On a Horizontal Conformal Killing Tensor of Degree p in a Sasakian Space.

SEIICHI YAMAGUCHI (Tokyo) (*)

Summary. – *We deal with a horizontal conformal Killing tensor of degree p in a Sasakian space. After some preparations we prove that a horizontal conformal Killing tensor of odd degree is necessarily Killing. Moreover, we consider horizontal conformal Killing tensor of even degree. The form of the associated tensor is determined completely and a decomposition theorem is proved. Then we give the examples of a conformal Killing tensor of even degree and a special Killing tensor of odd degree with constant l.*

Let M be an n-dimensional Riemannian space whose metric tensor is given by g_{ab} $(a, b, \ldots, r, s, \ldots = 1, 2, \ldots, n)$. We call a skew-symmetric tensor u_{ab} a conofrmal KILLING tensor of degree 2 [4] if it satisfies the equation

$$\nabla_a u_{bc} + \nabla_b u_{ac} = 2\theta_c g_{ab} - \theta_a g_{bc} - \theta_b g_{ac} ,$$

where ∇ denotes the operator of covariant derivative with respect to g_{ab}. Then we have $\theta_c = \nabla^r u_{rc}/(n-1)$ for the tensor u_{ab}. We call θ_c the associated vector (1) of u_{ab}.

Recently, the author [6] has studied a conformal KILLING tensor of degree 2 in a SASAKIAN space and obtained the followings:

THEOREM A. – *In a Sasakian space* $(n > 3)$, *any conformal Killing tensor* u_{ab} *is uniquely decomposed into the form:*

$$u_{ab} = w_{ab} + q_{ab} ,$$

where w_{ab} *is Killing and* q_{ab} *is a closed conformal Killing tensor. In this case* q_{ab} *is the form*

$$q_{ab} = -\nabla_a \theta_b ,$$

where θ_c *is the associated vector of* u_{ab}.

THEOREM B. – *Let* M *be a complete simply connected Sasakian space* $(n > 3)$ *admitting a conformal Killing tensor* u_{ab} *whose associated vector is* θ_a. *If the inner*

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(1) We adapt the identification between vector fields and 1-forms by virtue of Riemannian metric.

product $\langle\theta,\theta\rangle$ or $\langle\eta,\theta\rangle$ is not constant, where the vector η^a is a Sasakian structure, then M is isometric with a unit sphere.

In a Riemannian space, T. KASHIWADA [1] has defined a conformal KILLING tensor of degree $p\geqq 2$ and generalized some results about a conformal KILLING tensor of degree 2 to the case of degree $p\geqq 2$.

The purpose of this paper deals with a horizontal conformal KILLING tensor of degree p in a SASAKIAN space. After some preparations we prove in § 4 that a horizontal conformal KILLING tensor of odd degree is necessarily KILLING. In § 5 we shall consider horizontal conformal KILLING tensors of even degree. The form of the associated tensor is determined completely and a decomposition theorem is proved (cf. Theorem 5.1 and 5.3).

1. – Tensors.

In a Riemannian space M we call a p-form w with coefficient $w_{a_1\dots a_p}$ a KILLING tensor of degree p if it satisfies

$$\nabla_b w_{a_1\dots a_p}+\nabla_{a_1}w_{ba_2\dots a_p}=0\,.$$

If a KILLING tensor w satisfies

$$\nabla_c\nabla_d w_{a_1\dots a_p}+\alpha\Big(g_{cd}w_{a_1\dots a_p}+\sum_{i=1}^{p}(-1)^i g_{ca_i}w_{da_1\dots\hat{a}_i\dots a_p}\Big)=0\,,$$

where α is constant and $\hat{a}_i$ means that a_i is omitted, then it is called a special KILLING tensor of degree p with constant α [5].

Next we shall remember a conformal KILLING tensor of degree p. In M we call a p-form u with coefficient $u_{a_1\dots a_p}$ a conformal KILLING tensor of degree p, if there exists a $(p-1)$-form θ with coefficient $\theta_{a_2\dots a_p}$ such that

$$\nabla_b u_{a_1\dots a_p}+\nabla_{a_1}u_{ba_2\dots a_p}=2\theta_{a_2\dots a_p}g_{ba_1}-\sum_{i=2}^{p}(-1)^i(\theta_{a_1\dots\hat{a}_i\dots a_p}g_{ba_i}+\theta_{ba_2\dots\hat{a}_i\dots a_p}g_{a_1a_i})\,. \tag{1.1}$$

