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On a Hilbert-Type Integral Inequality with a Combination Kernel and Applications

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Abstract. By introducing some parameters and using the way of weight function and the technic of real analysis and complex analysis, a new Hilbert-type integral inequality with a best constant factor and a combination kernel involving two mean values is given, which is an extension of Hilbert's integral inequality. As applications, the equivalent form and the reverse forms are considered.

1. Introduction

If $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have[1]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}; \tag{1}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4 \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}, \tag{2}$$

where the constant factor π and 4 are all the best possible. We call (1) Hilbert's integral inequality. Both (1) and (2) are important in analysis and its applications[1, 2]. In recent years, by using the way of weight function, a number of extensions of (1) and (2) were given by Yang et al. [3, 4]. In 2006, Li et al. [5] gave a new inequality with a combination kernel as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}dxdy < c \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}, \tag{3}$$

where the constant factor $c = 2\sqrt{2}\arctan\frac{1}{\sqrt{2}}$ is the best possible. In 2007, Xie [6] gave a best extension of (3) and Guo et al. [7, 8] gave a similar form of (3) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{x+y+\min\{x,y\}} < \widetilde{c} \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}, \tag{4}$$

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where the constant factor $\widetilde{c} = 2\sqrt{2}\arctan\sqrt{2}$ is the best possible.

We know the following mean values inequalities:

$$\max\{x,y\} \geq \frac{x+y}{2} \geq \sqrt{xy} \geq \frac{2}{x^{-1}+y^{-1}} \geq \min\{x,y\}(x,y>0). \tag{5}$$

It means that $x+y-4(x^{-1}+y^{-1})^{-1} \geq 0$.

In this paper, by introducing some parameters and using the way of weight function and the technic of real analysis and complex analysis, we give a new Hilbert-type integral inequality with a combination kernel as $\frac{1}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}$ $(\lambda > 0, A > -4)$, which is an extension of (1). As applications, the equivalent form and the reverse forms are obtained.

2. Some Lemmas

Lemma 1. *If* $\lambda > 0, A > -4$, *then we have*

$$\widetilde{K}_\lambda(A) := \int_0^\infty \frac{u^{\frac{\lambda}{2}-1}du}{u^\lambda+1+A(u^{-\lambda}+1)^{-1}} = \frac{2\pi}{\lambda\sqrt{A+4}}; \tag{6}$$

$$\varpi_\lambda(y) := \int_0^\infty \frac{y^{\frac{\lambda}{2}}x^{\frac{\lambda}{2}-1}dx}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}} = \widetilde{K}_\lambda(A), \quad (y \in (0,\infty)). \tag{7}$$

Proof. Setting $v = u^{\lambda/2}$, by calculation, we find

$$\widetilde{K}_\lambda(A) = \frac{1}{\lambda}\int_{-\infty}^\infty \frac{v^2+1}{(v+1)^2+Av^2}dv.$$

Since for $A > -4$, it follows

$$\begin{aligned}(v+1)^2+Av^2 &= (v^2+1-\sqrt{-A}v)(v^2+1+\sqrt{-A}v) \\ &= (v-v_1)(v-v_2)(v-v_3)(v-v_4),\end{aligned}$$

where $v_1 = \frac{1}{2}(\sqrt{-A}+\sqrt{A+4}i)$, $v_2 = \frac{1}{2}(\sqrt{-A}-\sqrt{A+4}i), v_3 = \frac{1}{2}(-\sqrt{-A}+\sqrt{A+4}i)$, $v_4 = \frac{1}{2}(-\sqrt{-A}-\sqrt{A+4}i)$. Obviously, we can find that $Im\, v_1 > 0$ and $Im\, v_3 > 0$, for $-4 < A \leq 0$ and for $A > 0$. Setting a complex function as $f(z) = \frac{z^2+1}{(z-v_1)(z-v_2)(z-v_3)(z-v_4)}$, in view of the theorem of obtaining real integral by using residue[9], (a) if $A \neq 0$, then $v_1 \neq v_3$ and

$$\begin{aligned}\underset{z=v_1}{Res}\ f(z) &= \frac{v_1^2+1}{(v_1-v_2)(v_1-v_3)(v_1-v_4)} = \frac{1}{2\sqrt{A+4}i}; \\ \underset{z=v_3}{Res}\ f(z) &= \frac{v_3^2+1}{(v_3-v_1)(v_3-v_2)(v_3-v_4)} = \frac{1}{2\sqrt{A+4}i};\end{aligned}$$

$$k_\lambda(A) = \frac{1}{\lambda}\int_{-\infty}^{\infty} f(v)dv = \frac{2\pi i}{\lambda}\Big(\underset{z=v_1}{Res}\ f(z) + \underset{z=v_3}{Res}\ f(z)\Big) = \frac{2\pi}{\lambda\sqrt{A+4}};$$

