ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/233124688

On a higher dimensional Miura transform

Article in Complex Variables and Elliptic Equations · April 1999

DOI: 10.1080/17476939908815172

CITATIONS
21

READS
89

2 authors:

Swanhild Bernstein
Technische Universität Bergakademie Freiberg
100 PUBLICATIONS 577 CITATIONS

SEE PROFILE

Klaus Gürlebeck
Bauhaus-Universität Weimar
223 PUBLICATIONS 3,545 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

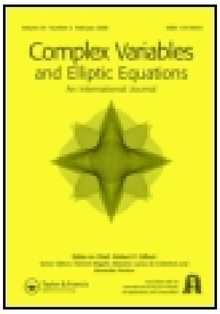
Project Beyond Wigner quantal systems View project

Project E-Learning, self assement by computer based problems View project

All content following this page was uploaded by Swanhild Bernstein on 01 June 2014.

The user has requested enhancement of the downloaded file.

This article was downloaded by: [TU Bergakademie Freiberg]
On: 28 February 2013, At: 04:40
Publisher: Taylor & Francis
Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



Complex Variables, Theory and Application: An International Journal: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcov19

On a higher dimensional miuratransform

S. Bernstein [a] & K. Gürlebeck [b]

[a] Institut für Angewandle Mathematik I, TU Bergakademie Freiberg, Freiberg, D-09596, Germany

[b] Fakultät B., Bauhaus-Universität Weimar, Weimar, D-99421, Germany

Version of record first published: 09 Jun 2010.

To cite this article: S. Bernstein & K. Gürlebeck (1999): On a higher dimensional miuratransform, Complex Variables, Theory and Application: An International Journal: An International Journal, 38:4, 307-319

To link to this article: http://dx.doi.org/10.1080/17476939908815172

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: http://www.tandfonline.com/page/terms-and-conditions

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae, and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand, or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.

Complex Variables, Vol. 38, pp. 307–319
Reprints available directly from the publisher
Photocopying permitted by license only

© 1999 OPA (Overseas Publishers Association) N.V.
Published by license under
the Gordon and Breach Science
Publishers imprint.
Printed in Malaysia.

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

On a Higher Dimensional Miura Transform

S. BERNSTEIN[a,*] and K. GÜRLEBECK[b]

[a] *Institut für Angewandte Mathematik I, TU Bergakademie Freiberg, D-09596 Freiberg, Germany;* [b] *Fakultät B, Bauhaus-Universität Weimar, D-99421 Weimar, Germany*

Communicated by R. Delanghe

(Received June 1997)

We investigate a spatial modified Miura transform. To describe this transform we have to solve a non-linear first-order system of partial differential equations. This investigation will be done by the help of quaternionic analysis. The main goal is to find a hypercomplex factorization of the Schrödinger equation. In one dimension Miura's transformation is needed to map solutions of the modified Korteweg–de Vries equation into solutions of Korteweg–de Vries equation.

Keywords: Quaternionic analysis; Schrödinger equation; factorization

Classification Categories: 1991 Mathematical Subject Classification Primary 30G35; Secondary 35J10, 35F30

1 INTRODUCTION

In one spatial dimension a non-linear transformation, the so-called Miura transformation, represents a connection between Korteweg–de Vries equations. These equations describe water waves. This famous transformation was introduced by Miura [14] in 1968. For more information about this transformation and the Korteweg–de Vries equations see for example [1]. The Miura transformation may be obtained by a factorization of the Schrödinger operator. We will show

* Corresponding author.

that using quaternionic or Clifford analysis an analogous factorization of the higher-dimensional Schrödinger operator is possible and leads to a non-linear system of first-order partial differential equations.

To get a factorization by solving the first-order system of partial differential equations it is necessary to have some boundary condition. If we state a Dirichlet-type boundary condition, for example that the boundary values should vanish, then the problem may be unsolvable! We will describe an admissible boundary condition and a suitable subspace of boundary values. Using an iterative procedure based on Banach's fixed-point principle we will solve the non-linear first-order system of partial differential equations.

Finally, we discuss some generalizations. We will outline that the used Sobolev space for the iteration depends on the spatial dimension and that the described method also works in the case of stronger nonlinearities.

