)$, $T \in \mathbb{R}_+$, the driving Brownian motion (or a more general process) W, and the volatility processes $(\beta(t,T))_{t\in[0,T]}$, $T \in \mathbb{R}_+$ of the forward rate curve. The drift process, however, is determined by the HJM drift condition

$$\alpha(t,T) = \beta(t,T) \int_t^T \beta(t,r)dr,$$

cf. [13]. Therefore, it should not be fixed beforehand. Moreover, the short rate is determined by the present forward rate through the consistency condition

$$r(t) = f(t,t).$$

Put differently, the initial forward rate curve $f(0,\cdot)$, the volatility process β and the driving process W constitute the “right” building blocks for a HJM model. In line with the original HJM approach, we suggest that the building blocks in our setup should include the initial state of the codebook $\Psi_0(T,u)$, $T \in \mathbb{R}_+$, $u \in \mathbb{R}$, a driving Lévy process M, and the volatility processes $(\beta_t(T,u))_{t\in[0,T]}$, $T \in \mathbb{R}_+$, $u \in \mathbb{R}$. Moreover, we fix the initial stock price X_0, which has no counterpart in interest rate theory. However, X_0, Ψ_0, β, M do not fully determine the model unless the local dependency of X and M is specified as well. This is done by also providing the dependent part $X^{\parallel}$ of X relative to M as defined in Section A.3.

The drift and consistency conditions in terms of $X_0, \Psi_0, \beta, M, X^\parallel$ read as

$$\begin{aligned} \alpha_t(T,u) &= \partial_T\left(\psi^{(X^\parallel,M)}\left(u,-i\int_t^T \beta_t(r,u)dr\right)\right), \\ \psi_t^X(u) &= \Psi_0(t-,u)+\int_0^t \alpha_s(t,u)ds+\int_0^{t-}\beta_s(t,u)dM_s, \end{aligned} \tag{4.2}$$

if the differentiation of the right hand side of the first equation exists, cf. Remark 3.5 and Lemma 4.2 below. Hence we propose $X_0, \Psi_0, \beta, M, X^\parallel$ as building blocks for our model.

Definition 4.1. We call a risk-neutral option surface model $(X, \Psi_0, \alpha, \beta, M)$ *consistent* with $X_0, \Psi_0, \beta, M, X^\parallel$ if X_0 is the initial value of X and $X^\parallel$ equals the dependent part of X relative to M.

Lemma 4.2 (Modified drift condition)**.** *The drift condition (3.5) is equivalent to*

$$\left(u,-i\int_t^T \beta_t(r,u)dr\right)_{t\in[0,T]} \in \mathscr{U}^{(X^\parallel,M)}$$

and

$$\int_t^T \alpha_t(r,u)dr = \psi_t^{X^\parallel}(u)-\psi_t^{(X^\parallel,M)}\left(u,-i\int_t^T\beta_t(r,u)dr\right) \tag{4.3}$$

outside some $dP\otimes dt$-null set for any $T\in\mathbb{R}_+$, $u\in\mathbb{R}$.

Proof. Let $X^\perp$ denote the independent part of X relative to M in the sense of Section A.3. Lemmas A.18 and A.21 yield

$$\begin{aligned} &\psi_t^{(X,M)}\left(u,-i\int_t^T\beta_t(r,u)dr\right) \\ &= \psi_t^{(X^\parallel,M,X^\perp,0)}\left(u,-i\int_t^T\beta_t(r,u)dr,u,-i\int_t^T\beta_t(r,u)dr\right) \\ &= \psi_t^{(X^\parallel,M,X^\perp)}\left(u,-i\int_t^T\beta_t(r,u)dr,u\right) \\ &= \psi_t^{(X^\parallel,M)}\left(u,-i\int_t^T\beta_t(r,u)dr\right)+\psi_t^{X^\perp}(u) \end{aligned}$$

and similarly

$$\psi_t^X(u)=\psi_t^{(X^\parallel,X^\perp)}(u,u)=\psi_t^{X^\parallel}(u)+\psi_t^{X^\perp}(u).$$

This implies

$$\psi_t^X(u)-\psi_t^{(X,M)}\left(u,-i\int_t^T\beta_t(r,u)dr\right)=\psi_t^{X^\parallel}(u)-\psi_t^{(X^\parallel,M)}\left(u,-i\int_t^T\beta_t(r,u)dr\right)$$

for any $T\in\mathbb{R}_+$, $t\in[0,T]$, $u\in\mathbb{R}$. □

4.2. **The minimum value condition.** Since our codebook space is a convex cone rather than a vector space, we must ensure that we do not leave the cone at any time. It turns out that the initial state of the codebook Ψ_0 must be 'large' enough for this purpose. Throughout this section let $(X, \Psi_0, \alpha, \beta, M)$ be a risk-neutral option surface model and denote by $X^\parallel, X^\perp$ the dependent resp. independent part of X relative to M as defined in Section A.3. $E_t^k :=$

$\operatorname{ess\,inf} M_t^k$ are the componentwise essential infimums of M_t, which may attain the value $-\infty$. As in Section 3.2 we set

$$\begin{aligned}\delta_t(T,u) &:= \int_t^T \alpha_t(r,u)dr - \psi_t^X(u),\\ \sigma_t(T,u) &:= \int_t^T \beta_t(r,u)dr,\\ \Gamma_t(T,u) &:= \int_0^T \Psi_0(r,u)dr + \int_0^t \delta_s(T,u)ds + \int_0^t \sigma_s(T,u)dM_s\end{aligned}$$

for any $T \in \mathbb{R}_+$, $t \in [0,T]$, $u \in \mathbb{R}$. Let Υ be the convex cone of all functions $f : \mathbb{R}_+ \to \Pi$ such that $f(0) = 0$ and $f(t) - f(s) \in \Pi$ for any $s \leq t$, denote by $\bar{\Upsilon}$ the vector space generated by Υ, and let $\leq$ be the partial order on $\bar{\Upsilon}$ which is generated by the convex cone Υ (i.e. $f \leq g$ if $f - g \in \Upsilon$). By slight abuse of notation, we also write $f \in \Upsilon$ (resp. $\bar{\Upsilon}$) almost surely if $f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ and $t \mapsto f(\omega, t, \cdot)$ is in Υ (resp. $\bar{\Upsilon}$) for almost all $\omega \in \Omega$. The notation $f \leq g$ a.s. is used accordingly. Since $u \mapsto \int_t^T \Psi_0(r,u)dr$ is in Π for any $T \in \mathbb{R}_+$, $t \in [0,T]$, we conclude that $\Gamma_0 \in \Upsilon$.

Remark 4.3. For any (deterministic) $f \in \Upsilon$ there is a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ and a PII L such that e^L is a martingale and $E(e^{iuL_T}) = \exp(f(T,u))$ for any $T \in \mathbb{R}_+$, $u \in \mathbb{R}$, cf. [15, III.2.16]. Conversely, if L is a PII which allows for local characteristics and such that e^L is a martingale, there is a (deterministic) $f \in \Upsilon$ such that $E(e^{iuL_T}) = \exp(f(T,u))$ for any $T \in \mathbb{R}_+$, $u \in \mathbb{R}$.

The following result shows that the initial state of the codebook has a nontrivial lower bound in any risk-neutral option surface model.

Proposition 4.4 (Minimum value condition)**.** *Let* $f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$,

$$(\omega, T, u) \mapsto \left(\int_0^T \psi_s^{(X^\parallel, M)}\left(u, -i\sigma_s(T,u)\right)ds - \int_0^T \sigma_s(T,u)dM_s\right)(\omega).$$

Then $f \in \bar{\Upsilon}$ *a.s. and* Γ_0 *satisfies the stochastic* minimum value condition

$$\Gamma_0 \geq f \text{ almost surely.}$$

Proof. Since e^X is a martingale, Lemma A.21 yields that $e^{X^\perp}$ is a martingale as well. Hence $\psi_t^{X^\perp}(-i) = 0$ by Proposition A.16. Together with Proposition A.8 we have that $u \mapsto \psi^{X^\perp}(u)$ is in Π. Let $T \in \mathbb{R}_+$, $t \in [0,t]$. Then $u \mapsto \int_t^T \psi_s^{X^\perp}(u)ds$ is in Π as well. Using the consistency condition, the definition of $\Psi_{t-}(t,\cdot)$, Fubini's theorem as in [20, Theorem IV.65], and the modified drift condition (4.3), we obtain

$$\begin{aligned}\int_0^T \psi_r^{X^\perp}(u)dr &= \int_0^T \left(\psi_r^X(u) - \psi_r^{X^\parallel}(u)\right)dr\\ &= \int_0^T \left(\Psi_0(r,u) + \int_0^r \alpha_s(r,u)ds + \int_0^r \beta_s(r,u)dM_s - \psi_r^{X^\parallel}(u)\right)dr\\ &= \Gamma_0(T,u) + \int_0^T\int_s^T \alpha_s(r,u)drds + \int_0^T \sigma_s(T,u)dM_s - \int_0^T \psi_s^{X^\parallel}(u)ds\\ &= \Gamma_0(T,u) - \int_0^T \psi_s^{(X^\parallel,M)}\big(u, -i\sigma_s(T,u)\big)ds + \int_0^T \sigma_s(T,u)dM_s.\\ &= \Gamma_0(T,u) - f(T,u).\end{aligned}$$

□

In more specific setups the stochastic lower bound f in the previous result can be replaced by a deterministic bound g. We will see in Theorem 4.11 that the corresponding condition (4.4) is also sufficient in some sense.

Proposition 4.5. *Assume that $E = (E^1, \dots, E^d) = 0$, the local exponent $\psi^{(X^\parallel, M)}$ is deterministic, M a Lévy process and $t \mapsto \sigma_t(T,u)$ is deterministic and bounded for any $T \in \mathbb{R}_+$, $u \in \mathbb{R}$. Furthermore we suppose that $u \mapsto \sigma_t(T,u)$ is continuous and that $u \mapsto \int_0^T \sigma_s(T,u)dM_s$ is continuous in 0. Set*

$$g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}, \quad (T,u) \mapsto \int_0^T \psi_s^{(X^\parallel, M)}\left(u, -i\sigma_s(T,u)\right) ds.$$

Then $g \in \bar{\Upsilon}$ and

$$\Gamma_0 \geq g. \tag{4.4}$$

Proof. Let f be the function defined in Proposition 4.4 and $T \in \mathbb{R}_+$, $u \in \mathbb{R}$. Then $\Gamma_0 - f \in \Upsilon$ a.s. and

$$\Gamma_0(T,u) - f(T,u) = \Gamma_0(T,u) - g(T,u) - \int_0^T \sigma_s(T,u)dM_s. \tag{4.5}$$

Corollary B.9 yields

$$\left| \int_0^T \sigma_s(T,u)dM_s \right|(\omega_n) \underset{n\to\infty}{\longrightarrow} 0$$

for some sequence $(\omega_n)_{n\in\mathbb{N}}$ in Ω. The proof actually shows that the same sequence can be chosen for all $u \in \mathbb{R}$. Hence there is a sequence $(\omega_n)_{n\in\mathbb{N}}$ in Ω such that

$$\left(\Gamma_0(T,\cdot) - f(T,\cdot)\right)(\omega_n) \in \Pi$$

for all $n \in \mathbb{N}$ and $\left(\Gamma_0(T,u) - f(T,u)\right)(\omega_n) \to \Gamma_0(T,u) - g(T,u)$ for all $u \in \mathbb{R}$. The mapping $u \mapsto \Gamma_0(T,u) - g(T,u)$ is continuous in 0 by Equation (4.5), Proposition 4.4 and assumption. We have shown that $u \mapsto (\Gamma_0(T,u) - g(T,u))$ is the pointwise limit of characteristic exponents of infinitely divisible distribution and thus [19, Theorem 5.3.3] yields that it is a characteristic exponent of an infinitely divisible distribution. Denote by Q_n resp. Q the infinitely divisible laws with Fourier transform $u \mapsto \left(\Gamma_0(T,\cdot) - f(T,\cdot)\right)(\omega_n)$ and $u \mapsto \Gamma_0(T,u) - g(T,u)$, respectively. [19, Corollary 3.5.2] yields

$$\int e^x Q(dx) = \lim_{n\to\infty} \int e^x Q_n(dx) = 1$$

and thus $\Gamma_0(T,\cdot) - g(T,\cdot) \in \Pi$. The same is true for $u \mapsto (\Gamma_0(T,u) - g(T,u) - (\Gamma_0(t,u) - g(t,u)))$ for all $t \in [0,T]$ and thus we have $\Gamma_0 - g \in \Upsilon$. □

Above we suggested that risk-neutral option surface models will typically be constructed from building blocks $X_0, \Psi_0, \beta, M, X^\parallel$. Proposition 4.4 shows that Ψ_0 must be sufficiently large for that purpose. In the corollary to the following lemma, we see that increasing Ψ_0 does not lead to problems as far as existence of $(X, \Psi_0, \alpha, \beta, M)$ is concerned.