(b) if $A = 0$, then $v_1 = v_3 = i$, $v_2 = v_4 = -i$ and

$$\begin{aligned} \underset{z=i}{Res}\ f(z) &= \left[\frac{z^2+1}{(z+i)^2}\right]'_{z=i} = \frac{1}{2i}; \\ k_\lambda(A) &= \frac{1}{\lambda}\int_{-\infty}^{\infty} f(v)dv = \frac{2\pi i}{\lambda}\ \underset{z=i}{Res}\ f(z) = \frac{\pi}{\lambda}. \end{aligned}$$

Hence (6) is valid. Setting $u = x/y$, we obtain (7). The lemma is proved. □

Lemma 2. *Assume that* $p > 0 (p \neq 1), |q| > 0, \lambda > 0, A > -4$ *and* $0 < \varepsilon < \frac{\lambda}{2}\min\{p, |q|\}$. *Then for* $\varepsilon \to 0^+$, *we have*

$$\int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} = \int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} + o_1(1); \tag{8}$$

$$\int_1^\infty \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} = \int_1^\infty \frac{u^{\frac{\lambda}{2}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} + o_2(1); \tag{9}$$

$$\int_0^1 \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} = \int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} + o_3(1). \tag{10}$$

Proof. (a) If $-4 < A < 0$, setting $\eta_A = \frac{1}{4}(A+4) > 0$, then we obtain

$$\begin{aligned} u^\lambda + 1 + A(u^{-\lambda}+1)^{-1} &= \eta_A(u^\lambda+1) + (1-\eta_A)(u^\lambda+1) + \frac{Au^\lambda}{1+u^\lambda} \\ &\geq \eta_A(u^\lambda+1) + \frac{4(1-\eta_A)u^\lambda}{1+u^\lambda} + \frac{Au^\lambda}{1+u^\lambda} \\ &= \eta_A(u^\lambda+1) \geq \begin{cases} \eta_A, & u\in(0,1] \\ \eta_A u^\lambda, & u\in(1,\infty); \end{cases} \end{aligned} \tag{11}$$

(b) if $A \geq 0$, setting $\eta_A = 1$, then we still have (11). For $\varepsilon \to 0^+$, we find

$$\begin{aligned}
0 &< \int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} - \int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} \\
&\leq \frac{1}{\eta_A}\int_0^1 u^{\frac{\lambda}{2}-1}(1-u^{\frac{\varepsilon}{q}})du \leq \frac{1}{\eta_A}\left(\frac{2}{\lambda} - \frac{1}{\frac{\lambda}{2}+\frac{\varepsilon}{q}}\right) \to 0; \\
0 &< \int_1^\infty \frac{u^{\frac{\lambda}{2}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} - \int_1^\infty \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} \\
&\leq \frac{1}{\eta_A}\int_1^\infty \frac{u^{\frac{\lambda}{2}-1}(1-u^{-\frac{\varepsilon}{p}})du}{u^\lambda} = \frac{1}{\eta_A}\left(\frac{2}{\lambda} - \frac{1}{\frac{\lambda}{2}+\frac{\varepsilon}{p}}\right) \to 0; \\
0 &< \int_0^1 \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} - \int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} \\
&\leq \frac{1}{\eta_A}\int_0^1 u^{\frac{\lambda}{2}-1}(u^{-\frac{\varepsilon}{p}}-1)du \leq \frac{1}{\eta_A}\left(\frac{1}{\frac{\lambda}{2}-\frac{\varepsilon}{p}} - \frac{2}{\lambda}\right) \to 0.
\end{aligned}$$