2 PRELIMINARIES

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis in $\mathbb{R}^n$. Consider the 2^n-dimensional real Clifford algebra $\mathcal{C}\ell_{0,n}$ generated by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ according to the multiplication rules $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}\mathbf{e}_0$, where $\mathbf{e}_0$ is the identity of $\mathcal{C}\ell_{0,n}$. The elements $\mathbf{e}_A$: $A = \{h_1, \ldots, h_k\} \subseteq \{1, \ldots, n\}$ define a basis of $\mathcal{C}\ell_{0,n}$, where $\mathbf{e}_A = \mathbf{e}_{h_1} \cdots \mathbf{e}_{h_k} = \mathbf{e}_{h_1 \ldots h_k}$, $1 \le h_1 < \cdots < h_k \le n$, and $\mathbf{e}_\emptyset = \mathbf{e}_0$. The main part of this paper is restricted to the case $n = 2$. The algebra $\mathcal{C}\ell_{0,2}$ will be generated by $\mathbf{e}_1$ and $\mathbf{e}_2$. We denote the product $\mathbf{e}_1\mathbf{e}_2$ by $\mathbf{e}_3$. Then $\mathcal{C}\ell_{0,2}$ can be identified with the algebra of real quaternions $\mathbb{H}$. Our multiplication rules look like $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}\mathbf{e}_0$ for $i, j \in \{1, 2, 3\}$.

An arbitrary element $q \in \mathbb{H}$ is given by $q = q_0 e_0 + \sum_{j=1}^{3} q_j \mathbf{e}_j$ and the conjugated quaternion by $\bar{q} = q_0\mathbf{e}_0 - \sum_{j=1}^{3} q_j\mathbf{e}_j$.

We suppose $\Omega \subset \mathbb{R}^3$ to be a domain with a smooth boundary Γ. The elements $(x_1, x_2, x_3) = \vec{x} \in \mathbb{R}^3$ will be identified with $x = \sum_{j=1}^{3} x_j \mathbf{e}_j \in \mathbb{H}$.

For each $x \in \mathbb{H}$ we have $x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2 = |x|^2$. Then, functions f defined in Ω with values in $\mathbb{H}$ are considered. These functions may be written as

$$f(x) = \sum_{k=0}^{3} \mathbf{e}_k f_k(x), \quad x \in \Omega.$$

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

Properties such as continuity, differentiability, integrability, and so on, which are ascribed to f have to be possessed by all components $f_k(x)$, $k = 0, \dots, 3$. In this way the usual Banach spaces of these functions are denoted by C^α, L_p and W_p^k. In the case of $p = 2$ we introduce in $L_2(\Omega)$ the $\mathbb{H}$-valued inner product

$$(u, v) = \int_\Omega \bar{u}(\xi)v(\xi)\,\mathrm{d}\Omega_\xi. \tag{1}$$

We now define the Dirac operator by

$$D = \sum_{k=1}^{3} \mathbf{e}_k \frac{\partial}{\partial x_k}.$$

For this operator we have the factorization

$$DD = -\Delta, \tag{2}$$

where Δ is the Laplacian in $\mathbb{R}^3$. We consider in Ω the equation

$$(Du)(x) = 0,$$

and look for its solutions which are called *left-monogenic* functions in Ω.

Now we define the Cauchy kernel in $\mathbb{R}^3$ by

$$e(x) = \frac{-x}{4\pi|x|^3}, \quad x \neq 0.$$

It is well known that $e(x)$ (a fundamental solution of D) is monogenic in $\mathbb{R}^3 \backslash \{0\}$. Using the function $e(x)$ we introduce the following integral operators:

$$(T_\Omega u)(x) := -\int_\Omega e(x-y)u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^3$$

(Teodorescu transform),

$$(F_\Gamma u)(x) := \int_\Gamma e(x-y)n(y)u(y)\,\mathrm{d}\Gamma_y, \quad x \notin \Gamma$$

(Cauchy type operator),

$$(S_\Gamma u)(x) := \int_\Gamma 2e(x-y)n(y)u(y)\,\mathrm{d}\Gamma_y, \quad x \in \Gamma$$

(singular integral operator),

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

where $n(y) = \sum_{i=1}^{3} \mathbf{e}_i n_i(y)$ is the outward pointing normal (unit) vector to Γ at the point y. The integral which defines the operator S_Γ has to be taken in the sense of Cauchy's principle value. From [8] we immediately get the following statements.

