Nihonkai Math. J.
Vol.5 (1994), 13-32

ON A KAEHLER MANIFOLD WHOSE TOTALLY REAL BISECTIONAL CURVATURE IS BOUNDED FROM BELOW

Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

Young Jin Suh*

Abstact The purpose of this paper is to show that a complete $n(\geq 3)$-dimensional Kaehler manifold with positively lower bounded totally real bisectional curvature and constant scalar curvature is globally isometric to a complex projective space $P_n(C)$ with Fubini-Study metric.

0. Introduction

R.L. Bishop and S.I Goldberg [2] introduced the notion of totally real bisectional curvature $B(X, Y)$ on a Kaehler manifold M. It is determined by a totally real plane $[X, Y]$ and its image $[JX, JY]$ by the complex structure J, where $[X, Y]$ denotes the plane spanned by linearly independent vector fields X, and Y. Moreover the above two planes $[X, Y]$ and $[JX, JY]$ are orthogonal to each other. And it is known that two orthonormal vectors X and Y span a totally real plane if and only if X, Y and JY are orthonormal.

C.S. Houh [7] showed that $(n\geq 3)$-dimensional Kaehler manifold with constant totally real bisectional curvature is congruent to a complex space form of constant

* Partially supported by the research grant of ANU,1993 and TGRC-KOSEF

holomorphic sectional curvature $H(X) = c$, where $H(X)$ is determined by the holomorphic plane $[X, JX]$.

On the other hand, S.I. Goldberg and S. Kobayashi [5] introduced the notion of holomorphic bisectional curvature $H(X, Y)$, which is determined by two holomorphic planes $[X, JX]$ and $[Y, JY]$, and asserted that a complex projective space $P_n(C)$ is the only compact Kaehler manifold with positive holomorphic bisectional curvature $H(X, Y)$ and constant scalar curvature. If we compare the notion of $B(X, Y)$ with $H(X, Y)$ and $H(X)$, it can be easily seen that the positiveness of $B(X, Y)$ is weaker than the positiveness of $H(X, Y)$, because $H(X, Y) > 0$ implies that both of $B(X, Y)$ and $H(X)$ are positive but neither $B(X, Y) > 0$ nor $H(X) > 0$ implies $H(X, Y) > 0$.

In section 1 we introduce a local formula for Kaehler manifolds, which will be used to prove our main result. And in section 2 let us find a relation between the totally real bisectional curvature and the sectional curvature of a Kaehler manifold M. Also the further relation between the totally real bisectional curvature and the holomorphic sectional curvature of M will be treated. Moreover in this section we calculate the totally real bisectional curvature of the complex quadric Q_n immersed in a complex projective space $P_{n+1}(c)$ with the constant holomorphic sectional curvature c. In section 3 we will prove that a complete Kaehler manifold M with positively lower bounded totally real bisectional curvature $B(X, Y) \geq b > 0$ and constant scalar curvature is congruent to a complex projective space $P_n(C)$. Before to obtain this result we should verify that a Kaehler manifold M with $B(X, Y) \geq b > 0$ is Einstein. Moreover we also show that the positive constant b in the above estimation is best possible, because we can find that there is a complete Kaehler manifold with non-negative totally real bisectional curvature $B(X, Y)\geq 0$ but not Einstein.

The present author would like to express his sincere gratitude to the referee for his valuable comments.

1. Local formulas.

This section is concerned with local formula for Kaehler manifolds. Let M be a complex n-dimensional connected Kaehler manifold. Then we can choose a local unitary frame field $\{E_A\} = \{E_1, \ldots, E_n\}$ on a neighborhood of M. With respect to this frame field, let $\{\omega_A\}$ be its local dual frame fields. Then the Kaehlerian metric tensor g of M is given by $g = 2\Sigma_A \omega_A \otimes \bar{\omega}_A$. The canonical forms ω_A and the connection forms ω_{AB} of M satisfy the following equations:

$$d\omega_A + \Sigma\omega_{AB}\wedge\omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{BA} = 0, \tag{1.1}$$

$$\begin{aligned} d\omega_{AB} + \Sigma\omega_{AC}\wedge\omega_{CB} &= \Omega_{AB}, \\ \Omega_{AB} &= \Sigma R_{\bar{A}BC\bar{D}}\,\omega_C\wedge\bar{\omega}_D, \end{aligned} \tag{1.2}$$

where Ω_{AB} (resp. $R_{\bar{A}BC\bar{D}}$) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) on M.