Lemma 4.6. *Let L be a PII with local exponent ψ^L such that e^L is a martingale and L is locally independent of (X, M). Then $(X + L, \Psi_0 + \psi^L, \alpha, \beta, M)$ is a risk-neutral option surface model.*

Proof. This is an immediate consequence of Theorem 3.6. □

Corollary 4.7. *Suppose that there exists a risk-neutral option surface model* $(X, \Psi_0, \alpha, \beta, M)$ *that is consistent with given building blocks* $X_0, \Psi_0, \beta, M, X^{\parallel}$. *Moreover, let* $\widetilde{\Psi}_0 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ *be such that*

$$\widetilde{\Gamma}_0 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}, (t, u) \mapsto \int_0^t \widetilde{\Psi}_0(r, u)dr$$

exists, is in Υ, *and satisfies* $\Gamma_0 \leq \widetilde{\Gamma}_0$. *On a possibly enlarged filtered probability space, there exists a risk-neutral option surface model that is consistent with* $X_0, \widetilde{\Psi}_0, \beta, M, X^{\parallel}$.

Proof. By Remark 4.3 there exists a PII U with characteristic function

$$E(e^{iuU_T}) = \exp\left(\widetilde{\Gamma}_0(T, u) - \Gamma_0(T, u)\right).$$

By possibly enlarging the probability space, we may assume U to be defined on Ω and to be locally independent (and in fact also independent) of (X, M). Lemma 4.6 now yields the claim. □

4.3. **Vanishing coefficient process** β**.** The simplest conceivable codebook model (2.12) is obtained for $\beta = 0$ or, equivalently, $M = 0$. Not surprisingly, it leads to constant codebook processes and hence to the simple model class that we used to motivate option surface models in Section 2.2.

Theorem 4.8. *Let* $X_0 \in \mathbb{R}$; $\beta_t(T, u) = 0$ *for* $t, T \in \mathbb{R}_+$, $u \in \mathbb{R}$; $\Psi_0 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ *with* $\int_t^T \Psi_0(r, \cdot)dr \in \Pi$ *for any* $T \in \mathbb{R}_+$, $t \in [0, T]$. *Moreover, let* $M = 0$ *and hence* $X^{\parallel} = 0$. *On a possibly enlarged filtered probability space, there exists a risk-neutral option surface model* $(X, \Psi_0, \alpha, \beta, M)$ *that is consistent with* $X_0, \Psi_0, \beta, M, X^{\parallel}$. *For any such model we have*

- $\alpha_t(T, u) = 0$ *for any* $T \in \mathbb{R}_+$, $t \in [0, T]$, $u \in \mathbb{R}$,
- $X - X_0$ *is a PII with characteristic function* $E(e^{iu(X_T - X_0)}) = \exp(\int_0^T \Psi_0(r, u)dr)$, $T \in \mathbb{R}_+$, $u \in \mathbb{R}$.

In particular, the law of X *is uniquely determined.*

Proof. By [15, III.2.16] and straightforward arguments there exists a PII X with local exponent $\psi^X = \Psi_0$. By possibly enlarging the probability space, we may assume that X is defined on the given filtered space. Since $M = 0$, the dependent part of X relative to M is obviously 0. We define $\alpha := 0$ and $\Psi_t(T, u) := \Psi_0(T, u)$ for any $T \in \mathbb{R}_+$, $t \in [0, T]$, $u \in \mathbb{R}$. Then $(X, \Psi_0, \alpha, \beta, M)$ is a risk-neutral option surface model because it satisfies the drift and consistency conditions.

If $(\widetilde{X}, \Psi_0, \widetilde{\alpha}, \beta, M)$ denotes another risk-neutral option surface model with $\widetilde{X}_0 = X_0$, the drift condition implies $\widetilde{\alpha} = 0 = \alpha$. Hence $\widetilde{\Psi} = \Psi$ and the consistency condition yields $\psi_t^{\widetilde{X}}(u) = \widetilde{\Psi}_{t-}(t, u) = \Psi_{t-}(t, u) = \psi_t^X(u)$ for any $t \in \mathbb{R}_+$, $u \in \mathbb{R}$. Therefore $\psi^{\widetilde{X}}$ is deterministic and $\widetilde{X}$ is a PII with the same law as X. □

Remark 4.9. In principle, one could start with an arbitrary process M. However, since M enters the model only through the trivial second integral in (3.1), this does not lead to a more general setup.

The Black-Scholes model is obtained for a particular choice of the initial state of the codebook.

Example 4.10 (Black-Scholes model)**.** If we choose $\Psi_0(T,u) := -\frac{iu+u^2}{2}\sigma^2$ in Theorem 4.8 for some $\sigma > 0$, we obtain $E(e^{iuX_T}) = \exp(iuX_0 - iu\frac{\sigma^2}{2}T - \frac{u^2}{2}\sigma^2 T)$, which means $X_T \sim N(X_0 - \frac{\sigma^2}{2}T, \sigma^2 T)$, $T \in \mathbb{R}_+$. Put differently, the return process X is Brownian motion with drift rate $-\frac{\sigma^2}{2}$ and volatility σ.

4.4. **Deterministic coefficient process β.** In this section we consider risk-neutral option surface models $(X, \Psi_0, \alpha, \beta, M)$ with deterministic β. Throughout this section let

- $X_0 \in (0,\infty)$,
- $\Psi_0 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ such that $u \mapsto \int_t^T \Psi_0(r,u)dr$ is in Π for any $T \in \mathbb{R}_+$, $t \in [0,T]$,
- $(X^\parallel, M)$ is a $1+d$-dimensional Lévy process such that M_1^k is bounded from below for $k = 1,\dots,d$ and $(X^\parallel)^\parallel = X^\parallel$ relative to M, i.e. $X^\parallel$ equals its dependent part relative to M,
- $f \in \Pi^d$ (i.e. $f = (f^1,\dots,f^d) : \mathbb{R} \to \mathbb{C}^d$ with $f^k \in \Pi$ for $k = 1,\dots,d$),
- $\lambda : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ is continuous with $\lambda(r)\lambda(s) = \lambda(s)\lambda(r)$ for any $r, s \in \mathbb{R}_+$,
- $\beta_t(T,u) := f(u)\exp(-\int_t^T \lambda(r)dr)$,
- $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$, $(T,u) \mapsto \int_0^T \psi^{(X^\parallel,M)}(u, -i\int_t^T \beta_t(r,u)dr)dt$ (cf. Proposition 4.5).

The Lévy exponent of $(X^\parallel, M)$ is denoted by $\psi^{(X^\parallel,M)}$. Moreover, we set

$$\Gamma_0(T,u) := \int_0^T \Psi_0(r,u)dr, \tag{4.6}$$

$$\sigma_t(T,u) := \int_t^T \beta_t(r,u)dr = \int_t^T \exp\Big(-\int_t^r \lambda(s)ds\Big)dr f(u) \tag{4.7}$$

for any $T \in \mathbb{R}_+$, $t \in [0,T]$, $u \in \mathbb{R}$. W.l.o.g. we assume that the one-dimensional Lévy processes M^k are subordinators with essential infimum $\operatorname{ess\,inf} M_t^k = 0$. Changing the drift of M leads to the same model as it is offset by the drift condition. Corollary A.9 and (A.2) imply that $-\partial_T\psi^{(X^\parallel,M)}(u, -i\sigma_t(T,u))$ exists for any $T \in \mathbb{R}_+$, $t \in [0,T]$, $u \in \mathbb{R}$. The key assumption in the above list is that β is deterministic and factorises with respect to dependence on (t,T) resp. u.

Theorem 4.11. (1) *On a possibly enlarged probability space, there exists a risk-neutral option surface model $(X, \Psi_0, \alpha, \beta, M)$ that is consistent with $X_0, \Psi_0, \beta, M, X^\parallel$ if and only if*

$$\Gamma_0 \geq g.$$

(2) *Let $(X, \Psi_0, \alpha, \beta, M)$ denote any risk-neutral option surface model that is consistent with $X_0, \Psi_0, \beta, M, X^\parallel$ and set*

$$Z_t := \int_0^t \exp\Big(-\int_s^t \lambda(r)dr\Big)dM_s, \quad t \in \mathbb{R}_+. \tag{4.8}$$

If $T \mapsto \Psi_0(T,u)$ is continuous in T for any fixed $u \in \mathbb{R}$, then (X, M, Z) is a time-inhomogeneous affine process in the sense of [12] *and its law is uniquely determined.*

In the following more specific setup, we provide an explicit representation of the stock price dynamics.

Theorem 4.12. *Let $d = 1$; $X^\parallel = \delta M$ for some $\delta \in \mathbb{R}_+$; $f(u) := -\frac{u^2+iu}{2}$; $\lambda(r) := \lambda$ for some $\lambda \in \mathbb{R}_+$, and suppose that $\Gamma_0 \geq g$.*

(1) *On a possibly enlarged probability space, there exists a risk-neutral option surface model $(X, \Psi_0, \alpha, \beta, M)$ which is consistent with $X_0, \Psi_0, \beta, M, X^\parallel$. Moreover,*

it can be chosen such that there is a standard Wiener process W and a time-inhomogeneous Lévy process L with characteristic function

$$E(e^{iuL_T}) = \exp(\Gamma_0(T,u) - g(T,u)), \quad T \in \mathbb{R}_+, u \in \mathbb{R}$$

such that W, L, M are independent and

$$dX_t = dL_t - \frac{1}{2}Z_t dt + \sqrt{Z_t}dW_t + \delta dM_t, \tag{4.9}$$

$$dZ_t = -\lambda Z_t dt + dM_t \tag{4.10}$$

with $Z_0 = 0$.