LEMMA 1 *Let $u \in C^1(\Omega, \mathbb{H}) \cap C(\bar{\Omega}, \mathbb{H})$. Then we have*

(i) $(F_\Gamma u)(x) + (T_\Omega D)u(x)$
$$= \begin{cases} u(x), & x \in \Omega \quad \text{(Borel–Pompeiu's formula)} \\ 0, & x \in \mathbb{R}^3 \backslash \Omega, \end{cases}$$

(ii) $(DT_\Omega u)(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^3 \backslash \bar{\Omega}, \end{cases}$

(iii) $(DF_\Gamma)u(x) = 0$ *in* $\Omega \cup (\mathbb{R}^3 \backslash \bar{\Omega})$.

LEMMA 2 (Plemelj–Sokhotzkij's formulas) *Let $u \in C^{0,\alpha}(\Omega, \mathbb{H})$, $0 < \alpha < 1$. Then we have*

(i) $\lim\limits_{\substack{x \to \xi \in \Gamma \\ x \in \Omega}} (F_\Gamma u)(x) = P_\Gamma u(\xi)$,

(ii) $\lim\limits_{\substack{x \to \xi \in \Gamma \\ x \in \mathbb{R}^3 \backslash \bar{\Omega}}} (F_\Gamma u)(x) = -Q_\Gamma u(\xi)$

for any $\xi \in \Gamma$.

COROLLARY 1 *Let $u \in C^{0,\alpha}(\Gamma, \mathbb{H})$. Then the equations* (i) $(S_\Gamma^2 u)(\xi) = u(\xi)$, (ii) $(F_\Gamma P_\Gamma u)(\xi) = F_\Gamma u(\xi)$, (iii) $(P_\Gamma^2 u)(\xi) = (P_\Gamma u)(\xi)$, (iv) $(Q_\Gamma^2 u)(\xi) = (Q_\Gamma u)(\xi)$ *are valid for any $\xi \in \Gamma$.*

The operator $P_\Gamma := 1/2(I + S_\Gamma)$ denotes the projection onto the space of all $\mathbb{H}$-valued functions which have a left monogenic extension into the domain Ω. $Q_\Gamma := 1/2(I - S_\Gamma)$ denotes the projection onto the space of all $\mathbb{H}$-valued functions which have a left monogenic extension into the domain $\mathbb{R}^3 \backslash \bar{\Omega}$ and vanish at infinity. We remark that the operators F_Γ, S_Γ, P_Γ, and Q_Γ are defined in spaces of Hölder continuous functions. It is possible to extend these operators to Sobolev spaces in the classical way by approximation (with Hölder continuous functions). We omit the detailed discussion here. We remark that then all the referred formulas have to be understood in the generalized sense. The restriction of an $\mathbb{H}$-valued function u to a function defined on the boundary Γ is expressed by tr u.

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

3 A MODIFIED MIURA TRANSFORMATION

After studying the conservation laws of the Korteweg–de Vries equation, and those associated with the modified Korteweg–de Vries equation, Miura (cf. [14]) discovered the following transform, nowadays known as Miura's transformation. If w is a solution of the modified Korteweg–de Vries equation, then

$$v = -(w^2 + w_x)$$

is a solution of the Korteweg–de Vries equation. Note that every solution of the mKdV equation maps, via Miura's transformation to a solution of the KdV equation, but the converse is not true.

It is posssible to obtain Miura's transformation by factorizing the Schrödinger equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}u - v(x)u = -\left(\frac{\mathrm{d}}{\mathrm{d}x} + \alpha(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha(x)\right)u.$$

We have

$$\begin{aligned}
&-\left(\frac{\mathrm{d}}{\mathrm{d}x} + \alpha(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha(x)\right)u \\
&= -\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x}(\alpha(x)u) + \alpha(x)\frac{\mathrm{d}u}{\mathrm{d}x} - \alpha^2(x)u\right) \\
&= -\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}\alpha}{\mathrm{d}x}u + \alpha(x)\frac{\mathrm{d}u}{\mathrm{d}x} - \alpha(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \alpha^2(x)u \\
&= -\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \left(-\frac{\mathrm{d}\alpha}{\mathrm{d}x} - \alpha^2(x)\right)u\right)
\end{aligned}$$

and thus

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \alpha^2(x) = -v(x).$$

This is a non-linear differential equation for $\alpha(x)$. To find a higher-dimensional analogy, we use the factorization (2) of the Laplacian into Dirac operators.