Hence Expressions (8), (9) and (10) are valid. The lemma is proved. □

3. Main results and applications

Theorem 1. *If* $\lambda, p > 0\,(p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, A > -4, \widetilde{K}_\lambda(A)$ *is expressed by* (6), $\phi_r(x) = x^{r(1-\frac{\lambda}{2})-1}(r = p, q), f, g \geq 0,\ 0 < ||f||_{p,\phi_p} := \{\int_0^\infty \phi_p(x) f^p(x)dx\}^{\frac{1}{p}} < \infty$ *and* $0 < ||g||_{q,\phi_q} = \{\int_0^\infty \phi_q(x) g^q(x)dx\}^{\frac{1}{q}} < \infty$, *then*
(a) *for* $p > 1$, *we have the following equivalent inequalities:*

$$I_\lambda := \int_0^\infty y^{\frac{p\lambda}{2}-1}\left[\int_0^\infty \frac{f(x)dx}{x^\lambda + y^\lambda + A(x^{-\lambda}+y^{-\lambda})^{-1}}\right]^p dy < \widetilde{K}_\lambda^p(A)||f||_{p,\phi_p}^p; \tag{12}$$

$$J_\lambda := \int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{x^\lambda + y^\lambda + A(x^{-\lambda}+y^{-\lambda})^{-1}} < \widetilde{K}_\lambda(A)||f||_{p,\phi_p}||g||_{q,\phi_q}; \tag{13}$$

(b) *for* $0 < p < 1$, *we have the equivalent reverse forms of* (12) *and* (13).

Proof. By Hölder's inequality[10] and (7), for $y \in (0, \infty)$, we obtain

$$\begin{aligned}
&\left[\int_0^\infty \frac{f(x)dx}{x^\lambda + y^\lambda + A(x^{-\lambda}+y^{-\lambda})^{-1}}\right]^p \\
&= \left\{\int_0^\infty \frac{1}{x^\lambda + y^\lambda + A(x^{-\lambda}+y^{-\lambda})^{-1}}\left[\frac{x^{(1-\frac{\lambda}{2})/q}}{y^{(1-\frac{\lambda}{2})/p}}f(x)\right]\left[\frac{y^{(1-\frac{\lambda}{2})/p}}{x^{(1-\frac{\lambda}{2})/q}}dx\right\}^p\right.
\end{aligned} \tag{14}$$

$$
\begin{aligned}
&\leq \left[\int_0^\infty \frac{x^{(1-\frac{\lambda}{2})(p-1)}y^{\frac{\lambda}{2}-1}f^p(x)dx}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}\right]\left[\int_0^\infty \frac{y^{(1-\frac{\lambda}{2})(q-1)}x^{\frac{\lambda}{2}-1}dx}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}\right]^{p-1} \\
&= \widetilde{K}_\lambda^{p-1}(A)y^{1-\frac{p\lambda}{2}}\int_0^\infty \frac{x^{(1-\frac{\lambda}{2})(p-1)}y^{\frac{\lambda}{2}-1}}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}f^p(x)dx.
\end{aligned}
$$

By (14), in view of Fubini's theorem[11] and (7), it follows

$$
\begin{aligned}
(15)\qquad I_\lambda &\leq \widetilde{K}_\lambda^{p-1}(A)\int_0^\infty\int_0^\infty \frac{x^{(1-\frac{\lambda}{2})(p-1)}y^{\frac{\lambda}{2}-1}f^p(x)}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}dxdy \\
&= \widetilde{K}_\lambda^{p-1}(A)\int_0^\infty \varpi(x)\phi_p(x)f^p(x)dx = \widetilde{K}_\lambda^{p}(A)||f||_{p,\phi_p}^p.
\end{aligned}
$$

If there exists a $y \in (0,\infty)$, such that (14) takes the form of equality, then[10] there exists constants a and b, such that they are not all zero and $x^{1-\frac{\lambda}{2}(p-1)}y^{\frac{\lambda}{2}-1}f^p(x) = y^{(1-\frac{\lambda}{2})(q-1)}x^{\frac{\lambda}{2}-1}$ *a.e.* in $(0,\infty)$. It means that $ax^{p(1-\frac{\lambda}{2})}f^p(x) = by^{q(1-\frac{\lambda}{2})}$ *a.e.* in $(0,\infty)$. We affirm that $a \neq 0$, otherwise $b = a = 0$. Hence it follows $x^{p(1-\frac{\lambda}{2})-1}f^p(x) = [by^{q(1-\frac{\lambda}{2})}]/(ax)$ *a.e.* in $(0,\infty)$, which contradicts the fact that $0 < ||f||_{p,\phi_p} < \infty$. Then (14) keeps the form of strict inequality; so does (15). And (12) is valid.