The second equation of (1.1) means the skew-hermitian symmetry of Ω_{AB}, which is equivalent to the symmetric conditions

$$R_{\bar{A}BC\bar{D}} = \bar{R}_{\bar{B}AD\bar{C}}.$$

The Bianchi identities $\Sigma_B \Omega_{AB} \wedge \omega_B = 0$ obtained by the exterior derivative of (1.1) and (1.2) give the further symmetric relations

$$R_{\bar{A}BC\bar{D}} = R_{\bar{A}CB\bar{D}} = R_{\bar{D}BC\bar{A}} = R_{\bar{D}CB\bar{A}}. \tag{1.3}$$

Now, with respect to the frame chosen above, the Ricci-tensor S of M can be expressed as follows;

$$S = \Sigma(S_{C\bar{D}}\omega_C \otimes \bar{\omega}_D + S_{\bar{C}D}\bar{\omega}_C \otimes \omega_D),$$

where $S_{C\bar{D}} = \Sigma_B R_{\bar{B}BC\bar{D}} = S_{\bar{D}C} = \bar{S}_{\bar{C}D}$. The scalar curvature r is also given by

$$r = 2\Sigma_D S_{D\bar{D}}.$$

The Kaehlerian manifold M is said to be *Einstein* if the Ricci tensor S is given by

$$S_{C\bar{D}} = \lambda\delta_{CD}, \quad \lambda = \frac{r}{2n},$$

for a constant λ, where λ is called the Ricci curvature of the Einstein manifold.

The component $R_{\bar{A}BC\bar{D}E}$ and $R_{\bar{A}BC\bar{D}\bar{E}}$ of the covariant derivative of the Riemannian curvature tensor R(resp. $S_{A\bar{B}C}$ and $S_{A\bar{B}\bar{C}}$ of the Ricci tensor S) are defined by

$$\Sigma_E(R_{\bar{A}BC\bar{D}E}\omega_E + R_{\bar{A}BC\bar{D}\bar{E}}\bar{\omega}_E) = dR_{\bar{A}BC\bar{D}} - \Sigma(R_{\bar{E}BC\bar{D}}\bar{\omega}_{EA}$$

$$+ R_{\bar{A}EC\bar{D}}\omega_{EB} + R_{\bar{A}BE\bar{D}}\omega_{EC} + R_{\bar{A}BC\bar{E}}\bar{\omega}_{ED}),$$

$$\Sigma_C(S_{A\bar{B}C}\omega_C + S_{A\bar{B}\bar{C}}\bar{\omega}_C) = dS_{A\bar{B}} - \Sigma(S_{C\bar{B}}\omega_{CA} + S_{A\bar{C}}\bar{\omega}_{CB}).$$

The second Bianchi formula is given by

$$(1.4) \qquad R_{\bar{A}BC\bar{D}E} = R_{\bar{A}BE\bar{D}C},$$

and hence we have

$$(1.5) \qquad S_{A\bar{B}C} = S_{C\bar{B}A} = \Sigma_D R_{\bar{B}AC\bar{D}D}, \quad r_A = 2\Sigma_C S_{B\bar{C}C},$$

where $dr = \Sigma_C(r_C\omega_C + \bar{r}_C\bar{\omega}_C)$. The components $S_{A\bar{B}CD}$ and $S_{A\bar{B}C\bar{D}}$ of the covariant derivative of $S_{A\bar{B}C}$ are expressed by

$$(1.6) \qquad \begin{aligned} \Sigma_D(S_{A\bar{B}CD}\omega_D + S_{A\bar{B}C\bar{D}}\bar{\omega}_D) = dS_{A\bar{B}C} - \Sigma_D(&S_{D\bar{B}C}\omega_{DA} \\ &+ S_{A\bar{D}C}\bar{\omega}_{DB} + S_{A\bar{B}D}\omega_{DC}). \end{aligned}$$

By the exterior differentiation of the definition of $S_{A\bar{B}C}$ and by taking account of (1.6) the Ricci formula for the Ricci tensor S is given as follows:

$$(1.7) \qquad S_{A\bar{B}C\bar{D}} - S_{A\bar{B}\bar{D}C} = \Sigma_E(R_{\bar{D}CA\bar{E}}S_{E\bar{B}} - R_{\bar{D}CE\bar{B}}S_{A\bar{E}}).$$

The sectional curvature of the holomorphic plane P spanned by u and Ju is called the *holomorphic sectional curvature*, which is denoted by $H(P) = H(u)$. A Kaehler manifold M is said to be of *constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for all P and for all points of M. Then M is called a complex space form, which is denoted by $M_n(c)$, provided that it is of constant holomorphic sectional curvature c, of complex dimension n. The standard models of complex space forms are the following three kinds: the complex Euclidean space C^n, the complex projective space $P_n(C)$ or the complex hyperbolic space $H_n(C)$, according as $c = 0, c > 0$ or $c < 0$.