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

It is easily seen that the Helmholtz operator may be factorized by using disturbed Dirac operators. We have

$$-\Delta - k^2 = (D+k)(D-k). \tag{3}$$

Factorizations of the Helmholtz operator have been studied several times. In [6] and later on in [7] and also in the book [8] the case of a real wave number was studied. Xu [18,19], Brackx and van Acker [4] and together with Delanghe and Sommen [5] considered the operators $D+k$ with k a complex number. Obolashvili [15,16] treated the case of purely vectorial k and later the same was done by Huang [9]. We also want to mention the related paper of Mitrea [13], where k is a real quaternion. The general quaternionic case that k is a complex quaternion was considered by Kravchenko and Shapiro [10,11] and a complete investigation can be found in their book [12]. The paper [3] is also related to this topic. In all these cases the wave number has to be a constant. We try to factorize the Schrödinger equation in the same way.

Let $x=(x_1,x_2,x_3)$. We consider $(-\Delta - V_0(x))u$ with $u=u(x)=u_0(x)\mathbf{e}_0$ and look for suitable functions α with

$$\begin{aligned}(-\Delta - V_0(x))u &= (D+\alpha(x))(D-\alpha(x))u \\ &= (D+\alpha(x))(Du-\alpha(x)u) \\ &= DDu - D(\alpha(x)u) + \alpha(x)Du - \alpha^2(x)u \\ &= -\Delta u \underline{- Du\cdot\alpha(x) + \alpha(x)Du} - D\alpha(x)\cdot u - \alpha^2(x)u.\end{aligned}$$

The underlined part will not vanish, because of the non-commutative multiplication of quaternions. Thus, we will change our approach using a multiplication operator $M^{\alpha(x)}$ defined by

$$M^{\alpha(x)}u(x) := u(x)\cdot\alpha(x).$$

Hence,

$$\begin{aligned}(-\Delta - V_0(x))u &= (D+M^{\alpha(x)})(D-M^{\alpha(x)})u \\ &= (D+M^{\alpha(x)})(Du-u\alpha(x)) \\ &= DDu - D(u\alpha(x)) + Du\cdot\alpha(x) - u\alpha^2(x) \\ &= -\Delta u - Du\cdot\alpha(x) + Du\cdot\alpha(x) - uD\alpha - u\alpha^2(x) \\ &= -\Delta u - (D\alpha + \alpha^2(x))u\end{aligned}$$

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

or

$$D\alpha + \alpha^2(x) = V_0(x). \tag{4}$$

Equation (4) will be called generalized Miura transformation.

It is a non-linear first-order partial differential equation. Some representation formulae for the solution of the Schrödinger equation are contained in [2].

4 THE FIRST-ORDER NON-LINEAR SYSTEM

The result of the previous factorization is an equation of the following type:

$$D\alpha + \alpha^2 = V_0(x), \quad \alpha \in W_2^1(\Omega).$$

Applying T_Ω and using the Plemelj–Sokhotzkij formula we obtain

$$\alpha - F_\Gamma\alpha = T_\Omega(V_0 - \alpha^2).$$

Because we are only interested in one special solution α we may state some additional assumptions. If we assume that $\alpha \in \operatorname{Im} Q_\Gamma$ then we have to solve

$$\alpha = T_\Omega(V_0 - \alpha^2).$$

If we additionally suppose that $\operatorname{Re}\alpha = 0$ then we have $\alpha^2 = -|\alpha|^2$ and our equation reads now as

$$\alpha = T_\Omega(V_0 + |\alpha|^2).$$

We will take $\alpha_0 \in W_2^1(\Omega)$ with $\operatorname{tr}\alpha_0 \in \operatorname{Im} Q_\Gamma$, $\operatorname{Re}\alpha_0 = 0$ and then we investigate the iteration

$$\alpha_n = T_\Omega(V_0 + |\alpha_{n-1}|^2), \quad n = 1, 2, \ldots \tag{5}$$

Obviously, $V_0(x) + |\alpha_{n-1}|^2 \in \mathbb{R}$ reproduces $\operatorname{Re}\alpha_n = 0$. Because $\operatorname{tr} T_\Omega f \in \operatorname{Im} Q_\Gamma$ our procedure preserves the additional assumptions.