(2) *Let $(X, \Psi_0, \alpha, \beta, M)$ denote any risk-neutral option surface model that is consistent with $X_0, \Psi_0, \beta, M, X^\|$ and set*

$$Z_t := \int_0^t e^{-\lambda(t-s)} dM_s, \quad t \in \mathbb{R}_+.$$

If $T \mapsto \Psi_0(T,u)$ is continuous in T for any fixed $u \in \mathbb{R}$, then (X, M, Z) is a time-inhomogeneous affine process in the sense of [12]. *Its law is uniquely determined by $X_0, \Psi_0, \lambda, \delta$ and the law of M.*

Remark 4.13. (1) Up to the additional time-inhomogeneous Lévy process L, the stock price model in (4.9, 4.10) is a special case of the so-called BNS model of [2]. If we consider more general functions f, then, again up to the additional PII L, we end up with the CGMY extension of the BNS model from [9], cf. also [17]. This follows from (4.13, 4.14) in the proof of Theorem 4.11.

(2) To be more precise, (X, M, Z) in Theorems 4.11, 4.12 are weakly regular affine time-inhomogeneous Markov processes in the language of [12]. Moreover, the conclusions of [12, Theorem 2.13] hold, in particular (Θ, X, M, Z) is a Feller process for $\Theta_t = t$. We do not know whether continuity of Ψ_0 is really needed for the uniqueness statement in Theorems 4.11, 4.12 to hold.

The proof of both theorems is based on the following

Lemma 4.14. *Define Z as in (4.8). Assume the existence of a semimartingale X such that $X^\|$ is the dependent part of X relative to M and*

$$\psi_t^{(X,M)}(u,v) = f(u)Z_{t-} + \psi^{(X^\|,M)}(u,v) \tag{4.11}$$

outside some $dP \otimes dt$-null set for any $u \in \mathbb{R}$, $v \in \mathbb{R}^d$. Then the option surface model $(X, \widetilde{\Psi}_0, \alpha, \beta, M)$ is risk neutral, where

$$\alpha_t(T,u) := -\partial_T \left(\psi^{(X^\|,M)}(u, -i\sigma_t(T,u))\right),$$

$$\widetilde{\Psi}_0(T,u) := \psi^{X^\|}(u) - \int_0^T \alpha_s(T,u)ds$$

for any $T \in \mathbb{R}_+$, $t \in [0,T]$, $u \in \mathbb{R}$. Moreover, we have $g = \widetilde{\Gamma}_0 \in \Upsilon$ for

$$\widetilde{\Gamma}_0(T,u) := \int_0^T \widetilde{\Psi}_0(r,u)dr, \quad T \in \mathbb{R}_+, u \in \mathbb{R}.$$

(Compare Section 4.2 for the definition of Υ.)

Proof. Observe that

$$\begin{aligned} Z_t &= \int_0^t \exp\Big(-\int_s^t \lambda(r)dr\Big)dM_s \\ &= \int_0^t \Big(1_{d\times d} - \int_s^t \lambda(r)\exp\Big(-\int_s^r \lambda(\tilde r)d\tilde r\Big)dr\Big)\,dM_s \\ &= M_t - \int_0^t\int_0^r \lambda(r)\exp\Big(-\int_s^r \lambda(\tilde r)d\tilde r\Big)dM_s dr \\ &= M_t - \int_0^t \lambda(r)Z_{r-}dr \end{aligned}$$

by Fubini's theorem as in [20, Theorem IV.65]. In view of Lemma A.19, we obtain

$$\begin{aligned} \psi_t^{(X,M,Z)}(u,v,w) &= \psi_t^{(X,M,M)}(u,v,w) - iw\lambda(t)Z_{t-} \\ &= \psi_t^{(X,M)}(u,v+w) - iw\lambda(t)Z_{t-} \\ &= \psi^{(X^\parallel,M)}(u,v+w) + (f(u) - iw\lambda(t))Z_{t-} \end{aligned} \tag{4.12}$$

and in particular

$$\begin{aligned} \alpha_t(T,u) &= -\partial_T\Big(\psi^{(X^\parallel,M)}(u,-i\sigma_t(T,u))\Big) \\ &= -\partial_T\big(\psi^{(X,M)}(u,-i\sigma_t(T,u)) - f(u)Z_{t-})\big) \\ &= -\partial_T\big(\psi^{(X,M)}(u,-i\sigma_t(T,u))\big) \end{aligned}$$

for any $t\in\mathbb{R}_+$, $u\in\mathbb{R}$, $v,w\in\mathbb{R}^d$. If we define $\widetilde\Psi$ as in (3.1) relative to $\widetilde\Psi_0$ instead of Ψ_0, we get

$$\begin{aligned} \widetilde\Psi_{t-}(t,u) &= \psi^{X^\parallel}(u) - \int_0^t \alpha_s(t,u)ds + \int_0^t \alpha_s(t,u)ds + \int_0^t \beta_s(t,u)dM_s \\ &= \psi^{X^\parallel}(u) + f(u)Z_{t-} \\ &= \psi_t^X(u) \end{aligned}$$

for any $t\in\mathbb{R}_+$, $u\in\mathbb{R}$. Thus the consistency and the drift condition hold. Theorem 3.6 completes the first part of the proof. Since $(X,\widetilde\Psi_0,\alpha,\beta,M)$ is risk neutral, we have $\widetilde\Gamma_0\in\Upsilon$. A short computation yields $\widetilde\Gamma_0 = g$. □

Proof of Theorem 4.11. (1) Step 1: By [15, III.2.16] there is a Lévy process Y on some probability space $(\Omega',\mathscr{G}',(\mathscr{G}'_\tau)_{\tau\in\mathbb{R}_+},P')$ such that its Lévy exponent equals $\psi^Y = f$. Let Z be as in (4.8). Define an extension $(\widetilde\Omega,\widetilde{\mathscr{F}},(\widetilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+},\widetilde P)$ of $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}_+},P)$ via $\widetilde\Omega := \Omega\times\Omega'$, $\widetilde{\mathscr{F}} := \mathscr{F}\otimes\mathscr{G}'$, $\widetilde P := P\otimes P'$. Let $\widetilde{\mathscr{F}}_t^0$ be the smallest σ-field such that the projection $\widetilde\Omega\to\Omega$, $(\omega,\omega')\mapsto\omega$ is $\widetilde{\mathscr{F}}_t^0$-$\mathscr{F}_t$-measurable and

$$X_s^\perp : \widetilde\Omega\to\mathbb{R},\quad (\omega,\omega')\mapsto X_s^\perp(\omega,\omega') := Y_{\int_0^s Z_r(\omega)dr}(\omega')$$

is $\widetilde{\mathscr{F}}_t^0$-measurable for any $s\le t$. As usual, we consider right-continuous filtrations by setting $\widetilde{\mathscr{F}}_t := \bigcap_{s>t}\widetilde{\mathscr{F}}_s^0$ for any $t\in\mathbb{R}_+$. By slight abuse of notation, we use the same letter for processes on Ω and their natural counterpart on $\widetilde\Omega$, i.e. $M(\omega,\omega') := M(\omega)$ etc.

Step 2: $(X^\parallel, M)$ is a Lévy process also relative to the filtration $(\widetilde{\mathscr{F}}_t^0)_{t\in\mathbb{R}_+}$ on $\widetilde\Omega$. Indeed, adaptedness is obvious. It remains to be shown that the increment $(X^\parallel,M)_t - (X^\parallel,M)_s$ is independent of $\widetilde{\mathscr{F}}_s^0$ for any $s\le t$. This follows from the fact that it is independent of the even larger σ-field $\mathscr{F}_s \otimes \mathscr{G}' \supset \widetilde{\mathscr{F}}_s^0$. A simple argument shows that the Lévy property of $(X^\parallel, M)$ still holds relative to the right-continuous extension $(\widetilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+}$ of $(\widetilde{\mathscr{F}}_t^0)_{t\in\mathbb{R}_+}$.

Step 3: We show that $X^\perp$ is a semimartingale with local exponent $\psi_t^{X^\perp}(u) = f(u)Z_{t-}$ for $t \in \mathbb{R}_+$, $u \in \mathbb{R}$. In view of [15, II.2.48] it suffices to show that $\exp(iuX^\perp - f(u)Z \bullet I)$ is a local martingale for any $u \in \mathbb{R}$. Define a filtration $(\widetilde{\mathscr{G}}_\tau)_{\tau\in\mathbb{R}_+}$ on $\widetilde{\Omega}$ by $\widetilde{\mathscr{G}}_\tau := \mathscr{F} \otimes \mathscr{G}'_\tau$ or, more precisely, its right-continuous extension. Relative to this filtration, the counterpart of Y on $\widetilde{\Omega}$ (i.e. $Y_\tau(\omega, \omega') := Y_\tau(\omega')$) is a Lévy process with Lévy exponent f, which means that $(L_\tau)_{\tau\in\mathbb{R}_+} := (\exp(iuY_\tau - f(u)\tau))_{\tau\in\mathbb{R}_+}$ is a $\widetilde{P}$-martingale relative to $(\widetilde{\mathscr{G}}_\tau)_{\tau\in\mathbb{R}_+}$. The random variables $\int_0^s Z_r(\omega)dr, \int_0^t Z_r(\omega)dr$ are $(\widetilde{\mathscr{G}}_\tau)_{\tau\in\mathbb{R}_+}$-stopping times for $s \leq t$. By Doob's optional stopping theorem (cf. [15, I.1.39]) we have that

$$E\left(L_{\int_0^t Z_r(\omega)dr\wedge\tau_n} - L_{\int_0^s Z_r(\omega)dr\wedge\tau_n} \middle| \widetilde{\mathscr{G}}_{\int_0^s Z_r dr}\right) = 0$$

for $\tau_n := \inf\{r \in \mathbb{R}_+ : \int_0^r Z_{\tilde{r}} d\tilde{r} \geq n\}$ and any $n \in \mathbb{N}$. Since $\widetilde{\mathscr{F}}_s \subset \widetilde{\mathscr{G}}_{\int_0^s Z_r dr}$, this implies

$$E\left(L_{\int_0^t Z_r(\omega)dr\wedge\tau_n} - L_{\int_0^s Z_r(\omega)dr\wedge\tau_n} \middle| \widetilde{\mathscr{F}}_s\right) = 0,$$

which means that the asserted local martingale property holds.

Step 4: Let $X := X^\parallel + X^\perp$. We show that $(X^\parallel, M)$ and $X^\perp$ are locally independent, which implies (4.11) by Steps 2 and 3. Indeed, $(X^\parallel, M)$ does not have a continuous local martingale part. By independence of $(X^\parallel, M)$ and Y, the processes $(X^\parallel, M)$ and $X^\perp$ never jump at the same time (up to an evanescent set). By Proposition A.11 this implies that local independence holds.

By Lemma A.20, $X^\parallel$ is the dependent part of X relative to M. From Lemma 4.14 we obtain that a consistent risk-neutral option surface model exists for $\Gamma_0 = g$. For $\Gamma_0 \geq g$, Corollary 4.7 and its proof yield that $(X + U, \Psi_0, \alpha, \beta, M)$ has the required properties for some PII U which is independent of (X, M).

The necessary condition $\Gamma_0 \geq g$ is shown in Proposition 4.5.