Now, the Riemannian curvature tensor $R_{\bar{A}BC\bar{D}}$ of $M_n(c)$ is given by

$$R_{\bar{A}BC\bar{D}} = \frac{c}{2}(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}). \tag{1.8}$$

First of all, let us introduce a fundamental property for the generalized maximal principal due to H. Omori [10] and S.T. Yau [12].

Theorem 1.1. *Let M be an n-dimensional Riemannian manifold whose Ricci curvature is bounded from below on M. Let F be a C^2-function bounded from below on M, then for any $\epsilon > 0$, there exists a point p such that*

$$|\nabla F(p)| < \epsilon, \quad \Delta F(p) > -\epsilon \quad and \quad \inf F + \epsilon > F(p).$$

2. Totally real bisectional curvature.

Let (M, g) be an n-dimensional Kaehlerian manifold with almost complex structure J. In this section, we consider a Kaehlerian manifold with totally real bisectional curvature, which is determined by a totally real plane $[u, v]$ and its image $[Ju, Jv]$ by the complex structure J. That is, the totally real bisectional curvature is defined by

$$B(u, v) = g(R(u, Ju)Jv, v), \tag{2.1}$$

where $[u, v]$ means the totally real plane section such that $g(u, u) = g(v, v) = 1$, $g(u, v) = 0$ and $g(u, Jv) = 0$. Then for a Kaehlerian manifold, using the first Bianchi-identity to (2.1), we get

$$\begin{aligned} B(u, v) &= g(R(u, Jv)Jv, u) + g(R(u, v)v, u) \\ &= K(u, v) + K(u, Jv), \end{aligned} \tag{2.2}$$

where $K(u, v)$ means the sectional curvature of the plane spanned by u and v.

Now if we put $u' = \frac{u+v}{\sqrt{2}}$ and $v' = \frac{J(u-v)}{\sqrt{2}}$, then it is easily seen that $g(u', u') = g(v', v') = 1$, and $g(u', Jv') = 0$. Thus $B(u', v') = g(R(u', Ju')Jv', v')$ implies that

$$(2.3) \qquad 4B(u', v') - 2B(u, v) = H(u) + H(v) - 4K(u, Jv),$$

where $H(u) = K(u, Ju)$, and $H(v) = K(v, Jv)$ means the holomorphic sectional curvatures of the plane $[u, Ju]$ and $[v, Jv]$ respectively.

If we put $u'' = \frac{u+Jv}{\sqrt{2}}$, and $v'' = \frac{Ju+v}{\sqrt{2}}$, then we get $g(u'', u'') = g(v'', v'') = 1$ and $g(u'', v'') = 0$. Using the similar method as in (2.3), we get

$$(2.4) \qquad 4B(u'', v'') - 2B(u, v) = H(u) + H(v) - 4K(u, v).$$

Summing up (2.3) and (2.4), we obtain

$$(2.5) \qquad 2B(u', v') + 2B(u'', v'') = H(u) + H(v).$$

Now we calculate the totally real bisectional curvatures of some manifolds.

Example 2.1 Let $M_n(c)$ be a complex space form of constant holomorphic sectional curvature c and $[u, v]$ be a totally real plane section. Then

$$\begin{aligned} B(u, v) &= g(R(u, Ju)Jv, v) \\ &= \frac{c}{4}\{g(u, v)g(Ju, Jv) - g(u, Jv)g(Ju, v) + g(Ju, v)g(-u, Jv) \\ &\quad - g(Ju, Jv)g(-u, v) - 2g(Ju, Jv)g(-u, v)\} \\ &= \frac{c}{2}. \end{aligned}$$

Thus $M_n(c)$ is a space of complex space form of constant totally real bisectional curvature $\frac{c}{2}$.

As a Kaehler manifold which is not of constant totally real bisectional curvature, we introduce the following example.