Let us look at the regularity.

$$V_0 \in L_2(\Omega) \implies T_\Omega V_0 \in W_2^1(\Omega),$$
$$\alpha_{n-1} \in W_2^1(\Omega) \implies \||\alpha_{n-1}|^2\|_{L_2} = \|\alpha_{n-1}\|_{L_4}^2 \leq C\|\alpha_{n-1}\|_{W_2^1}^2.$$

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

That means $\alpha_{n-1} \in W_2^1(\Omega) \Longrightarrow |\alpha_{n-1}|^2 \in L_2 \Longrightarrow T_\Omega|\alpha_{n-1}|^2 \in W_2^1$. All things together ensure that the sequence $\{\alpha_n\}_{n\in N}$ belongs to $W_2^1(\Omega)$. Here we used Sobolev's embedding theorems (see e.g. [17]) with the embedding constant $\sqrt{C}$.

Now we will try to apply Banach's fixed-point theorem. Therefore, at first we prove the boundedness of the sequence α_n.

From Eq. (5) we immediately obtain

$$\begin{aligned} \|\alpha_n\|_{W_2^1} &\le \|T_\Omega\|_{[L_2,W_2^1]}\left(\|V_0\|_{L_2} + \|\alpha_{n-1}^2\|_{L_2}\right) \\ &\le K_1\left(\|V_0\|_{L_2} + \|\alpha_{n-1}^2\|_{L_2}\right) \\ &\le K_1\left(\|V_0\|_{L_2} + C\|\alpha_{n-1}\|_{W_2^1}^2\right) \\ &\le K_1\|V_0\|_{L_2} + K_2\|\alpha_{n-1}\|_{W_2^1}^2, \end{aligned} \tag{6}$$

where $K_1 = \|T_\Omega\|_{[L_2,W_2^1]}$, $K_2 = K_1 C$, and C is the embedding constant from above.

LEMMA 3 *If*

$$\frac{1}{2K_2} - W \le \|\alpha_{n-1}\|_{W_2^1} \le \frac{1}{2K_2} + W \tag{7}$$

then $\|\alpha_n\|_{W_2^1} \le \|\alpha_{n-1}\|_{W_2^1}$. *Here W stands for* $\sqrt{1/4K_2^2 - K_1/K_2\|V_0\|_{L_2}}$.

Proof The inequality (7) ensures that

$$\|\alpha_{n-1}\|_{W_2^1}^2 - \frac{1}{K_2}\|\alpha_{n-1}\|_{W_2^1} + \frac{K_1}{K_2}\|V_0\|_{L_2} \le 0.$$

Using (6) we get

$$K_1\|V_0\|_{L_2} + K_2\|\alpha_{n-1}\|_{W_2^1}^2 \le \|\alpha_{n-1}\|_{W_2^1}.$$

Of course this condition requires that

$$\|V_0\|_{L_2} \le \frac{1}{4K_1K_2}.$$

LEMMA 4 *If* $\|\alpha_{n-1}\|_{W_2^1} \le 1/2K_2 - W$ *then we have* $\|\alpha_n\|_{W_2^1} \le 1/2K_2 - W$.

Proof This is a consequence of (6).

Therefore, we have proved that

$$\|\alpha_{n-1}\|_{W_2^1} \le \frac{1}{2K_2} + W \Longrightarrow \|\alpha_n\|_{W_2^1} \le \frac{1}{2K_2} + W.$$

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

If we start with $\alpha_0 \in W_2^1(\Omega), \operatorname{tr}\alpha_0 \in \operatorname{Im} Q_\Gamma, \operatorname{Re}\alpha_0 = 0, \|\alpha_0\|_{W_2^1} \leq 1/2K_2 + W$ then the sequence $\{\alpha_n\}_{n\in N}$ is bounded from above by $1/2K_2 + W$. This implies the existence of a subsequence $\{\alpha'_n\} \subset W_2^1(\Omega)$ with $\alpha'_n \rightharpoonup \alpha$ for $n \to \infty$ in $W_2^1(\Omega)$. Because of the continuity of $T_\Omega : L_2(\Omega) \to W_2^1(\Omega)$ we have that

$$\alpha = T_\Omega(V_0 + |\alpha|^2).$$

In this way we have proved a first existence result.

THEOREM 1 *Suppose that* $\|V_0\|_{L_2} \leq 1/4K_1K_2$. *Then the equation*

$$\alpha = T_\Omega(V_0 + |\alpha|^2) \tag{8}$$

has at least one solution with

$$\|\alpha\|_{W_2^1} \leq \frac{1}{2K_2} + \sqrt{\left(\frac{1}{4K_2^2}\right) - K_1K_2\|V_0\|_{L_2}^2}.$$

Proof The existence is clear from the above consideration. The norm estimate comes from the weak convergence of $\{\alpha'_n\}$ in a convex set.