(2) Step 1: Denote by $X^\parallel, X^\perp$ the dependent resp. independent part of X relative to M. The local exponent of $X^\perp$ satisfies

$$\begin{aligned}
\psi_t^{X^\perp}(u) &= \psi_t^X(u) - \psi^{X^\parallel}(u) \\
&= \Psi_0(t,u) + \int_0^t \alpha_s(t,u)ds + \int_0^{t-} \beta_s(t,u)dM_s - \psi^{X^\parallel}(u) \\
&= \Psi_0(t,u) + \int_0^t \alpha_s(t,u)ds + f(u)Z_{t-} - \psi^{X^\parallel}(u) \\
&= \Psi_0(t,u) - \int_0^t \partial_t \left(\psi^{(X^\parallel,M)}\left(u, -i\int_s^t \beta_s(r,u)dr\right)\right)ds \\
&\quad + f(u)Z_{t-} - \psi^{X^\parallel}(u),
\end{aligned}$$

where we used the consistency condition in the first and the drift condition (4.2) in the last step. As in (4.12) we obtain for the local exponent of (X, M, Z)

$$\begin{aligned}
\psi_t^{(X,M,Z)}(u,v,w) &= \psi_t^{(X,M)}(u, v+w) - iw\lambda(t)Z_{t-} \\
&= \psi^{(X^\parallel,M)}(u, v+w) - iw\lambda(t)Z_{t-} + \psi_t^{X^\perp}(u) \\
&= \Phi_0(t; u,v,w) + \Phi_1(t; u,v,w)Z_{t-}
\end{aligned}$$

with

$$\begin{aligned}\Phi_0(t;u,v,w) &:= \psi^{(X^\parallel,M)}(u,v+w)-\psi^{X^\parallel}(u)\\ &\quad+\Psi_0(t,u)-\int_0^t\partial_t\left(\psi^{(X^\parallel,M)}\left(u,-i\int_s^t\beta_s(r,u)dr\right)\right)ds, \quad (4.13)\\ \Phi_1(t;u,v,w) &:= f(u)-iw\lambda(t). \quad (4.14)\end{aligned}$$

which implies that $(u,v,w)\mapsto\Phi_0(t;u,v,w)+f(u)Z_{t-}(u)$ is a Lévy exponent on $\mathbb{R}^{1+2d}$. Since $\operatorname{ess\,inf}(M^k)=0$ for all $k\in\{1,\dots,d\}$ we have $\operatorname{ess\,inf}(Z^k)=0$ for all $k\in\{1,\dots,d\}$ by Corollary B.9. The same argument as in the proof of Proposition 4.5 yields that $(u,v,w)\mapsto\Phi_0(t;u,v,w)$ is a Lévy exponent for fixed t. The same is true for $\Phi_1^1,\dots,\Phi_1^d$. Relative to some truncation function h, denote by $(\beta_t^{(i)},\gamma_t^{(i)},\kappa_t^{(i)}), i=0,\dots,d$ Lévy Khintchine triplets on $\mathbb{R}^{1+2d}$ which correspond to $\Phi_0(t;\cdot),\Phi_1^1(t;\cdot),\dots,\Phi_1^d(t;\cdot)$ respectively. Observe that $\Phi_0(t;u,v,w)$ is continuous in t for fixed (u,v,w) and likewise $\Phi_1^1,\dots,\Phi_1^d$. Lévy continuity theorem and [15, VII.2.9] imply that $(\beta_t^{(i)},\gamma_t^{(i)},\kappa_t^{(i)}), i=0,\dots,d$ are continuous in t in the sense of $[\beta_1],[\gamma_1]$ and $[\delta_{1,3}]$ as in [15, VII.2.9]. A detailed inspection of the arguments shows this weaker continuity suffices for the proof of [12, Proposition 4.1]. The assertion follows now from [12, Proposition 5.4].

Proof of Theorem 4.12. (1) By Remark 4.3 there is a PII L with characteristic function

$$E(e^{iuL_T})=\exp(\Gamma_0(T,u)-g(T,u)),\quad T\in\mathbb{R}_+,u\in\mathbb{R}.$$

By possibly enlarging the probability space, we may assume that L and a standard Wiener process W are defined on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}_+},P)$ and that W,L,M are independent and thus locally independent. Define (X,Z) as in (4.9, 4.10). Lemma A.20 yields that $X^\parallel=\delta M$ is the dependent part of X relative to M. Consequently, $\psi^{(X,M)}=\psi^{(X^\parallel,M)}+\psi^{X^\perp}$ with

$$dX_t^\perp=dL_t-\frac{1}{2}Z_tdt+\sqrt{Z_t}dW_t.$$

Local independence and Lemmas A.18, A.19 yield

$$\psi^{X^\perp}(u)=f(u)Z_{t-}+\partial_t(\Gamma_0(t,u)-g(t,u))$$

and hence

$$\psi_t^{(X,M)}(u,v)=\psi^{(X^\parallel,M)}(u,v)+f(u)Z_{t-}+\partial_t(\Gamma_0(t,u)-g(t,u)).$$

We also get

$$\psi_t^{(X-L,M)}(u,v)=\psi^{(X^\parallel,M)}(u,v)+f(u)Z_{t-}=f(u)Z_{t-}+\psi^{((X-L)^\parallel,M)}(u,v),$$

where $(X-L)^\parallel$ is the dependent part of $X-L$ relative to M. By Lemma 4.14, $(X-L,\partial_tg,\alpha,\beta,M)$ is a risk neutral option surface model. Thus independence of L and Lemma 4.6 yield that $(X,\partial_tg+\psi^L,\alpha,\beta,M)$ is a risk neutral option surface model and $\psi^L=\partial_t(\Gamma_0-g)=\Psi_0-\partial_tg$.

(2) This follows from Theorem 4.11. □

5. Carmona & Nadtochiy's 'Tangent Lévy market models'

In [8], Carmona and Nadtochiy (CN) developed independently a HJM-type approach for option prices with substantial overlap to ours. Their simple model class in the sense of Step (4) in Section 2.1 is based on time-inhomogeneous pure jump Lévy processes. These can be described uniquely by their Lévy density because the drift is determined by the martingale condition for the stock under the risk neutral measure. Instead of the characteristic exponent from (2.3) CN use this Lévy density as the codebook $\kappa_t(T,x)$. Since we allow for a larger class of simple models, their framework can be embedded into ours. Indeed, there is a linear transformation A that converts their codebook into ours, given by

$$A(\kappa_t(T,x)) := \int (e^{iux} - 1 - iu(e^x - 1))\kappa_t(T,x)dx.$$

Up to this transformation, the drift conditions in both approaches coincide. However, the condition in the CN framework looks a little more complex because it involves convolutions and differential operators of higher order. As a side remark, the simple Black-Scholes model is not contained as a special case in the CN setup because of the slight limitation to pure jump processes.

CN focus on Itô processes for modelling the codebook process, which roughly corresponds to choosing M as Brownian motion in our setup. This assumption implies local independence of X and M in our terminology because X is a pure jump process in CN. By contrast and similarly as [7], we attach importance to allowing for a certain *leverage*, which here means local dependence between stock and codebook movements.

The positivity problem of the codebook is treated differently in CN and in our approach. Whereas CN ensure nonnegativity by stopping the codebook dynamics if necessary, we restrict the class of initial codebooks to sufficiently large ones. Otherwise the particularly tractable models in Theorems 4.11 and 4.12 would not fit in into the present setup.

With regards existence and uniqueness of models given basic building blocks, both approaches provide only partial answers. Our results in Section 4 are so far limited to vanishing or deterministic coefficient function β. By contrast, CN consider a more general situation in their Theorem 2. However, they do not prove uniqueness in law of the resulting stock price process. Moreover, they assume that the process β in their codebook dynamics

$$d\kappa_t = \alpha_t dt + \beta_t dB_t,$$

is given beforehand. This does not allow for the natural situation that β depends on the current state κ of the codebook itself, which occurs e.g. in Section 6 of CN.

Both CN and we provide basically one non-trivial example, both taking deterministic β as a starting point. In order to ensure positivity of the codebook, CN stop the process where appropriate, whereas we consider subordinators M rather than Brownian motion as driving process for the codebook dynamics. Both approaches have their respective advantages and disadvantages: while CN do not need to impose a largeness condition on the initial surface, our example in Theorem 4.12 leads to an established and very tractable model whose characteristic function is known in closed form.

APPENDIX A. LOCAL CHARACTERISTICS AND LOCAL EXPONENTS

In this section we define and recall some properties of local characteristics and local exponents.

A.1. **Local characteristics.** Let X be an $\mathbb{R}^d$-valued semimartingale with integral characteristics (B, C, ν) in the sense of [15] relative to some fixed truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. By [15, I.2.9] there exist a predictable $\mathbb{R}^d$-valued process b, a predictable $\mathbb{R}^{d\times d}$-valued process c, a kernel K from $(\Omega \times \mathbb{R}, \mathscr{P})$ to $(\mathbb{R}^d, \mathscr{B})$, and a predictable increasing process A such that

$$dB_t = b_t dA_t, \quad dC_t = c_t dA_t, \quad \nu(dt, dx) = K_t(dx)dA_t.$$

If $A_t = t$, we call the triplet (b, c, K) *local* or *differential characteristics* of X relative to truncation function h. Most processes in applications as e.g. diffusions, Lévy processes etc. allow for local characteristics. In this case b stands for a drift rate, c for a diffusion coefficient, and K for a local Lévy measure representing jump activity. If they exist, the local characteristics are unique up to a $dP \otimes dt$-null set on $\Omega \times \mathbb{R}_+$.

Proposition A.1 (Itô's formula for local characteristics)**.** *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b, c, K) and $f : \mathbb{R}^d \to \mathbb{R}^n$ a C^2-function. Then the triplet $(\widetilde{b}, \widetilde{c}, \widetilde{K})$ defined by*

$$\begin{aligned}
\widetilde{b}_t &= Df(X_{t-})b_t + \frac{1}{2}\sum_{j,k=1}^{n} \partial_j\partial_k f(X_{t-})c_t^{jk} \\
&\quad + \int \left(\widetilde{h}(f(X_{t-}+x) - f(X_{t-})) - Df(X_{t-})h(x)\right) K_t(dx), \\
\widetilde{c}_t &= Df(X_{t-})c_t(Df(X_{t-}))^\top, \\
\widetilde{K}_t(A) &= \int 1_A(f(X_{t-}+x) - f(X_{t-}))K_t(dx), \quad A \in \mathscr{B}^n \text{ with } 0 \notin A,
\end{aligned}$$

is a version of the local characteristics of $f(X)$ with respect to a truncation function $\widetilde{h}$ on $\mathbb{R}^n$. Here, ∂_j etc. denote partial derivatives relative to the j'th argument.

Proof. See [17, Proposition 2.5]. □

Proposition A.2. *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b, c, K) and let $\beta = (\beta^{ij})_{i\in\{1,\dots,n\},j\in\{1,\dots,d\}}$ be a $\mathbb{R}^{n\times d}$-valued predictable process such that $\beta^{i\cdot} \in L(X)$ for $i \in \{1, \dots, n\}$. Then the triplet $(\widetilde{b}, \widetilde{c}, \widetilde{K})$ defined by*

$$\begin{aligned}
\widetilde{b}_t &= \beta b_t + \int \left(\widetilde{h}(\beta_t x) - \beta_t h(x)\right) K_t(dx), \\
\widetilde{c}_t &= \beta_t c_t \beta_t^\top, \\
\widetilde{K}_t(A) &= \int 1_A(\beta_t x)K_t(dx), \quad A \in \mathscr{B}^n \text{ with } 0 \notin A,
\end{aligned}$$

is a version of the local characteristics of the $\mathbb{R}^n$-valued semimartingale $\beta \bullet X := (\beta^{1\cdot} \bullet X, \dots, \beta^{n\cdot} \bullet X)$ with respect to the truncation function $\widetilde{h}$ on $\mathbb{R}^n$,

Proof. See [17, Proposition 2.4]. □

A.2. **Local exponents.**

Definition A.3. Let (b, c, K) be a Lévy-Khintchine triplet on $\mathbb{R}^d$ relative to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. We call the mapping $\psi : \mathbb{R}^d \to \mathbb{C}$,

$$\psi(u) := iub - \frac{1}{2}ucu^\top + \int (e^{iux} - 1 - iuh(x))K(dx)$$

Lévy exponent corresponding to (b, c, K). By [15, II.2.44], the Lévy exponent determines the triplet (b, c, K) uniquely. If X is a Lévy process with Lévy-Khintchine triplet (b, c, K), we call ψ the *characteristic* or *Lévy exponent* of X.