Example 2.2 Let Q_n be a complex quadric in $P_{n+1}(c)$ and $[u, v]$ a totally real plane section. Since Q_n is represented as a Hermitian symmetric space of compact type, its sectional curvature is non-negative(cf [8]). Thus by (2.2) we know that the totally real bisectional curvature $B(u, v)$ of Q_n is non-negative. Now let us estimate the upper bounds of $B(u, v)$ of Q_n. For the action of $G = SO(n+2)$ on Q_n, the isotropy group H turns out to be $SO(2)\times SO(n)$, where $SO(n)$ denotes the group of special orthogonal $n\times n$-matrices.

The canonical decomposition of the Lie algebra of the group G is

$$\mathcal{G} = \mathcal{H} + \mathcal{M},$$

where $\mathcal{G} = \mathcal{O}(n+2)$, $\mathcal{H} = \mathcal{O}(2) + \mathcal{O}(n)$, $\mathcal{M} = \left\{ \begin{pmatrix} 0 & 0 & -{}^t\xi \\ 0 & 0 & -{}^t\eta \\ \xi & \eta & 0 \end{pmatrix} \middle| \xi, \eta \in R^n \right\}$, and $\mathcal{O}(n)$ denotes the Lie algebra of the special orthogonal group $SO(n)$.

Identifying $(\xi, \eta) \in R^n + R^n$ with the above matrix in $\mathcal{M}$, we define an inner product g on $\mathcal{M}\times \mathcal{M}$ by

$$g((\xi, \eta), (\xi', \eta')) = \frac{2}{c}\{< \xi, \xi' > + < \eta, \eta' >\},$$

where $<\xi, \xi'>$ is the standard inner product in R^n. We also define a complex structure J on $\mathcal{M}$ by

$$J(\xi, \eta) = (-\eta, \xi).$$

The curvature tensor R at the origin is given by the following

$$R((\xi, \eta), (\xi', \eta')) = ad\begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & B \end{pmatrix}, \quad B \in O(n),$$

where $\lambda = <\xi', \eta> - <\xi, \eta'>$, $B = \frac{c}{4}\{\xi \wedge \xi' + \eta \wedge \eta'\}$, and $(\xi \wedge \xi')\eta = \frac{4}{c}\{<\xi', \eta> \xi - <\xi, \eta> \xi'\}$. Thus for unit elements $u = (\xi, \eta)$, $v = (\xi', \eta')$ in $\mathcal{M}$, the holomorphic bisectional curvature is given by

$$(2.6) \quad \begin{aligned} H(u, v) = g(R(u, Ju)Jv, v) &= \frac{2}{c}\{< -B\eta', \xi' > + < B\xi', \eta' >\} + \frac{c}{2}g(v, v) \\ &= \frac{8}{c}\{<\xi, \xi'><\eta, \eta'> - <\xi, \eta'><\xi', \eta>\} + \frac{c}{2}. \end{aligned}$$

And the holomorphic sectional curvature $H(u)$ is given by

$$H(u) = g(R(u, Ju)Ju, u) = \frac{8}{c}(|\xi|^2|\eta|^2 - <\xi, \eta>^2) + \frac{c}{2} \geq \frac{c}{2}.$$

In fact, since the complex quadric Q_n is a Hermitian symmetric space of compact type with rank 2, by K. Ogiue and R. Takagi [9] the holomorphic sectional curvature $H(u)$ of Q_n is holomorphically pinched as $\frac{c}{2} \leq H(u) \leq c$.

Now we consider the totally real bisectional curvature of the complex quadric Q_n. Let [u, v] be a totally real plane section such that $u = (\xi, \eta), v = (\xi', \eta')$, and

$Jv = (-\eta', \xi')$. Then u, v, Ju and Jv constitute orthonormal unit elements in $\mathcal{M}$. That is

$$g(u, v) = \frac{2}{c}\{< \xi, \xi' > + < \eta, \eta' >\} = 0,$$

$$g(u, Jv) = \frac{2}{c}\{< \xi, -\eta' > + < \eta, \xi' >\} = 0.$$

From these together with (2.6) the totally real bisectional curvature is given by

$$B(u, v) = -\frac{8}{c}\{< \xi, \xi' >^2 + < \xi, \eta' >^2\} + \frac{c}{2}.$$

From this, using the elementary method of Lagrange multiplier rule, it can be easily seen that the totally real bisectional curvature $B(u, v)$ is bounded as

$$-\frac{3}{2}c \leq B(u, v) \leq \frac{1}{2}c,$$

where the upper equality holds if and only if ξ is orthogonal to ξ' and η' in R^n. Accordingly, it follows that

$$0 \leq B(u, v) \leq \frac{1}{2}c$$

for any totally real plane $[u, v]$ of M, because we have already known that the totally real bisectional curvature of the complex quadric Q_n is non- negative.