In the following we investigate the contractivity of the mapping $T_\Omega(V_0 + |\alpha|^2)$. At first we get

$$\|\alpha_n - \alpha_{n-1}\|_{W_2^1} = \|T_\Omega(|\alpha_{n-1}|^2 - |\alpha_{n-2}|^2)\|_{W_2^1} \leq K_1\||\alpha_{n-1}|^2 - |\alpha_{n-2}|^2\|_{L_2}, \tag{9}$$

Furthermore, we have

$$\begin{aligned}
&\||\alpha_{n-1}|^2 - |\alpha_{n-2}|^2\|_{L_2} \\
&\quad = \|(|\alpha_{n-1}| - |\alpha_{n-2}|)(|\alpha_{n-1}| + |\alpha_{n-2}|)\|_{L_2} \\
&\quad \leq \||\alpha_{n-1}| - |\alpha_{n-2}|\|_{L_4}\||\alpha_{n-1}| + |\alpha_{n-2}|\|_{L_4} \\
&\quad \leq \||\alpha_{n-1}| - |\alpha_{n-2}|\|_{L_4}(\|\alpha_{n-1}\|_{L_4} + \|\alpha_{n-2}\|_{L_4}) \\
&\quad \leq \||\alpha_{n-1}| - |\alpha_{n-2}|\|_{L_4}\left(\sqrt{C}\|\alpha_{n-1}\|_{W_2^1} + \sqrt{C}\|\alpha_{n-2}\|_{W_2^1}\right) \\
&\quad \leq 2\sqrt{C}\||\alpha_{n-1}| - |\alpha_{n-2}|\|_{L_4}\left(\frac{1}{2K_2} - W\right).
\end{aligned} \tag{10}$$

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

Now, we use

$$\||\alpha_{n-1}| - |\alpha_{n-2}|\|_{L_4} \leq \|\alpha_{n-1} - \alpha_{n-2}\|_{L_4} \leq \sqrt{C}\|\alpha_{n-1} - \alpha_{n-2}\|_{W_2^1}. \quad (11)$$

Collecting the estimates (9), (10) and (11) we have

$$\|\alpha_n - \alpha_{n-1}\|_{W_2^1} \leq 2K_1 C\left(\frac{1}{2K_2} - W\right)\|\alpha_{n-1} - \alpha_{n-2}\|_{W_2^1}.$$

Then, we can bound the contractivity constant L from above.

$$\begin{aligned} L &\leq 2K_1 C\left(\frac{1}{2K_2} - W\right) = 2K_2\left(\frac{1}{2K_2} - W\right) = 1 - 2K_2 W \\ &= 1 - \sqrt{1 - 4K_1K_2\|V_0\|L_2}. \end{aligned}$$

We have proved the following theorem.

THEOREM 2 *We assume that*

$$\|V_0\|_{L_2} < \frac{1}{4K_1K_2}, \quad \alpha_0 \in W_2^1(\Omega), \quad \operatorname{tr}\alpha_0 \in \operatorname{Im} Q_\Gamma,$$
$$\|\alpha_0\|_{W_2^1} \leq \frac{1}{2K_2} - \sqrt{\frac{1}{4K_2^2} - \frac{K_1}{K_2}\|V_0\|_{L_2}},$$

and define the sequence

$$\{\alpha_n\}_{n\in N} \text{ by } \alpha_n = T_\Omega(V_0 + |\alpha_{n-1}|^2), \quad n = 1, 2, \ldots.$$

Then, there exists a unique solution $\alpha \in W_2^1(\Omega)$ *of Eq.* (8) *with* $\operatorname{tr}\alpha \in \operatorname{Im} Q_\Gamma \cap W_2^{1/2}(\Gamma)$, $\operatorname{Re}\alpha = 0$, *and* $\{\alpha_n\}_{n\in N}$ *converges to* α *in* W_2^1. *The solution* α *fulfils the norm estimate*

$$\|\alpha\|_{W_2^1} \leq \frac{1}{2K_2} - \sqrt{\frac{1}{4K_2^2} - \frac{K_1}{K_2}\|V_0\|_{L_2}}.$$

Let us remark that there is no practical problem to find a suitable $\alpha_0 \in W_2^1(\Omega)$ with $\operatorname{tr}\alpha_0 \in \operatorname{Im} Q_\Gamma \cap W_2^1(\Gamma)$. We can start with $\alpha_0 \equiv 0$ or an arbitrarily chosen function $\beta \in L_2$ with $\operatorname{Im}\beta \equiv 0$, and $\|\beta\|_{L_2}$ small enough. Then, $\alpha_0 = T\beta$ fulfils all the necessary conditions.