In the same vein, local characteristics naturally lead to local exponents.

Definition A.4. If X is an $\mathbb{R}^d$-valued semimartingale with local characteristics (b, c, K), we write

$$\psi_t^X(u) := iub_t - \frac{1}{2}uc_tu^\top + \int (e^{iux} - 1 - iuh(x))K_t(dx), \quad u \in \mathbb{R}^d \qquad \text{(A.1)}$$

for the Lévy exponent corresponding to (b_t, c_t, K_t). We call the family of predictable processes $\psi^X(u) := (\psi_t^X(u))_{t\in\mathbb{R}_+}, u \in \mathbb{R}^d$ *local exponent* of X. (A.1) implies that $u \mapsto \psi_t^X(u)$ is the characteristic exponent of a Lévy process.

The name *exponent* is of course motivated by the following fact.

Remark A.5. If X is a semimartingale with deterministic local characteristics (b, c, K), it is a PII and we have

$$E(e^{iu(X_T-X_t)}|\mathscr{F}_t) = E(e^{iu(X_T-X_t)}) = \exp\left(\int_t^T \psi_s^X(u)ds\right)$$

for any $T \in \mathbb{R}_+$, $t \in [0, T]$, $u \in \mathbb{R}^d$, cf. [15, II.4.15].

We now generalize the notion of local exponents to complex-valued semimartingales and more general arguments.

Definition A.6. Let X be a $\mathbb{C}^d$-valued semimartingale and β a $\mathbb{C}^d$-valued X-integrable process. We call a predictable $\mathbb{C}$-valued process $\psi^X(\beta) = (\psi_t^X(\beta))_{t\in\mathbb{R}_+}$ *local exponent* of X at β if $\psi^X(\beta) \in L(I)$ and $(\exp(i\beta \bullet X_t - \int_0^t \psi_s^X(\beta)ds))_{t\in\mathbb{R}_+}$ is a complex-valued local martingale. We denote by $\mathscr{U}^X$ the set of processes β such that the local exponent $\psi^X(\beta)$ exists.

From the following lemma it follows that $\psi^X(\beta)$ is unique up to a $dP \otimes dt$-null set.

Lemma A.7. *Let X be a complex-valued semimartingale and A, B complex-valued predictable processes of finite variation with $A_0 = 0 = B_0$ and such that $\exp(X - A)$ and $\exp(X - B)$ are local martingales. Then $A = B$ up to indistinguishability.*

Proof. Set $M := e^{X-A}$, $N := e^{X-B}$, $V := e^{A-B}$. Integration by parts yields that

$$M_- \bullet V = MV - V \bullet M - M_0V_0 = N - V \bullet M - M_0$$

is a local martingale. Therefore $V = 1 + \frac{1}{M_-} \bullet (M_- \bullet V)$ is a predictable local martingale with $V_0 = 1$ and hence $V = 1$, cf. [15, I.3.16]. □

The following result shows that Definition A.6 truly generalizes Definition A.4.

Proposition A.8. *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b, c, K). Suppose that β is a $\mathbb{C}^d$-valued predictable and X-integrable process. If $i\beta$ is $\mathbb{R}^d$-valued for any $t \in \mathbb{R}_+$, then $\beta \in \mathscr{U}^X$. Moreover there is equivalence between*

(1) $\beta \in \mathscr{U}^X$,
(2) $\int_0^t \int 1_{\{-\mathrm{Im}(\beta_s x)>1\}} e^{-\mathrm{Im}(\beta_s x)} K_s(dx)ds < \infty$ *almost surely for any* $t \in \mathbb{R}_+$.

In this case we have

$$\psi_t^X(\beta) = i\beta_t b_t - \frac{1}{2}\beta_t c_t \beta_t^\top + \int (e^{i\beta_t x} - 1 - i\beta_t h(x))K_t(dx) \tag{A.2}$$

outside some $dP \otimes dt$-null set.

Proof. If $i\beta$ is $\mathbb{R}^d$-valued, then Statement (2) is obviously true. Thus we only need to prove the equivalence and (A.2). For real-valued $i\beta$ the equivalence follows from [18, Lemma 2.13]. The complex-valued case is derived similarly. For real-valued $i\beta$ (A.2) is shown in [18, Theorems 2.18(1,6) and 2.19]. The general case follows along the same lines. □

(A.2) implies that the local exponent of X at any $\beta \in \mathscr{U}^X$ is determined by the triplet (b, c, K) and hence by the local exponent of X in the sense of Definition A.4.

Corollary A.9. *Let (X, M) be a $1 + d$-dimensional semimartingale with local exponent $\psi^{(X,M)}$ such that M is a Lévy process whose components are subordinators. Then $\beta \in \mathscr{U}^{(X,M)}$ for any $\mathbb{R} \times (\mathbb{R} + i\mathbb{R}_+)^d$-valued (X, M)-integrable process β.*

Proof. This follows immediately from Proposition A.8. □

Definition A.10. Let $X^{(1)}, \dots, X^{(n)}$ be semimartingales which allow for local characteristics. We call them $X^{(1)}, \dots, X^{(n)}$ *locally independent* if

$$\mathscr{U}^{(X^{(1)},\dots,X^{(n)})} \cap (L(X^{(1)}) \times \cdots \times L(X^{(n)})) = \mathscr{U}^{X^{(1)}} \times \cdots \times \mathscr{U}^{X^{(n)}}$$

for

$$L(X^{(1)}) \times \cdots \times L(X^{(n)}) \;:=\; \{\beta = (\beta^{(1)}, \dots, \beta^{(n)}) \text{ complex-valued} : \beta^{(i)}\ X^{(i)}\text{-integrable for } i = 1, \dots, n\}$$

and

$$\psi^{(X^{(1)},\dots,X^{(n)})}(\beta) = \sum_{j=1}^{n} \psi^{X^{(j)}}(\beta^{(j)})$$

outside some $dP \otimes dt$-null set for any $\beta = (\beta^{(1)}, \dots, \beta^{(n)}) \in \mathscr{U}^{(X^{(1)},\dots,X^{(n)})}$.

The following lemma provides alternative characterisations of local independence. For ease of notation we consider two semimartingales but the extension to arbitrary finite numbers is straightforward.

Lemma A.11. *Let (X, Y) be an $\mathbb{R}^{m+n}$-valued semimartingale with local characteristics (b, c, K) and denote by (b^X, c^X, K^X) resp. (b^Y, c^Y, K^Y) local characteristics of X resp. Y. We have equivalence between*

(1) *X and Y are locally independent,*

(2)
$$\psi^{(X,Y)}(u, v) = \psi^X(u) + \psi^Y(v), \quad (u, v) \in \mathbb{R}^{m+n} \tag{A.3}$$
outside some $dP \otimes dt$-null set,

(3)
$$c = \begin{pmatrix} c^X & 0 \\ 0 & c^Y \end{pmatrix}$$
and
$$K(A) = K^X(\{x : (x, 0) \in A\}) + K^Y(\{y : (0, y) \in A\}), \quad A \in \mathscr{B}^{m+n}$$
outside some $dP \otimes dt$-null set.

Proof. (1)⇒(2): This is obvious by Proposition A.8.

(2)⇒(3): Both sides of (A.3) are Lévy exponents for fixed $(\omega, t) \in \Omega \times \mathbb{R}_+$. Indeed, the triplet corresponding to $(u, v) \mapsto (\psi_t^X(u) + \psi_t^Y(v))(\omega)$ is $(b_t, \widetilde{c}_t, \widetilde{K}_t)(\omega)$ with

$$\widetilde{c}_t = \begin{pmatrix} c_t^X & 0 \\ 0 & c_t^Y \end{pmatrix}$$

and

$$\widetilde{K}_t(A) = K_t^X(\{x : (x, 0) \in A\}) + K_t^Y(\{y : (0, y) \in A\}), \quad A \in \mathscr{B}^{m+n}.$$

Since the Lévy exponent determines the triplet uniquely (cf. [15, II.2.44]), the assertion follows.

(3)⇒(1): If β^X is X-integrable and β^Y is Y-integrable, then $\beta = (\beta^X, \beta^Y)$ is (X, Y)-integrable and $\beta \bullet (X, Y) = \beta^X \bullet X + \beta^Y \bullet Y$. The characterisation in Proposition A.8

yields $\beta \in \mathscr{U}^{(X,Y)}$ for such $\beta = (\beta^X, \beta^Y)$ if and only if $\beta^X \in \mathscr{U}^X$, $\beta^Y \in \mathscr{U}^Y$. In addition, $\psi^{(X,Y)}(\beta) = \psi^X(\beta^X) + \psi^Y(\beta^Y)$ follows from (A.2) □

Corollary A.12. *Let $X^{(1)}, \dots, X^{(n)}$ be locally independent semimartingales and $Q \overset{\text{loc}}{\ll} P$ another probability measure. Then $X^{(1)}, \dots, X^{(n)}$ are locally independent semimartingales relative to Q.*

Proof. This follows from Lemma A.11 and [15, III.3.24]. □

Corollary A.13. *If $(X^{(1)}, \dots, X^{(n)})$ is a Lévy process or, more generally, a PII allowing for local characteristics, then $X^{(1)}, \dots, X^{(n)}$ are independent if and only if they are locally independent.*

Proof. By Remark A.5 the characteristic function φ_{X_t} of $X_t := (X_t^{(1)}, \dots, X_t^{(n)})$ is given by

$$\varphi_{X_t}(u^1, \dots, u^n) = \exp\left(\int_0^t \psi_s^{(X^{(1)},\dots,X^{(n)})}(u^1, \dots, u^n) ds\right).$$

Thus independence of $X_t^{(1)}, \dots, X_t^{(n)}$ is equivalent to

$$\psi_t^{(X^{(1)},\dots,X^{(n)})}(u^1, \dots, u^n) = \sum_{k=1}^n \psi_t^{X^k}(u^k),$$

for Lebesgue-almost any $t \in \mathbb{R}_+$ and any $(u^1, \dots, u^n) \in \mathbb{R}^n$. By Lemma A.11 this in turn is equivalent to local independence of $X_t^{(1)}, \dots, X_t^{(n)}$. □

Lemma A.14. *If $f \in \Pi$, then $\mathrm{Re}(f(u)) \leq 0$ for any $u \in \mathbb{R}$, where Π is defined in Section 3.1.*

Proof. For $f \in \Pi$ we have

$$f(u) = -\frac{u^2 + iu}{2}c + \int (e^{iux} - 1 - iu(e^x - 1))K(dx) \tag{A.4}$$

with some Lévy measure K and some $c \in \mathbb{R}_+$. The real part of the first term is obviously negative and the real part of the integrand is negative as well. □

Remark A.15. If we extend the domain of f to $\mathbb{R} + i[-1, 0]$ by keeping the representation (A.4), then the conclusion of Lemma A.14 is still correct. However, this fact is not used in this paper.