3. Complete Kaehler manifolds with positive totally real bisectional curvature.

Let M be an n-dimensional Kaehler manifold with the complex structure J. We can choose a local field of orthonormal frames $u_1, ..., u_n, u_{1^*} = Ju_1, ..., u_{n^*} = Ju_n$

on a neighborhood on M. With respect to this frame field, let $\theta_1, \ldots, \theta_n, \theta_{1^*}, \ldots, \theta_{n^*}$ be the field of dual frames.

Let us denote by $\theta = (\theta_{AB}, \theta_{A^*B}, \theta_{AB^*}, \theta_{A^*B^*})$, $A, B = 1, \ldots, n$ the connection form of M. Then we have

$$(3.1) \qquad \theta_{AB} = \theta_{A^*B^*}, \theta_{AB^*} = -\theta_{A^*B}, \theta_{AB} = -\theta_{BA}, \textit{and} \quad \theta_{AB^*} = \theta_{BA^*}.$$

Now we set $e_A = \frac{1}{\sqrt{2}}(u_A - iu_A)$, $e_{\bar{A}} = \frac{1}{\sqrt{2}}(u_A + iu_A)$. Then $\{e_A, e_{\bar{A}}\}$ constitute a local field of unitary frames. And let us denote by $\omega_A = \theta_A + i\theta_{A^*}$ and $\bar{\omega}_A = \theta_A - i\theta_{A^*}$ its dual frame fields respectively. Then the components of Kaehler metric $g = 2\Sigma_A \omega_A \otimes \bar{\omega}_A$ and the metric components of the Riemannian curvature tensor are given by the following respectively

$$(3.2) \qquad g_{B\bar{C}} = g_{BC} + ig_{BC^*},$$

$$(3.3) \qquad R_{\bar{A}BC\bar{D}} = -\{K_{ABCD} + K_{A^*BC^*D} + i(-K_{ABC^*D} + K_{A^*BCD})\},$$

where $R_{\bar{A}BC\bar{D}} = g_{\bar{A}E}R^E{}_{BC\bar{D}}$. Thus for the case of $A = B$, $C = D$, $B \neq C$ in (3.3), the totally real bisectional curvature is given by

$$(3.4) \qquad R_{\bar{B}BC\bar{C}} = -K_{B^*BC^*C} = K_{BB^*C^*C} = B(u_B, u_C).$$

For the case of $A = B = C = D$ in (3.3), the holomorphic sectional curvature is given by

$$(3.5) \qquad R_{\bar{B}BB\bar{B}} = g(R(u_B, Ju_B)Ju_B, u_B) = H(u_B).$$

Remark 3.1 From (1.8) and (3.4) we know that for any totally real plane section $[u, v]$ the totally real bisectional curvature $B(u, v)$ of a complex space form $M_n(c)$ is $\frac{c}{2}$ which is the same value as in Example 2.1.

On the other hand, S.I. Goldberg and S. Kobayashi [5] showed that a Kaehler manifold with positive holomorphic bisectional curvature and constant scalar curvature is Einstein. It is well known that the Ricci 2-form is harmonic if and only if the scalar curvature is constant. In order to prove that the second Betti number of a compact connected Kaehler manifold M with positive holomorphic bisectional curvature $H(X, Y) > 0$ is one they have used the fact that $H(X) > 0$. Thus the Ricci 2-form is propotional to the Kaehler 2-form, so that M becomes to an Einstein manifold. But the condition $B(X, Y) > 0$ is weaker than the condition of $H(X, Y) > 0$ we can not use $H(X) > 0$ to obtain the above result. From this point of view by means of Theorem 1.1 we can obtain the following.