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

5 SOME GENERALIZATIONS

The above obtained results allow some generalizations. The first question is the possibility to prove similar results for all space dimensions. A second problem is to study other non-linear terms, e.g., general powers u^r.

To solve the first problem we have to work with general Clifford algebras instead of the special case of the quaternionic algebra. Because we have used the embedding theorems for Sobolev spaces the obtained results depend on the dimension of the space. We have only investigated for practical applications the most important case of space dimension 3. The same idea works for $n = 4$, too. For $n > 4$ the space $W_2^1(\Omega)$ can be embedded only in $L_p(\Omega)$ with $p < 2n/(n-2) < 4$ and the proof fails. Looking at the details again we see that the condition $p = 2$ is not necessary for our consideration. We have

$$V_0 \in L_p(\Omega) \implies T_\Omega V_0 \in W_p^1(\Omega) \quad \text{(see [8])},$$
$$\alpha_{n-1} \in W_p^1(\Omega) \implies \| |\alpha_{n-1}|^2 \|_{L_p} = \|\alpha_{n-1}\|_{L_{2p}}^2.$$

That means we have to look for an embedding $W_p^1(\Omega) \hookrightarrow L_{p^*}$ with $p^* \geq 2p$. Hence, we have the condition $np/(n-p) \geq 2p$ for p and, consequently, for $p < n$

$$np \geq 2np - 2p^2 \iff 2p^2 \geq np \iff 2p \geq n \iff p \geq \frac{n}{2}.$$

Therefore, for all p with $n/2 \leq p < n$ the proofs of Theorems 1 and 2 can be repeated.

In case of the equation with a more general non-linear item the whole consideration from Lemma 3 until Theorem 2 can be repeated. Some technical problems arise. We will give here only a sketch of the consideration and some hints concerning the new problems. If we study the equation

$$D\alpha + \alpha^r = V_0(x)$$

then we need the restriction that $r = 2l$ to ensure that our iteration procedure reproduces the properties of the initial function. For $[2l - 1/2l]\, n \leq p < n$ we have that from $\alpha_{n-1} \in W_p^1(\Omega)$ it follows that $|\alpha_{n-1}|^{2l} \in L_p(\Omega)$. Using the mapping properties of T_Ω and the additional assumption that $V_0 \in L_p(\Omega)$ we obtain that α_n again belongs to $W_p^1(\Omega)$.

Then, we consider the polynomial $x^{2l} - ax + b$ where $a = 1/K_2$ and $b = K_1 \| V_0 \|_p / K_2$. Here K_1 stands for $\| T_\Omega \|_{[L_p, W_p^1]}$, $K_2 = K_1 C_p$, and $C_p^{1/2l}$ is the embedding constant from the embedding $W_p^1 \hookrightarrow L_{2lp}$. This polynomial has at most two zeros $0 \le x_1 \le x_2$. We can prove that our sequence $\{\alpha_n\}$ completely belongs to a fixed ball with radius x_i in $W_p^1(\Omega)$ if $\|\alpha_0\|_{W_p^1} < x_i$, $i = 1, 2$. The existence of real zeros x_1 and x_2 is ensured if

$$\| V_0 \|_p \le \frac{1}{K_1} \left(1 - \frac{1}{2l} \right) \left(\frac{1}{2lK_2} \right)^{1/(2l-1)}.$$

To prove the contractivity we estimate

$$\begin{aligned} &\|\alpha_n - \alpha_{n-1}\|_{W_p^1} \\ &\le K_1 \| |\alpha_{n-1}| - |\alpha_{n-2}| \|_{L_{2lp}} (\|\alpha_{n-1}\|_{W_p^1} + \|\alpha_{n-2}\|_{W_p^1})^{2l-1} C^{(2l-1)/2l} \\ &\le 2^{2l-1} K_1 C \|\alpha_{n-1} - \alpha_{n-2}\|_{W_p^1} |x_1|^{2l-1} \end{aligned}$$

That means the contraction constant L may be estimated by

$$L \le 2^{2l-1} K_2 |x_1|^{2l-1}.$$

For $\| V_0 \|_p$ sufficiently small x_1 is small enough that $L < 1$ becomes true.