The following four statements follow immediately from the definition of local exponents.

Proposition A.16. *Let X be a $\mathbb{C}$-valued semimartingale that allows for local characteristics. Then there is equivalence between*

(1) $\exp(X)$ *is a local martingale,*
(2) $-i \in \mathscr{U}^X$ *and* $\psi^X(-i) = 0$ *outside some* $dP \otimes dt$*-null set,*

Lemma A.17. *Let X be a $\mathbb{C}^d$-valued semimartingale, β a $\mathbb{C}^d$-valued and X-integrable process and $u \in \mathbb{C}$. Then $u\beta \in \mathscr{U}^X$ if and only if $u \in \mathscr{U}^{\beta \bullet X}$. In that case we have*

$$\psi^X(u\beta) = \psi^{\beta \bullet X}(u).$$

Lemma A.18. *Let X, Y be $\mathbb{C}^d$-valued semimartingales and $u \in \mathbb{C}^d$. Then $u \in \mathscr{U}^{X+Y}$ if and only if $(u, u) \in \mathscr{U}^{(X,Y)}$. In this case we have*

$$\psi^{X+Y}(u) = \psi^{(X,Y)}(u, u)$$

outside some $dP \otimes dt$-null set.

Lemma A.19. *Let X, Z be $\mathbb{C}^d$-valued semimartingales and β, γ predictable $\mathbb{C}^d$-valued processes such that*

(1) *γ has I-integrable components,*
(2) *$\beta\gamma$ is I-integrable,*
(3) *$Z_t = Z_0 + \int_0^t \gamma_s ds + X_t$.*

Then $\beta \in \mathscr{U}^Z$ if and only if $\beta \in \mathscr{U}^X$. In this case $\psi^Z(\beta) = \psi^X(\beta) + i\beta\gamma$ outside some $dP \otimes dt$-null set.

A.3. **Semimartingale decomposition relative to a semimartingale.** Let (X, Y) be an $\mathbb{R}^{1+d}$-valued semimartingale with local characteristics (b, c, K), written here in the form

$$b = \begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \quad c := \begin{pmatrix} c^X & c^{X,Y} \\ c^{Y,X} & c^Y \end{pmatrix}. \tag{A.5}$$

Suppose that $\int_0^t \int_{(1,\infty)} e^x K_s(dx) ds < \infty$ for any $t \in \mathbb{R}_+$ or, equivalently, e^X is a special semimartingale. We set

$$X_t^{\parallel} \;:=\; \log \mathscr{E}\big((c^{X,Y}(c^Y)^{-1}) \bullet Y_t^c + f * (\mu^{(X,Y)} - \nu^{(X,Y)})_t\big)$$

for any $t \in \mathbb{R}_+$, where c^- denotes the pseudoinverse of a matrix c in the sense of [1], Y^c is the continuous local martingale part of Y, $\mu^{(X,Y)}$ resp. $\nu^{(X,Y)}$ are the random measure of jumps of (X, Y) and its compensator, and $f : \mathbb{R}^{1+d} \to \mathbb{R}, (x, y) \mapsto (e^x - 1)1_{\{y \neq 0\}}$. We call $X^{\parallel}$ and $X^{\perp} := X - X^{\parallel}$ the *dependent* resp. *independent part* of X relative to Y.

Lemma A.20. *$X \mapsto X^{\parallel}$ is a projection in the sense that $(X^{\parallel})^{\parallel} = X^{\parallel}$. Moreover, we have $(X + Z)^{\parallel} = X^{\parallel}$ if Z is a semimartingale such that Z, Y are locally independent.*

Proof. Observe that $(X^{\parallel})^c = (c^{X,Y}(c^Y)^{-1}) \bullet Y^c$ by [18, Lemma 2.6(2)]. Defining $c^{X^{\parallel},Y}$ similarly as $c^{X,Y}$ in (A.5), we have $c^{X^{\parallel},Y} = c^{X,Y}(c^Y)^{-1}c^Y = c^{X,Y}$. Moreover, $f(\Delta X_t^{\parallel}, \Delta Y_t) = f(\Delta X_t, \Delta Y_t)$ for any $t \geq 0$, which implies $f * (\mu^{(X^{\parallel},Y)} - \nu^{(X^{\parallel},Y)}) = f * (\mu^{(X,Y)} - \nu^{(X,Y)})$ by definition of stochastic integration relative to compensated random measures. Together, the first assertion follows.

Using the notation of (A.5), note that $c^{(X+Z),Y} = c^{X,Y} + c^{Z,Y} = c^{X,Y}$ by Lemma A.11. Lemma A.11 also implies that Z and Y do not jump together (outside some evanescent set) and hence $f(\Delta(X + Z)_t, \Delta Y_t) = f(\Delta X_t, \Delta Y_t)$. This implies $f * (\mu^{(X^{\parallel},Y)} - \nu^{(X^{\parallel},Y)}) = f * (\mu^{(X,Y)} - \nu^{(X,Y)})$ and hence $(X + Z)^{\parallel} = X^{\parallel}$. □

Lemma A.21. *$e^{X^{\parallel}}$ is a local martingale. Moreover, $X^{\perp}$ and $(X^{\parallel}, Y)$ are locally independent semimartingales. Finally, $e^{X^{\perp}}$ is a local martingale if and only if e^X is a local martingale.*

Proof. The first statement is obvious. The last statement follows from the first two and from Proposition A.16. It remains to prove local independence of $X^{\perp}$ and $(X^{\parallel}, Y)$. Denote the local characteristics of $(X^{\perp}, X^{\parallel}, Y)$ by $(b^{(X^{\perp},X^{\parallel},Y)}, c^{(X^{\perp},X^{\parallel},Y)}, K^{(X^{\perp},X^{\parallel},Y)})$ and accordingly for $(X, X^{\parallel}, Y)$, $X^{\perp}$ etc. Set $\bar{c} := c^{X,Y}(c^Y)^{-1}c^{Y,X}$. Since $(X^{\parallel})^c = (c^{X,Y}(c^Y)^{-1}) \bullet Y^c$, we have

$$c^{(X,X^{\parallel},Y)} = \begin{pmatrix} c^X & \bar{c} & c^{X,Y} \\ \bar{c} & \bar{c} & c^{X,Y} \\ c^{Y,X} & c^{Y,X} & c^Y \end{pmatrix}$$

and hence

$$c^{(X^{\perp},X^{\parallel},Y)} = \begin{pmatrix} c^X - \bar{c} & 0 & 0 \\ 0 & \bar{c} & c^{X,Y} \\ 0 & c^{Y,X} & c^Y \end{pmatrix}$$

e.g. by Proposition A.1. Moreover,

$$\begin{aligned}\Delta(X^\perp, X^\parallel, Y)_t &= 1_{\{\Delta Y_t=0\}}(\Delta X_t, 0, 0) + 1_{\{\Delta Y_t\neq 0\}}(0, \Delta X_t, \Delta Y_t)\\ &= \begin{cases}(\Delta X_t^\perp, 0, 0) & \text{if } \Delta X_t^\perp \neq 0,\\ (0, \Delta(X^\parallel, Y)_t) & \text{if } \Delta(X^\parallel, Y)_t \neq 0,\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

yields

$$K^{(X^\perp, X^\parallel, Y)}(A) = K^{X^\perp}(\{x : (x, 0, 0) \in A\}) + K^{(X^\parallel, Y)}(\{(x, z) : (0, x, z) \in A\})$$

for $A \in \mathscr{B}^{2+d}$. Lemma A.11 completes the proof. □

APPENDIX B. TECHNICAL PROOFS

B.1. **Option pricing by Fourier transform.** By

$$\mathscr{F} f(u) := \lim_{C\to\infty} \int_{-C}^{\infty} f(x) e^{iux} dx \tag{B.1}$$

we denote the (left-)improper Fourier transform of a measurable function $f : \mathbb{R} \to \mathbb{C}$ for any $u \in \mathbb{R}$ such that the expression exists. If f is Lebesgue integrable, then the improper Fourier transform and the ordinary Fourier transform (i.e. $u \mapsto \int f(x)e^{iux}dx$) coincide. In our application in Section 3 the improper Fourier transform exists for any $u \in \mathbb{R}\backslash\{0\}$. Moreover, we denote by

$$\mathscr{F}^{-1} g(x) := \frac{1}{2\pi}\left(\lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{\infty} e^{-iux} g(u) du + \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{-\varepsilon} e^{-iux} g(u) du\right) \tag{B.2}$$

an improper inverse Fourier transform, which will be suitable to our application in Section 3.

Lemma B.1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ a sub-σ-field. Furthermore suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is $\mathscr{F} \otimes \mathscr{B}$-measurable, $m : \Omega \to \mathbb{R}$ is $\mathscr{F}$-measurable and*

$$H(x) := 1_{[0,m]}(x) f(x) - 1_{[m,0)}(x) f(x)$$

is nonnegative with $E(\int_0^m f(x)dx) < \infty$. (Note that $\int_0^m f(x)dx$ is always nonnegative.) Then we have

$$\mathscr{F}\{x \mapsto E(H(x)|\mathscr{G})\}(u) = E\left(\int_0^m f(x) e^{iux} dx \middle| \mathscr{G}\right), \quad u \in \mathbb{R} \tag{B.3}$$

where the improper Fourier transform coincides with the ordinary Fourier transform.

Proof. Let $u \in \mathbb{R}$. From

$$\begin{aligned}\int_0^\infty H(x) e^{iux} dx &= 1_{\{m\geq 0\}} \int_0^m f(x) e^{iux} dx,\\ \int_{-\infty}^0 H(x) e^{iux} dx &= 1_{\{m<0\}} \int_0^m f(x) e^{iux} dx\end{aligned}$$

it follows that $\int_{-\infty}^\infty H(x)e^{iux}dx = \int_0^m f(x)e^{iux}dx$. This implies

$$E\left(E\left(\int_{-\infty}^\infty H(x)dx \middle| \mathscr{G}\right)\right) = E\left(\int_{-\infty}^\infty H(x)dx\right) = E\left(\int_0^m f(x)dx\right) < \infty$$

and hence

$$\int_{-\infty}^\infty E(H(x)|\mathscr{G})dx = E\left(\int_{-\infty}^\infty H(x)dx \middle| \mathscr{G}\right) < \infty.$$

Now we can apply Fubini's theorem and we get

$$\mathscr{F}\{x \mapsto E(H(x)|\mathscr{G})\}(u) = E\bigg(\int_{-\infty}^{\infty} H(x)e^{iux}dx \bigg| \mathscr{G} \bigg) = E\bigg(\int_{0}^{m} f(x)e^{iux}dx \bigg| \mathscr{G} \bigg).$$

□

The next proposition is a modification of [3, Proposition 1], cf. also [10].