By Hölder's inequality[10], we find

$$
(16)\quad J_\lambda = \int_0^\infty \left[\int_0^\infty \frac{y^{\frac{-1}{p}+\frac{\lambda}{2}}f(x)dx}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}\right]\left[y^{\frac{1}{p}-\frac{\lambda}{2}}g(y)\right]dy \leq I_\lambda^{\frac{1}{p}}||g||_{q,\phi_q}.
$$

In view of (12), we have (13).

On the other-hand, suppose that (13) is valid. There exists $n_0 \in N$, such that for $n \geq n_0, 0 < \int_{1/n}^n \phi_p(x)[f(x)]_n^p dx < \infty$, where $[f(x)]_n = n$, for $f(x) \geq n$; $[f(x)]_n = f(x)$, for $f(x) < n$. For $n \geq n_0$, setting

$$
(17)\qquad g_n(y) := y^{\frac{p\lambda}{2}-1}\left[\int_{1/n}^n \frac{[f(x)]_n dx}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}\right]^{p-1}, y \in (0,n],
$$

by (13), we find

$$
\begin{aligned}
(18)\quad 0 < &\int_{\frac{1}{n}}^n \phi_q(y)g_n^q(y)dy = \int_{\frac{1}{n}}^n y^{\frac{p\lambda}{2}-1}\left[\int_{\frac{1}{n}}^n \frac{[f(x)]_n dx}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}\right]^p dy \\
= &\int_{1/n}^n\int_{1/n}^n \frac{[f(x)]_n g_n(y)}{x^\lambda+y^\lambda+A(x^{-\lambda}+y^{-\lambda})^{-1}}dxdy \\
< &\widetilde{K}_\lambda(A)\left\{\int_{1/n}^n \phi_p(x)[f(x)]_n^p dx\right\}^{\frac{1}{p}}\left\{\int_{1/n}^n \phi_q(y)g_n^q(y)dy\right\}^{\frac{1}{q}} < \infty;
\end{aligned}
$$

$$(19)\qquad \left\{\int_{1/n}^{n}\phi_q(y)g_n^q(y)dy\right\}^{\frac{1}{p}} < \widetilde{K}_\lambda(A)\left\{\int_0^\infty \phi_p(x)f^p(x)dx\right\}^{\frac{1}{p}} < \infty.$$

Hence $0 < \int_0^\infty \phi_q(y)g_\infty^q(y)dy < \infty$, and then (18) and (19) are valid for $n \to \infty$ by using (13). Therefore we obtain (12), which is equivalent to (13).

(b) By the reverse Hölder's inequality and the same way, we can obtain the reverse forms of (12) and (16) and then deduce the reverse form of (13). Setting $g_n(y)$ as (17), by the reverse form of (13), we obtain the reverse forms of (18) and (19), and then deduce the reverse form of (13), which is equivalent to the reverse form of (12). The theorem is proved. □

Theorem 2. *As the assumption of Theorem 1, all the constant factors in* (12), (13) *and the reverse forms are the best possible.*

Proof. For $0 < \varepsilon < \frac{\lambda}{2}\min\{p, |q|\}$, we set $f_\varepsilon, g_\varepsilon$ as: $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0,1)$; $f_\varepsilon(x) = x^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}, g_\varepsilon(x) = x^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}$, for $x \in [1,\infty)$.