This very short outline shows (independently of physical interpretations) that the method demonstrated above works also in the case of more general nonlinearities in our first-order differential equation.

References

[1] Ablowitz, M.J. and Clarkson, P.A. (1991) Solitons, nonlinear evolution equations and inverse scattering. *London Math. Soc. Lect. Note Series*, No. 149, Cambridge University Press, Cambridge.
[2] Bernstein, S. (1996) Factorization of solutions of the Schrödinger equation. In W. Sprößig and K. Gürlebeck (eds.), Proceedings of the Symposium "*Analytical and Numerical Methods in Quaternionic and Clifford Analysis*", Seiffen, Germany, University of Mining and Technology Freiberg, pp. 207–216.
[3] Bernstein, S. (1996) Fundamental solutions for Dirac-type operators, In J. Lawrynowicz (ed.), *Generalizations of Complex Analysis and their Applications in Physics*, Banach Center Publ., vol. 37, 159–172.
[4] Brackx, F. and Van Acker, N. (1993) Boundary value theory for eigenfunctions of the Dirac operator, *Bull. Soc. Math. Belg., Ser. B*, **45**(2), 113–123.

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

[5] Brackx, F., Delanghe, R., Sommen, F. and Van Acker, N. (1993) Reproducing kernels on the unit sphere, In R.S. Pathak (ed.), *Generalized Functions and their Applications*. Proceedings of the international symposium, December 23–26, 1991, Varanasi, India. New York: Plenum Press, pp. 1–10.
[6] Gürlebeck, K. (1986) Hypercomplex factorization of the Helmholtz equation, *Zeitschrift für Analysis und ihre Anwendungen*, **5**(2), 125–131.
[7] Gürlebeck, K. (1988) Grundlagen einer diskreten räumlich verallgemeinerten Funktionen-theorie und ihre Anwendungen, Diss. (B), TU Karl-Marx-Stadt (Chemnitz).
[8] Gürlebeck, K. and Sprößig, W. (1990) *Quaternionic Analysis and Elliptic Boundary Value Problems*. Birkhäuser Verlag, Basel.
[9] Huang, L. (1990) The existence and uniqueness theorems of the linear and nonlinear R.-H. Problems for the generalized holomorphic vector of the second kind, *Acta Math. Sci. Engl. Ed.*, **10**(2), 185–199.
[10] Kravchenko, V.V. and Shapiro, M.V. (1993) Helmholtz operator with a quaternionic wave number and associated function theory II. Integral representations, *Acta Applicandae Mathematicae*, **32**(3), 243–265.
[11] Kravchenko, V.V. and Shapiro, M.V. (1994) Helmholtz operator with a quaternionic wave number and associated function theory, In J. Lawrynowicz (ed.), *Deformations of Mathematical Structures, II*, Kluwer Academic Publishers, pp. 101–128.
[12] Kravchenko, V.V. and Shapiro, M.V. (1996) Integral representations for spatial models of mathematical physics. Pitman Research Notes in Mathematics Series 351.
[13] Mitrea, M. (1996) Boundary value problems and Hardy spaces associated to the Helmholtz equation in Lipschitz domains, *J. Math. Anal. Appl.*, **202**(3), 819–842.
[14] Miura, R.M. (1968) Korteweg–de Vries equations and generalizations I. A remarkable explicit nonlinear transformation. *J. Math. Phys.*, **9**, 1202–1204.
[15] Obolashvili, E. (1975) Space generalized holomorphic vectors, *Diff. Urav.* T. XI. 1, 108–115 (Russian).
[16] Obolashvili, E. (1988) Effective solutions of some boundary value problems in two and three dimensional cases, Functional analytical methods in complex analysis and applications to PDE, Trieste, pp. 149–172.
[17] Wloka, J. (1982) Partielle Differentialgleichungen. Teubner, Stuttgart.
[18] Xu, Z. (1991) A function theory for the operator $D - \lambda$. *Complex Variables Theory and Appl.*, **16**, 27–42.
[19] Xu, Z. (1992) Helmholtz equations and boundary value problems, In H. Begehr and A. Jeffrey (eds.), *Partial Differential Equations with Complex Analysis. Pitman Research Notes in Mathematics Series*, 262, pp. 204–214.

Downloaded by [TU Bergakademie Freiberg] at 04:40 28 February 2013

View publication stats