Lemma B.2. *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ a sub-σ-field. Let Y be a random variable such that $E(e^Y) < \infty$ and consider*

$$\mathscr{O}(x) := \begin{cases} E((e^{Y-x}-1)^+|\mathscr{G}) & \text{if } x \geq 0, \\ E((1-e^{Y-x})^+|\mathscr{G}) & \text{if } x < 0. \end{cases}$$

Then we have

$$\mathscr{F}\{x \mapsto \mathscr{O}(x)\}(u) = \frac{1}{iu} - \frac{E(e^Y|\mathscr{G})}{iu-1} - \frac{E(e^{iuY}|\mathscr{G})}{u^2+iu}$$

and

$$\begin{aligned} &\mathscr{F}\{x \mapsto 1_{\{x \geq -C\}}\mathscr{O}(x)\}(u) \\ &= \frac{1}{iu} - \frac{E\left(e^y|\mathscr{G}\right)}{iu-1} - \frac{1-E\left(e^{iu(Y \vee -C)}\left(1+iu\left(e^{0 \wedge (Y+C)}-1\right)\right)|\mathscr{G}\right)}{u^2+iu} \end{aligned}$$

for any $C \in \mathbb{R}_+$, $u \in \mathbb{R}\backslash\{0\}$. If $E(e^Y|\mathscr{G}) = 1$, then in particular

$$\mathscr{F}\{x \mapsto \mathscr{O}(x)\}(u) = \frac{1-E(e^{iuY}|\mathscr{G})}{u^2+iu}$$

for any $u \in \mathbb{R}\backslash\{0\}$.

Proof. Let $C \in \mathbb{R}_+$, $u \in \mathbb{R}\backslash\{0\}$. We define $m := (Y \vee -C)$, $f(x) := e^{Y-x} - 1$, and $H(x) := 1_{[0,m]}(x)f(x) - 1_{[m,0)}(x)f(x)$. Then we have $1_{\{x \geq -C\}}\mathscr{O}(x) = E(H(x)|\mathscr{G})$, $H \geq 0$, and

$$E\left(\int_0^m f(x)dx\right) = E(e^Y - m - e^{Y-m}) < \infty.$$

Hence Lemma B.1 yields

$$\begin{aligned} \int_{-C}^{\infty} \mathscr{O}(x)e^{iux}dx &= \mathcal{F}\{x \mapsto E(H(x)|\mathscr{G})\}(u) \\ &= E\left(\int_0^m (e^{Y-x}-1)e^{iux}dx \bigg| \mathscr{G} \right) \\ &= E\left(\left[\frac{e^{Y+(iu-1)x}}{iu-1} - \frac{e^{iux}}{iu} \right]_{x=0}^{m} \bigg| \mathscr{G} \right) \\ &= \frac{1}{iu} - \frac{E(e^Y|\mathscr{G})}{iu-1} - \frac{E\left(e^{ium}(1+iu(e^{Y-m}-1))|\mathscr{G}\right)}{u^2+iu}. \end{aligned}$$

Since $|e^{ium}(1+iu(e^{Y-m}-1))| \leq 1+|u|$, we can apply Lebesgue's theorem and get

$$E\left(e^{ium}(1+iu(e^{Y-m}-1))|\mathscr{G}\right) \overset{C\to\infty}{\longrightarrow} E\left(e^{iuY}|\mathscr{G}\right).$$

□

Corollary B.3. *Let Q be a probability measure on $\mathbb{R}$ satisfying $K := \int e^y Q(dy) < \infty$. Define*

$$O(x) := \begin{cases} \int (e^{y-x}-1)^+ Q(dy) & \text{if } x \geq 0, \\ \int (1-e^{y-x})^+ Q(dy) & \text{if } x < 0. \end{cases}$$

Then we have

$$\left| \int_{-C}^{\infty} O(x) e^{iux} dx \right| \leq K + \frac{1+2|u|}{u^2}$$

for any $C \in \mathbb{R}_+$, $u \in \mathbb{R} \backslash \{0\}$.

Proof. Apply Lemma B.2 with $(\Omega, \mathscr{F}, P) = (\mathbb{R}, \mathscr{B}, Q), \mathscr{G} = \{\varnothing, \mathbb{R}\}, Y = \mathrm{id}$. □

Proposition B.4. *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ a sub-σ-field. Let Y be a random variable with $E(e^Y|\mathscr{G}) = 1$ and define*

$$\mathscr{O}(x) := \begin{cases} E((e^{Y-x}-1)^+|\mathscr{G}) & \text{if } x \geq 0, \\ E((1-e^{Y-x})^+|\mathscr{G}) & \text{if } x < 0. \end{cases}$$

Then we have

$$\begin{aligned} \mathscr{O}(x) &= \mathscr{F}^{-1}\left\{ u \mapsto \frac{1 - E(e^{iuY}|\mathscr{G})}{u^2 + iu} \right\}(x), \\ E(e^{iuY}|\mathscr{G}) &= 1 - (u^2 + iu)\mathscr{F}\{x \mapsto \mathscr{O}(x)\}(u) \end{aligned}$$

for any $u, x \in \mathbb{R}$.

Proof. The second equation is a restatement of Lemma B.2. Let $0 < \alpha < 1$ and define $O(x) := e^{\alpha x}\mathscr{O}(x)$, $m := Y$, $f(x) := e^{\alpha x}(e^{Y-x} - 1)$, and $H(x) := 1_{[0,m]}(x)f(x) - 1_{[m,0)}(x)f(x)$ for any $x \in \mathbb{R}$. Then H is nonnegative, $O(x) = E(H(x)|\mathscr{G})$, and

$$E\left(\int_0^m f(x)dx \right) = E\left(\frac{e^{\alpha Y}}{\alpha^2 - \alpha} + \frac{e^Y}{1-\alpha} + \frac{1}{\alpha} \right) < \infty.$$

Lemma B.1 yields

$$\begin{aligned} \mathcal{F}\{x \mapsto O(x)\}(u) &= E\left(\int_0^m f(x)e^{iux}dx \middle| \mathscr{G} \right) \\ &= \frac{E\left(e^{(\alpha+iu)Y}|\mathscr{G}\right) - 1}{(\alpha+iu)^2 - (\alpha+iu)}. \end{aligned}$$

We have $E(|e^{(\alpha+iu)Y}|) \leq E(1 + e^Y) = 2$ and thus $u \mapsto \mathcal{F}\{x \mapsto O(x)\}(u)$ is integrable. The Fourier inversion theorem yields

$$O(x) = \mathcal{F}^{-1}\{u \mapsto \mathcal{F}\{\tilde{x} \mapsto O(\tilde{x})\}(u)\}(x)$$

because the ordinary inverse Fourier transform coincides with the improper inverse Fourier transform for Lebesgue-integrable functions. Define

$$g : \{z \in \mathbb{C} \setminus \{0\} : -1 < \mathrm{Re}(z) \leq 0\} \to \mathbb{C}, \quad z \mapsto \frac{E(e^{-zY}|\mathscr{G}) - 1}{z^2 + z}.$$

g is continuous and holomorphic in the interior of its domain. Let $0 < \varepsilon < \frac{1}{2} =: \alpha$ and define

$$\begin{aligned}
\gamma_{(1,\varepsilon)} &: [-1,1] \to \mathbb{C}, \qquad t \mapsto i\frac{t}{\varepsilon} - \frac{1}{2},\\
\gamma_{(2,\varepsilon)} &: [0,1] \to \mathbb{C}, \qquad t \mapsto i\frac{1}{\varepsilon} - \frac{1-t}{2},\\
\gamma_{(3,\varepsilon)} &: [0,1] \to \mathbb{C}, \qquad t \mapsto i(1-t)\left(\frac{1}{\varepsilon} - \varepsilon\right) + i\varepsilon,\\
\gamma_{(4,\varepsilon)} &: [0,\pi] \to \mathbb{C}, \qquad t \mapsto i\varepsilon e^{it},\\
\gamma_{(5,\varepsilon)} &: [0,1] \to \mathbb{C}, \qquad t \mapsto it\left(\varepsilon - \frac{1}{\varepsilon}\right) - i\varepsilon,\\
\gamma_{(6,\varepsilon)} &: [0,1] \to \mathbb{C}, \qquad t \mapsto -i\frac{1}{\varepsilon} - \frac{t}{2}
\end{aligned}$$

as well as $\Gamma_\varepsilon := \sum_{k=1}^{6} \gamma_{(k,\varepsilon)}$. Cauchy's integral theorem yields

$$\int_{\Gamma_\varepsilon} g(z)e^{xz}dz = 0.$$

Moreover we have

$$\frac{1}{2\pi i}\int_{\gamma_{(1,\varepsilon)}} g(z)e^{xz}dz \overset{\varepsilon\to 0}{\longrightarrow} O(x)e^{-\frac{1}{2}x} = \mathscr{O}(x)$$

and

$$\int_{\gamma_{(k,\varepsilon)}} g(z)e^{xz}dz \overset{\varepsilon\to 0}{\longrightarrow} 0$$

for $k \in \{2,6\}$ and even for $k=4$ because $zg(z) \to 0$ for $z \to 0$. Thus we conclude

$$\begin{aligned}
&\frac{1}{2\pi}\left(\int_{-1/\varepsilon}^{-\varepsilon} g(-iu)e^{-iux}du + \int_{\varepsilon}^{1/\varepsilon} g(-iu)e^{-iux}du\right)\\
&= \frac{1}{2\pi i}\int_{-\gamma_{(3,\varepsilon)}-\gamma_{(5,\varepsilon)}} g(z)e^{xz}dz\\
&= \frac{1}{2\pi i}\int_{\gamma_{(1,\varepsilon)}-\Gamma_\varepsilon+\gamma_(2,\varepsilon)+\gamma_(4,\varepsilon)+\gamma_(6,\varepsilon)} g(z)e^{xz}dz\\
&\overset{\varepsilon\to 0}{\longrightarrow} \mathscr{O}(x).
\end{aligned}$$

Since

$$\int_{-\infty}^{-1/\varepsilon} g(-iu)e^{-iux}du + \int_{1/\varepsilon}^{\infty} g(-iu)e^{-iux}du \overset{\varepsilon\to 0}{\longrightarrow} 0,$$

we have

$$\frac{1}{2\pi}\left(\int_{-\infty}^{-\varepsilon} g(-iu)e^{-iux}du + \int_{\varepsilon}^{\infty} g(-iu)e^{-iux}du\right) \overset{\varepsilon\to 0}{\longrightarrow} \mathscr{O}(x)$$

and hence

$$\mathscr{F}^{-1}\left\{u \mapsto \frac{1-E(e^{iuY}|\mathscr{G})}{u^2+iu}\right\}(x) = \mathscr{O}(x).$$

□

Proposition B.5. *Let $(N(x))_{x\in\mathbb{R}}$ be a family of nonnegative local martingales, and $(\tau_n)_{n\in\mathbb{N}}$ a common localising sequence for all $N(x)$ such that*

(1) $(\omega, x) \mapsto N_t(x)(\omega)$ *is* $\mathscr{F} \otimes \mathscr{B}$*-measurable for all* $t \in \mathbb{R}_+$,
(2) $x \mapsto N_t(x)(\omega)$ *is right-continuous and* $\int_{-C}^{\infty} N_t(x)(\omega)dx < \infty$ *for all* $t \in \mathbb{R}_+$,
(3) $\lim_{C\to\infty} \int_{-C}^{\infty} e^{iux} N_t(x)(\omega)dx$ *exists for all* $\omega \in \Omega$, $t \in \mathbb{R}_+$, $u \in \mathbb{R}\backslash\{0\}$,
(4) *for any* $n \in \mathbb{N}, t \in \mathbb{R}, u \in \mathbb{R}\backslash\{0\}$ *there is an integrable random variable* Z *such that* $|\int_{-C}^{\infty} e^{iux} N_t^{\tau_n}(x)(\omega)dx| \leq Z$ *for any* $C \in \mathbb{R}_+$.