This form θ is called the associated tensor of u. For a conformal KILLING tensor u of degree p, the following identities are known:

$$\nabla^r u_{ra_2\dots a_p}=(n-p+1)\theta_{a_2\dots a_p}\,, \tag{1.2}$$

$$\begin{aligned}(p-1)\,l_{ba_2\dots a_p}+\sum_{i=2}^{p}l_{a_ia_2\dots b\dots a_p}\\ =-\frac{1}{n-p}\Big[(p-1)R_b{}^e u_{ea_2\dots a_p}+(p-2)\sum_{i=2}^{p}R_b{}^d{}_{a_i}{}^e u_{da_2\dots e\dots a_p}\\ +\sum_{i=2}^{p}R_{a_i}{}^e u_{ea_2\dots b\dots a_p}-\sum_{2\leq j<k}^{p}R_{a_ja_k}{}^{de}u_{da_2\dots e\dots b\dots a_p}\Big]\,,\end{aligned} \tag{1.3}$$

$$
\begin{aligned}
(1.4) \qquad & -(p-1)\sum_{i=1}^{p} R_{bca_i}{}^{e} u_{a_1\ldots e\ldots a_p} - \sum_{j<k}^{p} R_{a_j a_k b}{}^{e} u_{a_1\ldots e\ldots c\ldots a_p} + \sum_{j<k}^{p} R_{a_j a_k c}{}^{e} u_{a_j\ldots e\ldots b\ldots a_p} \\
& -\frac{1}{n-p}\sum_{i=1}^{p}(-1)^i g_{a_i c}\Big[(p-1)R_b{}^{e} u_{e a_1\ldots \hat{a}_i\ldots a_p} + (p-2)\sum_{j\neq i} R_b{}^{d}{}_{a_j}{}^{e} u_{d a_1\ldots e\ldots \hat{a}_i\ldots a_p} \\
& \qquad + \sum_{j\neq i} R_{a_j}{}^{e} u_{e a_1\ldots b\ldots \hat{a}_i\ldots a_p} - \sum_{\substack{j<k\\ j,k\neq i}} R_{a_j a_k}{}^{de} u_{d a_1\ldots e\ldots b\ldots \hat{a}_i\ldots a_p}\Big] \\
& +\frac{1}{n-p}\sum_{i=1}^{p}(-1)^i g_{a_i b}\Big[(p-1)R_c{}^{e} u_{e a_1\ldots \hat{a}_i\ldots a_p} + (p-2)\sum_{j\neq i} R_c{}^{d}{}_{a_j}{}^{e} u_{d a_1\ldots e\ldots \hat{a}_i\ldots a_p} \\
& \qquad + \sum_{j\neq i} R_{a_j}{}^{e} u_{e a_1\ldots c\ldots \hat{a}_i\ldots a_p} - \sum_{\substack{j<k\\ j,k\neq i}} R_{a_j a_k}{}^{de} u_{d a_1\ldots e\ldots c\ldots \hat{a}_i\ldots a_p}\Big],
\end{aligned}
$$

where $l_{a_1\ldots a_p} = \nabla_{a_1}\theta_{a_2\ldots a_p}$ and the indices e and c in $u_{a_1\ldots e\ldots c\ldots a_p}$ appear at the j-th and k th position respectively.

2. – Sasakian space and operators.

An n-dimensional Riemannian space M is called a SASAKIAN space if it admits a unit KILLING vector field η^a such that

$$\nabla_a\nabla_b\eta_c = \eta_b g_{ac} - \eta_c g_{ab}\,.$$

Then we have

$$R_{abc}{}^{r}\eta_r = \eta_a g_{bc} - \eta_b g_{ac}\,. \tag{2.1}$$

In a SASAKIAN space M, n is necessarily odd $(=2m+1)$ and M is orientable. With respect to a local coordinates system $\{x^a\}$, if we define a 2-form $\varphi = (\frac{1}{2})\varphi_{ab}\cdot\cdot dx^a\wedge dx^b$ by $\varphi_{ab} = \nabla_a\eta_b$, then we have $d\eta = 2\varphi$, where $d\eta$ denotes the exterior differential of η.

In the following, let M be an $n(=2m+1)$-dimensional SASAKIAN space. We shall remember some operators and identities in M which have been used in Y. OGAWA [2]. We denote by $i(\eta)$ and Λ (resp. $e(\eta)$ and L) the inner product (resp. exterior product) of 1-form η and 2-form $d\eta$. Then, for any p-form u the operators $i(\eta)$, Λ, $e(\eta)$ and L (²) are defined by

$$(i(\eta)u)_{a_2\ldots a_p} = \eta^r u_{r a_2\ldots a_p} \qquad (p\geqq 1)\,,$$

$$i(\eta)u = 0 \qquad (p=0)\,,$$

(²) These definitions of Λ and L are different from the definitions of S. TACHIBANA [3]. S. TACHIBANA denotes by Λ (resp. L) the inner product (resp. exterior product) of 2-form φ $(=\frac{1}{2}d\eta)$.

$$(e(\eta)u)_{a_1\ldots a_{p+1}}=\sum_{i=1}^{p+1}(-1)^{i+1}\eta_{a_i}u_{a_1\ldots\hat{a}_i\ldots a_{p+1}} \qquad (p\geqq 1)\,,$$

$$(e(\eta)u)_{a_1}=u\eta_{a_1} \qquad (p=0)\,,$$

$$(\Lambda u)_{a_3\ldots a_p}=\varphi^{rs}u_{rsa_3\ldots a_p} \qquad (p\geqq 2)\,,$$

$$\Lambda u=0 \qquad (p=0,1)\,,$$

$$(Lu)_{a_1\ldots a_{p+2}}=2\sum_{j<i}(-1)^{i+j+1}\varphi_{a_ja_i}u_{a_1\ldots\hat{a}_j\ldots\hat{a}_i\ldots a_{p+2}} \qquad (p\geqq 1)\,,$$

$$(Lu)_{ab}=2u\varphi_{ab} \qquad (p=0)\,.$$

Then we have for any p-form u [2]:

$$Lu=e(\eta)du+de(\eta)u\,, \tag{2.2}$$

$$(\Lambda L^k-L^k\Lambda)u=4k[(m-p-k+1)L^{k-1}u+e(\eta)i(\eta)L^{k-1}u]\,, \tag{2.3}$$

where k is non-negative integer and $L^{-1}u=0$. We shall call a form u to be horizontal (resp. effective) if it satisfies $i(\eta)u=0$ (resp. $\Lambda u=0$). If a p-form u satisfies

$$du=0\,, \qquad \delta u=e(\eta)\Lambda u \qquad (\text{resp. } \delta u=0,\ du=i(\eta)Lu)\,,$$

where δu denotes the codifferential of u, then we call u to be C-harmonic (resp. C^*-harmonic).

Moreover, the operators ∇_η, Φ and D for any p-form u are defined by

$$(\nabla_\eta u)_{a_1\ldots a_p}=\eta^r\nabla_r u_{a_1\ldots a_p} \qquad (p\geqq 0)\,, \tag{2.4}$$

$$(\Phi u)_{a_1\ldots a_p}=\sum_{i=1}^{p}\varphi_{a_i}{}^r u_{a_1\ldots r\ldots a_p} \qquad (p\geqq 1)\,, \tag{2.5}$$

$$(Du)_{a_2\ldots a_p}=\varphi^{rs}\nabla_r u_{sa_2\ldots a_p} \qquad (p\geqq 1)\,. \tag{2.6}$$

Then the following relation holds good for any p-form u [2]:

$$Du=\delta\nabla_\eta u-\nabla_\eta\delta u+(n-p)i(\eta)u\,. \tag{2.7}$$

3. – Horizontal conformal Killing tensor.

Let u be a horizontal conformal KILLING tensor of degree p whose associated tensor is θ. We shall prove a series of lemmas about u.

First we have

LEMMA 3.1. – *A horizontal conformal Killing vector (i.e., tensor of degree* 1) *is necessarily Killing.*

PROOF. – Let v^a be a conformal KILLING tensor of degree 1. Then it holds that

$$\nabla_a v_b + \nabla_b v_a = 2\varrho g_{ab}\,, \qquad (\varrho = \nabla_r v^r/n)\,.$$

Contracting this by $\eta^a\eta^b$ and making use of $i(\eta)v=0$, we have $\varrho=0$, which means that the lemma is true.

Next, we study the nature of the associated tensor θ of u $(p>1)$.

LEMMA 3.2. – *The associated tensor θ of u $(p>1)$ is Killing.*

PROOF. – By contraction with $\eta^c\eta^{a_1}$, the equation (1.4) turns to

$$(p-1)\,R_b{}^e u_{ea_2\dots a_p} + (p-2)\sum_{i=2}^{p} R_b{}^d{}_{a_2}{}^e u_{da_2\dots e\dots a_p} - \sum_{i=2}^{p} R_{a_i}{}^e u_{ba_2\dots e\dots a_p} - \sum_{2\leqslant j<k}^{p} R_{a_j a_k}{}^{de} u_{ba_2\dots d\dots e\dots a_p} = 0\,,$$

where we have used $i(\eta)u=0$. We can obtain by substitution this into (1.3)

$$(p-1)\,l_{ba_2\dots a_p} + \sum_{i=2}^{p} l_{a_i a_2\dots b\dots a_p} = 0\,,$$

and therefore we get

$$l_{ba_2\dots a_p} + l_{a_2 b a_3\dots a_p} = 0\,,$$

which means that θ is KILLING.

LEMMA 3.3. – *The associated tensor θ of u $(p>1)$ satisfies the equation $e(\eta)\theta=0$.*

PROOF. – Differentiating $i(\eta)u=0$ covariantly, we have

$$\varphi_{a_1}{}^r u_{ra_2\dots a_p} + \eta^r\nabla_{a_1} u_{ra_2\dots a_p} = 0\,.$$

Hence if we add this to the equation obtained by interchanging the indice a_1 and a_2 and take account of (1.1), then we find

$$\begin{aligned}(3.1)\qquad & -\varphi_{a_1}{}^r u_{ra_2\dots a_p} + \varphi_{a_2}{}^r u_{a_1 r a_3\dots a_p} + 2\theta'_{a_3\dots a_p} g_{a_1 a_2} - \theta_{a_2\dots a_p}\eta_a \\ & \qquad - \theta_{a_1 a_3\dots a_p}\eta_{a_2} + \sum_{i=3}^{p}(-1)^i\big(g_{a_1 a_i}\theta'_{a_2\dots\hat{a}_i\dots a_p} + g_{a_2 a_i}\theta'_{a_1 a_3\dots\hat{a}_i\dots a_p}\big) = 0\,,\end{aligned}$$

where we put $\theta' = i(\eta)\theta$. Contracting this with η^{a_1}, by virtue of $i(\eta)u=0$ it follows that

$$\theta_{a_2\dots a_p} = \sum_{i=2}^{p}(-1)^i\eta_{a_i}\eta^r\theta_{ra_2\dots\hat{a}_i\dots a_p}\,,$$

that is,

$$(3.2)\qquad \theta = e(\eta)\,i(\eta)\theta\,.$$

Since $e(\eta)^2\theta=0$, we get $e(\eta)\theta=0$.