Theorem 3.1 *Let M be a complete n-dimensional Kaehler manifold with constant scalar curvature. Assume that the totally real bisectional curvature is lower bounded for some positive constant b. Then M is Einstein.*

Proof. Since $(S_{B\bar{C}})$ is a Hermitian matrix, it can be diagonalizable. Thus $S_{B\bar{C}} = \lambda_B \delta_{BC}$, where λ_B is a real valued function. From this it follows that $r = 2\Sigma_B S_{B\bar{B}} = 2\Sigma_B \lambda_B$. Now we put $S_2 = \Sigma_{B,C} S_{B\bar{C}} S_{C\bar{B}}$. Then it yields easily that

$$S_2 - \frac{r^2}{4n} = \Sigma\lambda_B^2 - \frac{(\Sigma\lambda_B)^2}{n} = \frac{1}{2n}\Sigma_{B,C}(\lambda_B - \lambda_C)^2. \tag{3.6}$$

Since we have assumed that the scalar curvature r of M is constant, from (1.5) it follows $\Sigma_B S_{B\bar{B}C} = \Sigma_B S_{C\bar{B}B} = 0$. Together with this fact using (1.5) and the Ricci formula (1.7) we have that

$$\Delta S_{B\bar{C}} = \Sigma_D S_{B\bar{C}D\bar{D}} = \Sigma_D S_{D\bar{C}B\bar{D}}$$

$$= \Sigma_{E,D}(R_{\bar{D}BD\bar{E}} S_{E\bar{C}} - R_{\bar{D}BE\bar{C}} S_{D\bar{E}}),$$

from which, if we use the first Bianchi-identity (1.3) to the final term, we have

$$\Delta S_{B\bar{C}} = \Sigma_E(S_{B\bar{E}} S_{E\bar{C}} - \Sigma_D R_{\bar{D}EB\bar{C}} S_{D\bar{E}})$$

$$= \lambda_B S_{B\bar{C}} - \Sigma_A \lambda_A R_{\bar{A}AB\bar{C}}.$$

Thus we get

(3.7) $$\frac{1}{2}\Delta S_2 = \frac{1}{2}|\nabla S|^2 + \Sigma_{B,C} S_{\bar{C}B}(\lambda_B S_{B\bar{C}} - \Sigma_A \lambda_A R_{\bar{A}AB\bar{C}}),$$

where $|\nabla S|^2 = 2\Sigma S_{A\bar{B}C}\bar{S}_{A\bar{B}C}$. Since the second term of the right hand side is reduced to

$$\Sigma_{A,B}(\lambda_B^2 R_{\bar{A}AB\bar{B}} - \lambda_A \lambda_B R_{\bar{A}AB\bar{B}}) = \frac{1}{2}\Sigma_{A,B}(\lambda_A - \lambda_B)^2 R_{\bar{A}AB\bar{B}},$$

we get the following inequality by (3.7)

(3.8) $$\Delta S_2 \geq \Sigma(\lambda_A - \lambda_B)^2 R_{\bar{A}AB\bar{B}},$$

where the above equality holds if and only if the Ricci tensor S is parallel on M.

Now let us consider a non-negative function $f = S_2 - \frac{r^2}{4n}$. Then from (3.6),(3.8) and the assumption it follows that

(3.9) $$\Delta f \geq 2nbf,$$

where the above equality holds if and only if the Ricci tensor S is parallel on M. In order to prove this theorem, we need the following lemma.

Lemma 3.2 *Under the same assumption as stated in Theorem 3.1 the Ricci-curvature is bounded from below.*

Proof. From the assumption and (2.5) it follows that

$$H(u) + H(v) \geq 4b.$$

Using (3.5) to the above equation for $u = u_A, v = u_B, A \neq B$, then we can rewrite the above inequality as the following

$$R_{\bar{A}AA\bar{A}} + R_{\bar{B}BB\bar{B}} \geq 4b.$$

If we put $R_A = R_{\bar{A}AA\bar{A}}$, then

(3.10) $$R_A + R_B \geq 4b \qquad (A \neq B).$$

Thus $\Sigma_{A<B}(R_A + R_B) \geq 2n(n-1)b$ implies that

(3.11) $$\Sigma_A R_A \geq 2nb,$$

where the equality holds if and only if $R_A = 2b$ for any A.

On the other hand, from the fact that

$$r = 2\Sigma_A S_{A\bar{A}} = 2\Sigma_{A,B} R_{\bar{A}AB\bar{B}} = 2(\Sigma_A R_A + \Sigma_{A\neq B} R_{\bar{A}AB\bar{B}})$$

$$\geq 2\Sigma_A R_A + 2n(n-1)b$$

it follows

(3.12) $$\Sigma_A R_A \leq \frac{r}{2} - n(n-1)b,$$

where the equality holds if and only if $R_{\bar{A}AB\bar{B}} = b$ for any A, B ($A\neq B$). In this case due to C.S.Houh [7] M is congruent to $M_n(2b)$. From (3.11) and (3.12) we know that $r \geq 2n(n+1)b$. Thus from the assumption the scalar curvature r is positive constant. Also (3.10) gives $\Sigma_{B=2}^n (R_1 + R_B) \geq 4(n-1)b$, so that