Define the (improper, cf. (B.1)) Fourier transform of N *by*

$$X_t(u) := \mathscr{F}\{x \mapsto N_t(x)\}(u).$$

If $X(u)$ *has càdlàg paths, then it is a local martingale for all* $u \in \mathbb{R}\backslash\{0\}$ *with common localising sequence* $(\tau_n)_{n\in\mathbb{N}}$.

Proof. For any $C \in \mathbb{R}_+$, $u \in \mathbb{R}\backslash\{0\}, \omega \in \Omega, t \in \mathbb{R}_+$ define

$$X_t^C(u)(\omega) := \int_{-C}^{\infty} e^{iux} N_t(x)(\omega)dx.$$

Fix $n \in \mathbb{N}, C \in \mathbb{R}_+$, $u \in \mathbb{R}\backslash\{0\}$. Then $X^C_{t\wedge\tau_n(\omega)}(u)(\omega) = \int_{-C}^{\infty} e^{iux} N_{t\wedge\tau_n(\omega)}(x)(\omega)dx$ for any $t \in \mathbb{R}_+, \omega \in \Omega$. Setting

$$\begin{aligned} c(k,x) &:= \begin{cases} 1_{\cos(x)>0}\cos(x) & \text{for } k=0, \\ 1_{\sin(x)>0}\sin(x) & \text{for } k=1, \\ -1_{\cos(x)<0}\cos(x) & \text{for } k=2, \\ -1_{\sin(x)<0}\sin(x) & \text{for } k=3, \end{cases} \\ I_t^C(k) &:= \int_{-C}^{\infty} c(k,ux) N_{t\wedge\tau_n}(x)dx \end{aligned}$$

yields

$$X^C_{t\wedge\tau_n}(u) = \sum_{k=0}^{3} i^k I_t^C(k).$$

Since $c(k,\cdot) : \mathbb{R} \to \mathbb{R}_+$ and hence $I^C(k)$ are positive, we can apply Tonelli's theorem and conclude that $I^C(k)$ is a martingale up to the càdlàg property for all $k \in \{0,1,2,3\}$. Thus $(t,\omega) \mapsto X^C_{t\wedge\tau_n}(u)(\omega)$ is a martingale up to the càdlàg property as well. By the definitions of X^C and X we have $X_t(u)(\omega) = \lim_{C\to\infty} X_t^C(u)(\omega)$ and thus we get

$$X_{t\wedge\tau_n(\omega)}(u)(\omega) = \lim_{C\to\infty} X^C_{t\wedge\tau_n(\omega)}(u)(\omega).$$

The fourth assumption on N and Lebesgue's theorem yield

$$\begin{aligned} E(X_{t\wedge\tau_n}(u)|\mathcal{F}_s) &= E\left(\lim_{C\to\infty} X^C_{t\wedge\tau_n}(u)|\mathcal{F}_s\right) \\ &= \lim_{C\to\infty} E\left(X^C_{t\wedge\tau_n}(u)|\mathcal{F}_s\right) \\ &= X_{s\wedge\tau_n}(u) \end{aligned}$$

for $s \leq t$. Thus $(t,\omega) \to X_t(u)(\omega)$ is a local martingale. □

B.2. **Essential infimum of a subordinator.**

Definition B.6. Let ψ be a deterministic local exponent (of some semimartingale X). We define the *infimum process* E of ψ by $E_t := \text{ess inf } X_t$ for any $t \in \mathbb{R}_+$.

By [15, III.2.16], X in the previous definition is a PII whose law is determined by ψ. Since E_t is in turn determined by the law of X_t, the infimum process E does not depend on the particular choice of X.

Lemma B.7. *Let X, Y be independent random variables. Then*

$$\text{ess inf }(X+Y) = \text{ess inf }\ X + \text{ess inf }\ Y.$$

Proof. Let $x := \text{ess inf }\ X$ and $y := \text{ess inf }\ Y$. We obviously have $\text{ess inf }(X+Y) \geq x+y$ because $X+Y \geq x+y$ almost surely. Independence yields

$$\begin{aligned} P(X+Y \leq x+y+\varepsilon) &\geq P\left(X \leq x+\frac{\varepsilon}{2}, Y \leq y+\frac{\varepsilon}{2}\right) \\ &= P\left(X \leq x+\frac{\varepsilon}{2}\right) P\left(Y \leq y+\frac{\varepsilon}{2}\right) \\ &> 0 \end{aligned}$$

for any $\varepsilon > 0$. □

Proposition B.8. *Let X be a subordinator (i.e. an increasing Lévy process) and $E_t := \text{ess inf }X_t$ for any $t \in \mathbb{R}_+$. Then $E_t = tE_1 \geq 0$ for any $t \geq 0$ and E is the drift part of X relative to the "truncation" function $h = 0$. Moreover, $X - E$ is a subordinator.*

Proof. Since X is a subordinator, we have $E_t \geq 0$ for any $t \in \mathbb{R}_+$. Moreover, $\text{ess inf }(X_t - X_s) = E_{t-s}$ because X_{t-s} has the same distribution as $X_t - X_s$. Since X_s and $X_t - X_s$ are independent, Lemma B.7 yields $E_s + E_{t-s} = E_t$. The mapping $t \mapsto E_t$ is increasing because X is a subordinator. Together we conclude $E_t = tE_1$. This implies that $X - E$ is a positive Lévy process and hence a subordinator. By [21, Theorem 21.5] the Lévy-Khintchine triplet (b, c, K) relative to "truncation" function $h = 0$ exists and satisfies $c = 0$. Moreover, K and the random measure of jumps μ^X of X are concentrated on $\mathbb{R}_+$. In view of [15, II.2.34] we have $X_t = x * \mu^X_t + bt$ and thus we get $E_t = \text{ess inf }X_t \geq bt$. According to [21, Theorem 21.5], $X - E$ is a subordinator only if its drift rate $\widetilde{b}$ relative to $h = 0$ is greater or equal 0. Hence $bt - E_t = \widetilde{b}t \geq 0$, which implies $bt = E_t$. □

Corollary B.9. *Let X be a d-dimensional semimartingale whose components X^k are subordinators with essential infimum $E^k_t = \text{ess inf }X^k_t$ for $k = 1, \dots, d$. For componentwise nonnegative bounded predictable processes φ we have*

$$\text{ess inf }(\varphi \bullet (X-E)_t) = 0.$$

Moreover, for bounded predictable $\mathbb{C}^d$-valued processes φ we have

$$\text{ess inf }\ |\varphi \bullet (X-E)_t| = 0$$

for any $t \in \mathbb{R}_+$.

Proof. The second statement is an application of the first statement because

$$0 \leq |\varphi \bullet (X-E)_t| \leq (|\varphi^1|, \dots, |\varphi^d|) \bullet (X-E)_t.$$

Suppose that φ is a componentwise nonnegative bounded predictable process. Proposition B.8 yields that $X^k - E^k$ is a subordinator with essential infimum 0. Hence we may assume w.l.o.g. that $E^k = 0$. Since φ is bounded, there is a constant $c \in \mathbb{R}_+$ such that $\varphi^k \leq c$ for $k = 1, \dots, d$. Hence $\varphi \bullet X_t \leq c \sum_{k=1}^d X^k_t$ for any $t \in \mathbb{R}_+$. By Proposition B.8 the drift part of X^k is 0 relative to the truncation function $h = 0$. Consequently, the drift part of the subordinator $L := c\sum_{k=1}^d X^k$ is also 0 relative to the truncation function $h = 0$. Proposition B.8 yields $\text{ess inf }L_t = 0$ for any $t \in \mathbb{R}_+$. Thus we conclude

$$0 \leq \text{ess inf }(\varphi \bullet X_t) \leq \text{ess inf }L_t = 0.$$

□

REFERENCES

1. A. Albert, *Regression and the Moore-Penrose pseudoinverse*, Academic Press, New York, 1972.
2. O. Barndorff-Nielsen and N. Shephard, *Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics*, Journal of the Royal Statistical Society, Series B **63** (2001), 167–241.
3. D. Belomestny and M. Reiß, *Spectral calibration of exponential Lévy models*, Finance & Stochastics **10** (2006), 449–474.
4. N. Bennani, *The forward loss model: a dynamic term structure approach for the pricing of portfolios of credit derivatives*, Preprint, 2005.
5. H. Bühler, *Consistent variance curve models*, Finance & Stochastics **10** (2006), 178–203.
6. R. Carmona, *HJM: A unified approach to dynamic models for fixed income, credit and equity markets*, Paris-Princeton Lectures in Mathematical Finance, 2005 (R. Carmona et al., ed.), Lecture Notes in Mathematics, vol. 1919, Springer, Berlin, 2009, pp. 3–45.
7. R. Carmona and S. Nadtochiy, *Local volatility dynamic models*, Finance & Stochastics **13** (2009), 1–48.
8. ——, *Tangent Lévy market models*, Finance & Stochastics (2009), to appear.
9. P. Carr, H. Geman, D. Madan, and M. Yor, *Stochastic volatility for Lévy processes*, Mathematical Finance **13** (2003), 345–382.
10. P. Carr and D. Madan, *Option valuation using the fast Fourier transform*, The Journal of Computational Finance **2** (1999), 61–73.
11. M. Davis and D. Hobson, *The range of traded option prices*, Mathematical Finance **17** (2007), 1–14.
12. D. Filipović, *Time-inhomogeneous affine processes*, Stochastic Processes and their Applications **115** (2005), 639–659.
13. D. Heath, R. Jarrow, and A. Morton, *Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation*, Econometrica **60** (1992), 77–105.
14. J. Jacod and P. Protter, *Risk neutral compatibility with option prices*, Finance & Stochastics (2006), To appear.
15. J. Jacod and A. Shiryaev, *Limit theorems for stochastic processes*, second ed., Springer, Berlin, 2003.
16. J. Kallsen, *σ-localization and σ-martingales*, Theory of Probability and Its Applications **48** (2004), 152–163.
17. ——, *A didactic note on affine stochastic volatility models*, From Stochastic Calculus to Mathematical Finance (Yu. Kabanov, R. Liptser, and J. Stoyanov, eds.), Springer, Berlin, 2006, pp. 343–368.
18. J. Kallsen and A. Shiryaev, *The cumulant process and Esscher's change of measure*, Finance & Stochastics **6** (2002), 397–428.
19. E. Lukacs, *Characteristic Functions*, Griffin, London, 1970.
20. P. Protter, *Stochastic integration and differential equations*, second ed., Springer, Berlin, 2004.
21. K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.
22. P. Schönbucher, *Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives*, Preprint, 2005.
23. M. Schweizer and J. Wissel, *Arbitrage-free market models for option prices: The multi-strike case*, Finance & Stochastics **12** (2008), 469–505.
24. ——, *Term structures of implied volatilities: Absence of arbitrage and existence results*, Mathematical Finance **18** (2008), 77–114.
25. J. Wissel, *Arbitrage-free market models for liquid options*, Ph.D. thesis, ETH Zürich, 2008.

MATHEMATISCHES SEMINAR, CHRISTIAN-ALBRECHTS-UNIVERSITÄT ZU KIEL, WESTRING 383, 24118 KIEL, GERMANY, (E-MAIL: KALLSEN@MATH.UNI-KIEL.DE, KRUEHNER@MATH.UNI-KIEL.DE).