THEOREM 3.1. – *Let u be a conformal Killing tensor of degree p in a Sasakian space. If u is horizontal and effective, then it is necessarily Killing.*

PROOF. – For $p=1$, the theorem is true by Lemma 3.1. Let us consider the case of $p \geqq 2$. By contraction (3.1) with $g^{a_1a_2}$, we find

$$\Lambda u = -(n-p+1)\, i(\eta)\,\theta\,, \tag{3.3}$$

and we have $i(\eta)\theta = 0$ because $\Lambda u = 0$. Consequently we obtain $\theta = 0$ from (3.2). This completes the proof.

On the other hand, Y. OGAWA [2] has proved:

THEOREM C. – *If a Killing tensor w of degree p satisfies $e(\eta)w = 0$, then w is C^*-harmonic, $i(\eta)w$ is C-harmonic and the equations*

$$\nabla_\eta w = 0 \qquad \text{and} \qquad \Phi w = 0$$

hold good.

Owing to Lemma 3.2, 3.3 and Theorem C, it follows that the associated tensor θ of u $(p>1)$ is C^*-harmonic and satisfies

$$\nabla_\eta \theta = 0\,, \qquad \Phi\theta = 0\,. \tag{3.4}$$

Let $A_{a_1a_2\ldots a_p}$ be a tensor field of degree p and skew-symmetric with respect to the indices $a_2, a_3, \ldots, a_p$. We set

$$\bar{A}_{a_1\ldots a_p} = \sum_{i=1}^{p} (-1)^{i+1} A_{a_i a_1 \ldots \hat{a}_i \ldots a_p}\,,$$

then $\bar{A}_{a_1\ldots a_p}$ is skew-symmetric with respect to all the indices $a_1, a_2, \ldots, a_p$.

Now we shall make some preparations for Lemma 3.4 below.

Covariant differentiation of (3.3) yields

$$2\eta^r u_{ra_2\ldots a_p} + \varphi^{rs}\nabla_{a_2} u_{rsa_3\ldots a_p} = -(n-p+1)(\varphi_{a_2}{}^r\theta_{ra_3\ldots a_p} + \eta^r\nabla_{a_2}\theta_{ra_3\ldots a_p})\,,$$

from which, by virtue of $i(\eta)u = 0$ and (3.4) it holds that

$$\varphi^{rs}\nabla_{a_2} u_{rsa_3\ldots a_p} = -(n-p+1)\,\varphi_{a_2}{}^r\theta_{ra_3\ldots a_p}\,. \tag{3.5}$$

So, we have

$$2\theta^*_{a_4\ldots a_p} g_{a_2a_3} + (n-p+3)(\varphi_{a_2}{}^r\theta_{ra_3\ldots a_p} - \varphi_{a_3}{}^r\theta_{a_2ra_4\ldots a_p}) - \sum_{i=4}^{p}(-1)^i[g_{a_2a_i}\theta^*_{a_3\ldots\hat{a}_i\ldots a_p} + g_{a_3a_i}\theta^*_{a_2a_4\ldots\hat{a}_i\ldots a_p}] = 0\,, \tag{3.6}$$

be making use of (1.1) if we set $\theta^* = \Lambda\theta$. We shall take the skew-symmetric part of (3.6) with respect to the indices $a_3, a_4, \ldots, a_{p-1}, a_p$. Since each term of (3.6) is skew-symmetric with respect to the indices $a_4, a_5, \ldots, a_p$, we may apply the above method to (3.6). We have

$$(n-p+3)[(p-1)\varphi_{a_2}{}^r\theta_{ra_3\ldots a_p} - (\Phi\theta)_{a_2\ldots a_p}] = (p-1)\sum_{i=3}^{p}(-1)^i g_{a_2a_i}\theta^*{}_{a_3\ldots\hat{a}_i\ldots a_p},$$

from which

$$(n-p+3)\varphi_{a_2}{}^r\theta_{ra_3\ldots a_p} = \sum_{i=3}^{p}(-1)^i g_{a_2a_i}\theta^*_{a_3\ldots\hat{a}_i\ldots a_p} \tag{3.7}$$

by taking account of $\Phi\theta = 0$ and $p>1$. Therefore we get

$$(n-p+3)(-\theta_{a_2\ldots a} + \eta_{a_2}\theta'_{a_3\ldots a_p}) = \sum_{i=3}^{p}(-1)^i\varphi_{a_2a_i}\theta^*_{a_3\ldots\hat{a}_i\ldots a_p} \tag{3.7$'$}$$

Furthermore we take the skew-symmetric part of (3.7)′ with respect to the indices $a_2, a_3, \ldots, a_p$, therefore using $\theta = e(\eta)i(\eta)\theta$ and $c = (p-2)(n-p+3)$ we have

$$c\theta - L\Lambda\theta = 0\,. \tag{3.8}$$

On the other hand, from (2.7) we find

$$\varphi^{rs}\nabla_r\theta_{sa_3\ldots a_p} = (n-p+1)\theta'_{a_3\ldots a_p} \tag{3.9}$$

by making use of $\nabla_\eta\theta = 0$ and $\delta\theta = 0$.