(3.13) $$(n-2)R_1 + \Sigma_B R_B \geq 4(n-1)b.$$

From this and (3.12) it follows

$$(n-2)R_1 \geq 4(n-1)b - \Sigma_B R_B \geq 4(n-1)b - \{\frac{r}{2} - n(n-1)b\}.$$

Thus if we use the similar method to the other index, we can assert the following

$$(n-2)R_B \geq (n-1)(n+4)b - \frac{r}{2}$$

for any index B, so that R_B is bounded from below for $n \geq 3$. Moreover the above equality holds for some index B if and only if M is congruent to $M_n(2b)$. Accordingly the Ricci-curvature is given by

$$\lambda_A = S_{A\bar{A}} = \Sigma_B R_{\bar{A}AB\bar{B}} = R_A + \Sigma_{A\neq B} R_{\bar{A}AB\bar{B}}$$

$$\geq R_A + (n-1)b.$$

Thus the Ricci-curvature is also bounded from below. Now Lemma 3.2 is proved.

Now we will complete the proof of Theorem 3.1. For a constant $a > 0$, we consider a smooth positive function $F = (f + a)^{-\frac{1}{2}}$. Thus, from Lemma 3.2 we can apply Theorem 1.1(H.Omori [10] and S.T.Yau [12]) to the function $F = (f + a)^{-\frac{1}{2}}$ for the given f. Given any positive number $\epsilon > 0$, there exists a point p such that

(3.15) $$|\nabla F|(p) < \epsilon, \quad \Delta F(p) > -\epsilon, \quad F(p) < inf\ F + \epsilon.$$

On the other hand, the Laplacian of the function F can be calculated by

$$\Delta F = \Sigma_k\{(f + a)^{-\frac{1}{2}}\}_{k\bar{k}} = \frac{3}{4}F^5\Sigma_k f_{\bar{k}} f_k - \frac{1}{2}F^3\Delta f,$$

where f_k and $f_{\bar{k}}$ denote $\frac{\partial f}{\partial z_k}$ and $\frac{\partial f}{\partial \bar{z}_k}$ respectively. From this and (3.15), together with the fact that

$$|\nabla F| = |grad\ F|^2 = 2\Sigma_k F_{\bar{k}} F_k = \frac{1}{2}F^6\Sigma_k f_{\bar{k}} f_k$$

it follows that

(3.16) $$\epsilon(3\epsilon + 2F(p)) > F(p)^4\Delta f(p) \geq 0.$$

Thus for a convergent sequence $\{\epsilon_m\}$ such that $\epsilon_m > 0$ and $\epsilon_m \to 0$ as $m \to \infty$, there is a point sequence $\{p_m\}$ so that the sequence $\{F(p_m)\}$ satisfies (3.15) and converges to F_0, by taking a subsequence, if necessary, because the sequence $\{F(p_m)\}$ is bounded. From the definition of the infimum and (3.15) we have $F_0 = infF$ and hence $f(p_m) \to f_0 = supf$. It follows from (3.16) that we have

$$\epsilon_m\{3\epsilon_m + 2F(p_m)\} > F(p_m)^4\Delta f(p_m)$$

and the left hand side converges to 0 because the function F is bounded. Thus we get

$$F(p_m)^4 \Delta f(p_m) \to 0 \quad (m \to \infty).$$

As is already seen, the Ricci-curvature is bounded from below i.e., so is any λ_B. Since $r = 2\Sigma_B \lambda_B$ is constant, λ_B is bounded from above. Hence $F = (f + a)^{-\frac{1}{4}}$ is bounded from below by a positive constant. From (3.17) it follows that $\Delta f(p_m) \to 0$ as $m \to \infty$. Since $b > 0$, by (3.9) we have that

$$\Delta f(p_m) \geq \frac{n}{2} b f(p_m) \geq 0.$$

Thus we have $f(p_m) \to 0 = inf\ f$. Since $f(p_m) \to sup\, f$, we have $sup\, f = inf\ f = 0$. Hence $f = 0$ on M. That is, M is Einstein. This completes the above proof of Theorem 3.1.