(a) For $p > 1$, if there exists a constant $0 < k \le \widetilde{K}_\lambda(A)$, such that (13) is still valid as we replace $\widetilde{K}_\lambda(A)$ by k, then in particular, we have

$$\begin{aligned} k &= \varepsilon k||f_\varepsilon||_{p,\phi_p}||g_\varepsilon||_{q,\phi_q} > \varepsilon\int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)dxdy}{x^\lambda + y^\lambda + A(x^{-\lambda}+y^{-\lambda})^{-1}} \\ &= \varepsilon\int_1^\infty x^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}\left[\int_1^\infty \frac{y^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}dy}{x^\lambda + y^\lambda + A(x^{-\lambda}+y^{-\lambda})^{-1}}\right]dx. \end{aligned}$$

Setting $u = x/y$ in the above integral, by Fubini's theorem, we find

$$\begin{aligned} k &> \varepsilon\int_1^\infty x^{-\varepsilon-1}\left[\int_0^x \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}}du\right]dx \\ &= \int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} + \int_1^\infty x^{-\varepsilon-1}\left[\int_1^x \frac{\varepsilon u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}}\right]dx \\ &= \int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} + \varepsilon\int_1^\infty \frac{(\int_u^\infty x^{-\varepsilon-1}dx)u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}}du \\ &= \int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}} + \int_1^\infty \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}}{u^\lambda + 1 + A(u^{-\lambda}+1)^{-1}}du. \end{aligned}$$

For $\varepsilon \to 0^+$, in view of (8) and (9), we find $k \ge \widetilde{K}_\lambda(A)$. Hence $k = \widetilde{K}_\lambda(A)$ is the best constant factor of (13). If the constant factor in (12) is not the best possible, then by (16), we may get a contradiction that the constant factor in (13) is not the best possible.

(b) For $0 < p < 1$, if there exists $K \geq \widetilde{K}_\lambda(A)$, such that the reverse form of (13) is valid as we replace $\widetilde{K}_\lambda(A)$ by K, then we have

$$\begin{aligned} K &= \varepsilon K ||f_\varepsilon||_{p,\phi_p} ||g_\varepsilon||_{q,\phi_q} < \int_0^\infty \int_0^\infty \frac{\varepsilon f_\varepsilon(x) g_\varepsilon(y) dx dy}{\max\{x^\lambda, y^\lambda\} + A(x^{-\lambda} + y^{-\lambda})^{-1}} \\ &\leq \varepsilon \int_1^\infty y^{\frac{\lambda}{2} - \frac{\varepsilon}{q} - 1} \left[\int_0^\infty \frac{x^{\frac{\lambda}{2} - \frac{\varepsilon}{p} - 1} dx}{x^\lambda + y^\lambda + A(x^{-\lambda} + y^{-\lambda})^{-1}} \right] dy \\ &= \int_0^1 \frac{u^{\frac{\lambda}{2} - \frac{\varepsilon}{p} - 1} du}{u^\lambda + 1 + A(u^{-\lambda} + 1)^{-1}} + \int_1^\infty \frac{u^{\frac{\lambda}{2} - \frac{\varepsilon}{p} - 1} du}{u^\lambda + 1 + A(u^{-\lambda} + 1)^{-1}} \\ &\leq \int_0^1 \frac{u^{\frac{\lambda}{2} - \frac{\varepsilon}{p} - 1} du}{u^\lambda + 1 + A(u^{-\lambda} + 1)^{-1}} + \int_1^\infty \frac{u^{\frac{\lambda}{2} - 1} du}{u^\lambda + 1 + A(u^{-\lambda} + 1)^{-1}}. \end{aligned}$$

For $\varepsilon \to 0^+$, in view of (10), we obtain $K \leq \widetilde{K}_\lambda(A)$. Hence $K = \widetilde{K}_\lambda(A)$ is the best constant factor of the reverse form of (13). If the constant factor in the reverse form of (12) is not the best possible, then by the reverse form of (16), we may get a contradiction that the constant factor in the reverse form of (13) is not the best possible. The theorem is proved. □

Remark. (a) For $p = q = 2, \lambda = 1, A = 0$ in (13), it deduces to (1). Hence inequality (13) is an extension of (1);
(b) for $p > 1, A = -3, \lambda > 0$ in (13), we obtain a new inequality with the best constant factor $\frac{2\pi}{\lambda}$ as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{\max\{x^\lambda, y^\lambda\} - 3(x^{-\lambda} + y^{-\lambda})^{-1}} < \frac{2\pi}{\lambda} ||f||_{p,\phi_p} ||g||_{q,\phi_q}, \tag{20}$$

and it follows that the revere form of (20) with the same best constant factor for $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$ and $0 < ||f||_{p,\phi_p}, ||g||_{q,\phi_q} < \infty$ is valid.

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