Let us prove

LEMMA 3.4. – *The associated tensor θ of u $(p>1)$ is a special Killing tensor with constant* 1.

PROOF. – Differentiating (3.7)′ covariantly, we obtain with the aid of $\nabla_\eta\theta = 0$ and (3.9)

$$\begin{aligned}(n-p+3)(-\nabla_{a_1}\theta_{a_2\ldots a_p} + \varphi_{a_1a_2}\theta'_{a_3\ldots a_p} + \eta_{a_2}\varphi_{a_1}{}^r\theta_{ra_3\ldots a_p})\\ -\sum_{i=1}^{p}(-1)^i[(\eta_{a_2}g_{a_1a_i} - \eta_{a_i}g_{a_1a_2})\theta^*_{a_3\ldots\hat{a}_i\ldots a_p} + (n-p+3)\varphi_{a_2a_i}\theta'_{a_1a_3\ldots\hat{a}_i\ldots a_p}] = 0\,,\end{aligned}$$

and hence it holds that

$$\begin{aligned}(n-p+3)(-\nabla_{a_1}\theta_{a_2\ldots a_p} + \varphi_{a_1a_2}\theta'_{a_3\ldots p})\\ -\sum_{i=3}^{p}(-1)^i[-\eta_{a_i}g_{a_1a_2}\theta^*_{a_3\ldots\hat{a}_i\ldots a_p} + (n-p+3)\varphi_{a_2a_i}\theta'_{a_1a_3\ldots\hat{a}_i\ldots a_p}] = 0\end{aligned}$$

by virtue of (3.7). Again, if we apply ∇_{a_0} to this, then we find by taking account of $\nabla_\eta \theta = 0$ and (3.9)

$$(3.10) \qquad (1)+(2)+\dots+(9)=0\,,$$

where we have set

$$(1)=-(n-p+3)\nabla_{a_0}\nabla_{a_1}\theta_{a_2\cdots a_p}\,, \qquad (2)=(n-p+3)g_{a_0a_2}\eta_{a_1}\theta'_{a_3\cdots a_p}\,,$$

$$(3)=-(n-p+3)g_{a_0a_1}\eta_{a_2}\theta'_{a_3\cdots a_p}\,, \qquad (4)=(n-p+3)\varphi_{a_1a_2}\varphi_{a_0}{}^r\theta_{ra_3\cdots a_p}\,,$$

$$(5)=g_{a_1a_2}\sum_{i=3}^p(-1)^i\varphi_{a_0a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots a_p}\,, \qquad (6)=(n-p+3)g_{a_1a_2}\sum_{i=3}^p(-1)^i\eta_{a_i}\theta'_{a_0a_3\cdots\hat{a}_i\cdots a_p}\,,$$

$$(7)=-(n-p+3)\eta_{a_2}\sum_{i=3}^p(-1)^i g_{a_0a_i}\theta'_{a_1a_3\cdots\hat{a}_i\cdots a_p}\,,$$

$$(8)=(n-p+3)g_{a_0a_2}\sum_{i=3}^p(-1)^i\eta_{a_i}\theta'_{a_1a_3\cdots\hat{a}_i\cdots a_p}\,,$$

$$(9)=-(n-p+3)\sum_{i=3}^p(-1)^i\varphi_{a_2a_i}\varphi_{a_0}{}^r\theta_{ra_1a_3\cdots\hat{a}_i\cdots a_p}\,.$$

The equations (3), (4) and (6)$\sim$(9) can be rewritten as follows with the aid of (3.7) and (3.7)$'$:

$$(3)=-(n-p+3)g_{a_0a_1}\theta_{a_2\cdots a_p}-g_{a_0a_1}\sum_{i=3}^p(-1)^i\varphi_{a_2a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots a_p}\,,$$

$$(4)=\varphi_{a_1a_2}\sum_{i=3}^p(-1)^i g_{a_0a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots p}\,,$$

$$(6)=g_{a_1a_2}\Big[(p-2)(n-p+3)\theta_{a_0a_3\cdots a_p}+\sum_{i=3}^p(-1)^i\varphi_{a_0a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots p}+2\sum_{3\leqq j<k}^p(-1)^{j+k}\varphi_{a_ja_k}\theta^*_{a_0a_3\cdots\hat{a}_j\cdots\hat{a}_k\cdots p}\Big],$$

$$(7)=(n-p+3)\sum_{i=3}^p(-1)^i g_{a_0a_i}\theta_{a_1a_2a_3\cdots\hat{a}_i\cdots a_p}-\varphi_{a_1a_2}\sum_{i=3}^p(-1)^i g_{a_0a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots a_p}$$
$$+\sum_{j=3}^p\sum_{j\neq k}(-1)^{j+k}\varphi_{a_2a_j}g_{a_0a_k}\theta^*_{a_1a_3\cdots\hat{a}_j\cdots\hat{a}_k\cdots a_p}\,,$$

$$(8)=(n-p+3)g_{a_0a_2}(\theta_{a_1a_3\cdots a_p}-\eta_{a_1}\theta'_{a_3\cdots a_p})\,,$$

$$(9)=g_{a_0a_1}\sum_{i=3}^p(-1)^i\varphi_{a_2a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots a_p}-\sum_{j=3}^p\sum_{j\neq k}(-1)^{j+k}\varphi_{a_2a_j}g_{a_0a_k}\theta^*_{a_1a_3\cdots\hat{a}_j\cdots\hat{a}_k\cdots p}\,.$$