Remark 3.2 The positive constant $b > 0$ in Theorem 3.1 is best possible. Because there is a complete Kaehler manifold with non-negative totally real bisectional curvature $B(u, v) \geq 0$ but not Einstein as follows: Consider a product manifold $M = P_{n^1}(c_1) \times P_{n^2}(c_2)$. Then from (3.8) we know that its totally real bisectional curvature is given by

$$R_{\bar{A}AB\bar{B}} = \begin{cases} R_{\bar{a}ab\bar{b}} = \frac{c_1}{2} & (A = a,\ B = b), \\ 0 & (A = a,\ B = s), \\ R_{\bar{r}rs\bar{s}} = \frac{c_2}{2} & (A = r,\ B = s), \end{cases}$$

where indices $A, B (A \neq B), \ldots; 1, \ldots, n_1, n_1 + 1, \ldots, n_2$, and $a, b, \ldots; 1, \ldots, n_1, r, s, \ldots; n_1 + 1, \ldots, n_2$.

And its Ricci-tensor is given by the following

$$S_{A\bar{B}} = \Sigma_C R_{\bar{B}AC\bar{C}} = \Sigma_a R_{\bar{B}Aa\bar{a}} + \Sigma_r R_{\bar{B}Ar\bar{r}}$$

$$= \begin{cases} \frac{n_1+1}{2} c_1 \delta_{bc} & (B = c, A = b), \\ 0 & (B = s, A = b), \\ \frac{n_2+1}{2} c_2 \delta_{ts} & (B = s, A = t). \end{cases}$$

Thus for the case where $(n_1+1)c_1 \neq (n_2+1)c_2$, $M = P_{n^1}(c_1) \times P_{n^2}(c_2)$ is not Einstein.

Since a complete Kaehler manifold M with the assumption in Theorem 3.1 is known to be Einstein and its scalar curvature r is positive constant, its Ricci-tensor is positive definite. Thus by using a theorem of Myers we can assert that M is compact [8]. Now let us introduce a theorem of S.I. Goldberg and S. Kobayashi [5], which is slight different from the original one.

Theorem A. *An n-dimensional compact connected Kaehler manifold with an Einstein metric of totally real bisectional curvature is globally isometric to $P_n(C)$ with Fubini-Study metric.*

Though the original theorem in [5] are assumed with positive holomorphic bisectional curvature, the above result in Theorem A also holds for the assumption with positive totally real bisectional curvature. Thus combining Theorem A and Theorem 3.1 we can assert the following.

Theorem 3.3 *Let M be a complete $n(\geq 3)$-dimensional Kaehler manifold with constant scalar curvature. Assume that the totally real bisectional curvature is lower bounded for some positive constant b. Then M is globally isometric to $P_n(C)$ with Fubini-Study metric.*

References

1. R. Aiyama, H. Nakagawa and Y.J. Suh, Semi-Kaehlerian submanifolds of an indefinite complex space form, Kodai Math. J.11(1988),325-343.

2. R.L. Bishop and S.I. Goldberg, Some implications of the generalized Gauss-Bonnet Theorem, Trans. A.M.S.,112(1964),508-535.

3. R.L. Bishop and S.I. Goldberg, On the second cohomology group of a Kaehler manifold of positive curvature, Proc. Amer. Math. Soc. 16(1965),119 -122.

4. R.L. Bishop and S.I. Goldberg, On the topology of positively curved Kaehler manifolds II, Tohoku Math. J.17(1965),310-318.

5. S.I. Goldberg and S. Kobayashi, Holomorphic bisectional curvature, J. Diff. Geometry 1(1967),33-43.

6. A. Gray, Compact Kaehler manifolds with non-negative sectional curvature. Invent. Math. 4(1977),33-43.

7. C.S. Houh, On totally real bisectional curvature, Proc. Amer. Math. Soc. 56(1976),261-263.

8. S. Kobayashi and K. Nomizu, Foundations of differential geometry II (1969) Interscience, Publ. New York,1969.

9. K. Ogiue and R. Takagi, A geometric meaning of the rank of Hermitian symmetric space, Tsukuba, J. Math. 5(1981),33-37.

10. H. Omori, Isometric immersions of Riemannian manifolds. J. Math. Soc. Japan, 19(1967),205-211.

11. J.A. Wolf, Spaces of constant curvatures, MaGrew-Hill, New-York(1967)

12. S.T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure and Appl. Math., 28(1975), 201-228.

Department of Mathematics
Andong University
Andong, Kyungpook,
760-749, KOREA

Received September 10, 1993, Revised February 23, 1994