On the other hand, we get

$$(5)+(6)=g_{a_1a_2}\sum_{i=3}^p(-1)^i\varphi_{a_0a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots p}+g_{a_1a_2}\Big[(p-2)(n-p+3)\theta_{a_0a_3\cdots a_p}$$
$$+\sum_{i=3}^p(-1)^i\varphi_{a_0a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots p}+2\sum_{3\leqq j<k}^p(-1)^{j+k}\varphi_{a_ja_k}\theta^*_{a_0a_3\cdots\hat{a}_j\cdots\hat{a}_k\cdots a_p}\Big]=g_{a_1a_2}[c\theta_{a_0a_3\cdots a_p}-(L\Lambda\theta)_{a_0a_3\cdots a_p}]=0$$

by taking account of (3.8), and therefore the equation (3.10) reduces to

$$\nabla_{a_0}\nabla_{a_1}\theta_{a_2\cdots a_p}+\sum_{i=2}^{p}(-1)^i g_{a_0a_i}\theta_{a_1a_2\cdots\hat{a}_i\cdots a_p}=0\,, \tag{3.11}$$

which means that the Lemma is true.

From (1.1) and (3.11) we get

LEMMA 3.5. – *A horizontal conformal Killing tensor u of degree p $(1<p<n)$ is uniquely decomposed into the form:*

$$u_{a_1\cdots a_p}=w_{a_1\cdots a_p}+q_{a_1\cdots a_p}\,,$$

where $w_{a_1\cdots a_p}$ is Killing and $q_{a_1\cdots a_p}$ is a closed horizontal conformal Killing tensor. In this case, $q_{a_1\cdots a_p}$ is the form

$$q_{a_1\cdots a_p}=-\nabla_{a_1}\theta_{a_2\cdots a_p}\,,$$

where $\theta_{a_2\cdots a_p}$ is the associated tensor of u.

PROOF. – We find from (1.1) and (3.11)

$$\nabla_{a_0}(u_{a_1\cdots a_p}+\nabla_{a_1}\theta_{a_2\cdots a_p})+\nabla_{a_1}(u_{a_0a_2\cdots a_p}+\nabla_{a_0}\theta_{a_2\cdots a_p})=0\,.$$

Consequently, u is decomposed in the form stated in Lemma. Next, let $l_{a_1\cdots a_p}$ be a closed KILLING tensor of degree p. Then we have

$$(dl)_{a_1\cdots a_{p+1}}=0\,,\qquad \nabla_{a_1}l_{a_2\cdots a_{p+1}}+\nabla_{a_2}l_{a_1a_3\cdots a_{p+1}}=0\,.$$

Hence we obtain $\nabla_{a_1}l_{a_2\cdots a_{p+1}}=0$. Regarding to the following Lemma [6], we have proved the uniqueness of the decomposition.

LEMMA 3.6. – *There are no covariant constant p-forms on M for $1\leqq p\leqq n-1$.*

LEMMA 3.7. – *Let u be a horizontal conformal Killing tensor of degree $p(>2)$ whose associated tensor is θ. Then the tensor Λu is a horizontal conformal Killing tensor of degree $p-2$ whose associated tensor is $((n-p+1)/(n-p+3))\theta^*$.*

PROOF. – As u is horizontal, we have from (3.5)

$$\nabla_{a_2}(\Lambda u)_{a_3\cdots a_p}=-(n-p+1)\,\varphi_{a_2}{}^{r}\theta_{ra_3\cdots a_p}\,,$$

and by virtue of (3.7) it follows that

$$\nabla_{a_2}(\Lambda u)_{a_3\cdots a_p}=-\frac{n-p+1}{n-p+3}\sum_{i=3}^{p}(-1)^i g_{a_2a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots a_p}\,,$$

from which

$$\nabla_{a_2}(\Lambda u)_{a_3\cdots a_p} + \nabla_{a_3}(\Lambda u)_{a_2 a_4\cdots a_p} = \frac{n-p+1}{n-p+3}\left[2g_{a_2a_3}\theta^*_{a_4\cdots a_p} - \sum_{i=4}^{p}(-1)^i(g_{a_2a_i}\theta^*_{a_3\cdots\hat{a}_i\cdots a_p} + g_{a_3a_i}\theta^*_{a_2a_4\cdots\hat{a}_i\cdots a_p})\right]. \tag{3.12}$$

The tensor Λu is horizontal, because of $i(\eta)u = 0$.

In the last place, for any p-form u we shall prove

LEMMA 3.8. – *For any p-form u in an n $(=2m+1)$-dimensional M, we have*

$$(\Lambda^k L - L\Lambda^k)u = 4k[(m-p+k-1)\Lambda^{k-1}u + e(\eta)\,i(\eta)\Lambda^{k-1}u], \tag{3.13}$$

where k is non-negative integer and $\Lambda^{-1}u = 0$.

PROOF. – The theorem is trivial for $k = 0$. Proceeding inductively, assume its validity for $0, 1, \ldots, k$, and consider $(k+1)$-case. Then we have

$$(\Lambda^{k+1}L - L\Lambda^{k+1})u = \Lambda^k(\Lambda Lu - L\Lambda u) + (\Lambda^k L - L\Lambda^k)\Lambda u = 4(k+1)[(m-p+k)\Lambda^k u + e(\eta)\,i(\eta)\Lambda^k u]$$

for any p-form u, which asserts that Lemma is true for all non-negative integer k.

Taking account of Lemma 3.8, we shall show the following for a horizontal conformal KILLING tensor u:

LEMMA 3.9. – *Let θ be the associated tensor of u $(p > 1)$. Then we have*

$$\alpha_1\alpha_2 \ldots \alpha_s\theta = L^s\Lambda^s\theta, \tag{3.14}$$

where we put $\alpha_t = (p-2t)(n-p+2t+1)$ for the integer t $(1 \leqq t \leqq s)$.

PROOF. – Making use of (3.8), for $s = 1$ the equation (3.14) holds good. Now suppose that it is true for all $1, 2, \ldots, s$ and condier $(s+1)$-case. Since the $(p-1)$-form θ satisfies $\theta = e(\eta)i(\eta)\theta$, we have from (3.13)

$$(\Lambda^k L - L\Lambda^k)\theta = 4k(m-p+k+1)\Lambda^{k-1}\theta. \tag{3.15}$$

We put $k = 1$ here and hence we get by virtue of (3.8)

$$\Lambda L\theta = [\alpha_1 + 4(m-p+2)]\theta. \tag{3.16}$$

Next, set $k = s+1$ in (3.15), it follows that

$$(\Lambda^{s+1}L - L\Lambda^{s+1})\theta = 4(s+1)(m-p+s+2)\Lambda^s\theta.$$

Operating L^s to this and making use of (3.14) and (3.16), we find

$$L^{s+1}\Lambda^{s+1}\theta = L^s\Lambda^{s+1}L\theta - 4(s+1)(m-p+s+2)L^s\Lambda^s\theta$$
$$= [\alpha_1 + 4(m-p+2) - 4(s+1)(m-p+s+2)]L^s\Lambda^s\theta = \alpha_1\alpha_2 \dots \alpha_{s+1}\theta \,.$$

This completes the proof.

4. – The case of odd degree.

Let u be a horizontal conformal KILLING tensor of odd degree p whose associated tensor is θ.

First we show

LEMMA 4.1. – *Let u be a horizontal conformal Killing tensor of degree 3. Then the vector Λu is necessarily Killing.*

PROOF. – Putting $p = 3$ in Lemma 3.7, Λu is a horizontal conformal KILLING tensor. Therefore, by virtue of Lemma 3.1, we know that Λu is KILLING.

Next we have by virtue of Lemma 3.7 and 4.1.

LEMMA 4.2. – *Let u be a horizontal conformal Killing tensor of odd degree p. Then the vector $\Lambda^{(p-1)/2}u$ is necessarily Killing.*

Lastly, we shall prove

THEOREM 4.1. – *In a Sasakian space, a horizontal conformal Killing tensor of odd degree is necessarily Killing.*

PROOF. – For $p = 1$, this theorem is true by making use of Lemma 3.1. For $p > 1$, we can take account of Lemma 3.9. First, we shall show that α_t in (3.14) is non-zero constant. In fact, if $\alpha_t = 0$, then we obtain with the aid of $p - 2t \neq 0$

$$p = n + 2t + 1 \,.$$

As p is odd, this does not hold. Consequently, we have $\alpha_t \neq 0$ $(t = 1, 2, \dots, s)$. Therefore, the form θ vanishes identically if it satisfies $\Lambda^s\theta = 0$ for some s. On the other hand, the vector $\Lambda^{(p-1)/2}u$ is KILLING by Lemma 4.2 and hence $\Lambda^{(p-1)/2}\theta = 0$ holds good. Thus we have $\theta = 0$, which means that u is KILLING.

5. – The case of even degree $p(=2q)$ (³).

Let us prove

LEMMA 5.1. – *The associated vector θ of a horizontal conformal Killing tensor of degree 2 satisfies $\theta = (i(\eta)\theta)e(\eta)\cdot 1$, where $i(\eta)\theta$ is constant.*

PROOF. – By Lemma 3.3, we get $\theta = (i(\eta)\theta)e(\eta)\cdot 1$, that is,

$$\theta_a = \theta' \eta_a .$$

Covariant differentiation of this yields

$$\nabla_b \theta_a = \nabla_b \theta' \eta_a + \theta' \varphi_{ba} ,$$

from which, we find

$$\eta_a \nabla_b \theta' + \eta_b \nabla_a \theta' = 0$$

by making use of $\nabla_a \theta_b + \nabla_b \theta_a = 0$. Contracting this with η^a and $\eta^a \eta^b$ respectively, we get

$$\nabla_b \theta' + \eta_b \nabla_\eta \theta' = 0 , \qquad \nabla_\eta \theta' = 0 ,$$

from which, $\nabla_b \theta' = 0$. This means that θ' is constant.

LEMMA 5.2. – *Let θ be the associated tensor of a horizontal conformal Killing tensor u of even degree $2q$ (>2). If u is non-Killing, then the associated tensor θ turns to the form:*

$$\theta = \beta e(\eta) L^{q-1}\cdot 1 ,$$

where β $(= \Lambda^{q-1} i(\eta)\theta / \alpha_1 \alpha_2 \dots \alpha_{q-1})$ is constant.

PROOF. – We have from Lemma 3.9

$$\alpha_1 \alpha_2 \dots \alpha_{q-1} \theta = L^{q-1} \Lambda^{q-1} \theta .$$

Operating $e(\eta) i(\eta)$ to this and making use of $\theta = e(\eta) i(\eta) \theta$, it holds that

$$\alpha_1 \alpha_2 \dots \alpha_{q-1} \theta = (\Lambda^{q-1} i(\eta)\theta)\, e(\eta) L^{q-1}\cdot 1 .$$

Suppose that $\alpha_1 \alpha_2 \dots \alpha_{q-1} \neq 0$, then $\beta (= \Lambda^{q-1} i(\eta)\theta / \alpha_1 \alpha_2 \dots \alpha_{q-1})$ is constant from Lem-

(³) In this Section, suppose that M is connected.

ma 3.7 and 5.1. Next we shall consider the case of $\alpha_1\alpha_2 \ldots \alpha_{q-1} = 0$. Then we get

$$\Lambda^{q-1} i(\eta)\theta = 0 \ . \tag{5.1}$$

In fact, if this is not true, then the $(p-1)$-form $e(\eta)L^{q-1}\cdot 1$ vanishes identically. However, this contradicts that M is SASAKIAN. Applying $e(\eta)$ to (5.1) and using $\theta = e(\eta)i(\eta)\theta$, we have $\Lambda^{q-1}\theta = 0$. Differentiating this covariantly, we get $\Lambda^{q-2}\theta = 0$ with the aid of (2.7), Lemma 3.2 and 3.3. Moreover, covariant differentiation of this yields $\Lambda^{q-3}\theta = 0$. By the same method we obtain $\theta = 0$ at last. Therefore the lemma is proved.

Combining Lemma 5.1 and 5.2, we have immediately

THEOREM 5.1. – *In a Sasakian space, the associated tensor of a horizontal conformal Killing tensor of even degree 2q which is non-Killing turns to the form*:

$$\theta = ce(\eta)L^{q-1}\cdot 1 \ , \tag{5.2}$$

where c is constant.

As a corollary of Theorem 5.1, we get by virtue of Lemma 3.4

THEOREM 5.2. – *In a Sasakian space, the* $(2p+1)$*-form* $e(\eta)L^p\cdot 1$ *is a special Killing tensor with constant* 1.

Lastly, we prove

THEOREM 5.3. – *In a Sasakian space, a horizontal conformal Killing tensor u of even degree 2q which is non-Killing is uniquely decomposed into the form*:

$$u_{a_1\ldots a_{2q}} = w_{a_1\ldots a_{2q}} + q_{a_1\ldots a_{2q}} \ ,$$

where $w_{a_1\ldots a_{2q}}$ *is a horizontal Killing tensor and* $q_{a_1\ldots a_{2q}}$ *is a closed horizontal conformal Killing tensor. In this case,* $q_{a_1\ldots a_{2q}}$ *is the form*

$$q_{a_1\ldots a_{2q}} = h(L^q\cdot 1)_{a_1\ldots a_{2q}} \ ,$$

where h is constant.

PROOF. – We have $q_{a_1\ldots a_{2q}} = -\nabla_{a_1}\theta_{a_2\ldots a_{2q}}$ from Lemma 3.5. By substitution this into (5.2) we get

$$q_{a_1\ldots a_{2q}} = -c\nabla_{a_1}\big(e(\eta)L^{q-1}\cdot 1\big)_{a_2\ldots a_{2q}} \ . \tag{5.3}$$

On the other hand, since the form $e(\eta)L^{q-1}\cdot 1$ is KILLING, it follows that

$$\big(de(\eta)L^{q-1}\cdot 1\big)_{a_1\ldots a_{2q}} = (2q+1)\nabla_{a_1}\big(e(\eta)L^{q-1}\cdot 1\big)_{a_2\ldots a_{2q}} \ , \tag{5.4}$$

and hence by taking account of (2.3) we obtain

$$de(\eta)L^{q-1}\cdot 1 = \big(L - e(\eta)d\big)L^{q-1}\cdot 1 = L^{q}\cdot 1 - e(\eta)dL^{q-1}\cdot 1 \, .$$

Now, as d commutes with L (4), the above equation becomes

$$de(\eta)L^{q-1}\cdot 1 = L^{q}\cdot 1 \, .$$

Therefore we can obtain from (5.3), (5.4) and the last equation

$$q_{a_1 \dots a_{2q}} = h(L^{q}\cdot 1)_{a_1 \dots a_{2q}} \, .$$

The form $L^{q}\cdot 1$ is horizontal, because of $i(\eta)\cdot 1 = 0$. Hence we see $i(\eta)w = 0$. Consequently, this completes the proof.

As a corollary of Theorem 5.3, we get

THEOREM 5.4. – *In a Sasakian space, the $2p$-form $L^{p}\cdot 1$ is a closed horizontal conformal Killing tensor of degree $2p$.*

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(4) See Y. OGAWA [2].

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