

# On a kinetic model for a simple market economy

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## Abstract

In this paper, we consider a simple kinetic model of economy involving both exchanges between agents and speculative trading. We show that the kinetic model admits non trivial quasi-stationary states with power law tails of Pareto type. In order to do this we consider a suitable asymptotic limit of the model yielding a Fokker-Planck equation for the distribution of wealth among individuals. For this equation the stationary state can be easily derived and shows a Pareto power law tail. Numerical results confirm the previous analysis.

**Keywords.** Econophysics, Boltzmann equation, wealth and income distributions, Pareto distribution.

## 1 Introduction

Microscopic models of simple market economies [6, 7, 11, 13, 14] and financial markets [12, 16, 22] have been recently introduced by several authors. The main idea is that an economic system composed by a sufficiently large number of agents can be described using the laws of statistical mechanics as it happens in a physical system composed of many interacting particles. The details of the economical interactions between agents characterize their wealth distribution.

The study of wealth distributions has a long history going back to more than one century ago with the Italian sociologist and economist Vilfredo Pareto which studied the distribution of income among people of different western countries and found an inverse power law [21]. More precisely if  $f(w)$  is the probability density function of agents with wealth  $w$  we have

$$F(w) = \int_w^\infty f(w_*) dw_* \sim w^{-\mu}.$$

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Pareto mistakenly believed that power laws apply to the whole distribution with a universal exponent  $\mu$  approximatively equal to 1.5. Later, Mandelbrot, proposed a weak Pareto law that applies only to high incomes [18].

Income data from developed countries show Pareto-like behaviors with order 1 exponents for large values of incomes (USA  $\sim 1.6$ , Japan  $\sim 1.8 - 2.2$  [11]). It is common, in fact, that 90% of the total wealth is owned by only 5% of the population. According to data, across the full range of income, we should expect the probability density function  $f(w)$  to increase at low income, reach a maximum and, finally decrease with increasing wealth [11].

In our discussion we use the terms distribution of income (money) and distribution of wealth interchangeably (although the two distributions may not exactly coincide there is a strong dependence on one another). Income, in fact, is only one part of wealth, the other part being material wealth. Material products have no conservation law and their monetary value is not constant (they can be manufactured, destroyed, consumed, etc.). The distinction between the two will not influence the conclusion drawn here from the dynamical model of economic interactions we consider.

Following this line of thought, we consider here a very simple model of an open market economy involving both assets exchanges between individuals and speculative trading. In our non-stationary economy the average wealth is not conserved due to a random dynamics which describes the spontaneous growth or decrease of wealth due to investments in the stock market, housing, ect. It is important to note that this mechanism corresponds to the effects of an open market economy where the investments cause the total economy to growth (more precisely the rich would get richer and the poor would get poorer). The conservative exchanges dynamic between individuals redistributes the wealth among people.

We emphasize that the mathematical analysis of the kinetic model is not as simple as it may look due to the constraint that debts are not allowed. One immediately sees that in general the average wealth should grow yet it is not obvious to prove that this grow occurs exponentially. As usual in similar situations, the best way to extract information on the large-time behavior of the solution relies in the scaling the solution itself to keep the average wealth constant. Nevertheless, the determination of an explicit form of the limit distribution of the kinetic equation remains extremely difficult and requires the use of suitable numerical methods.

A complementary method to extract information on the tails is linked to the possibility to obtain particular asymptotics, which hopefully maintain the characteristics of the solution to the original problem for large times. Along this line, we shall show that the kinetic model converges in a suitable asymptotic limit (hereafter called continuous trading limit) towards a partial differential equation of Fokker-Planck type for the distribution of money among individuals. The same Fokker-Planck equation was obtained in [4] as the mean field limit of a stochastic dynamical equation, as well as in [24, 17] in the context of a generalized Lotka-Volterra dynamic.

The equilibrium state of the Fokker-Planck equation can be computed explicitly and is of Pareto type, namely it is characterized by a power-law tail for the richest individuals.

The mathematical methods we use are close to those used in the context of kinetic

theory for granular flows, where the limit procedure is known as quasi-elastic asymptotics [25, 19]. We mention here a similar asymptotic analysis performed on a kinetic model of a simple market economy with a constant growth mechanism [23].

The paper is organized as follows. In the next section, we introduce the binary interaction between agents, which is at the basis of the kinetic model. The main properties of the model are discussed in section 3. These properties justify the continuous trading limit procedure, performed in section 4. The limit is illustrated by several numerical results in section 5.

## 2 A kinetic model of money asset exchanges

The goal of a kinetic model of simple market economies, is to describe the evolution of the distribution of money by means of *microscopic* interactions among agents or individuals which perform exchange of money. Each trade can indeed be interpreted as an interaction where a fraction of the money changes hands. We will assume that this wealth after the interaction is non negative, which corresponds to impose that no debts are allowed. This rule emphasizes the difference between economic interactions, where not all outcomes are permitted, and the classical interactions between molecules.

From a microscopic view point, the binary interaction is described by the following rules

$$\begin{aligned} w' &= (1 - \gamma)w + \gamma w_* + \eta w \\ w'_* &= (1 - \gamma)w_* + \gamma w + \eta_* w_* \end{aligned} \tag{1}$$

where  $(w, w_*)$  denote the (positive) money of two arbitrary individuals before the trade and  $(w', w'_*)$  the money after the trade. In (1) we will not allow agents to have debts, and thus the interaction takes place only if  $w' \geq 0$  and  $w'_* \geq 0$ . In (1) the transaction coefficient  $\gamma \in (0, 1/2)$  is a given constant, while  $\eta$  and  $\eta_*$  are random variables with the same distribution (for example normal) with variance  $\sigma^2$  and zero mean.

Let us describe the three terms in the right hand side. The first term is related to the marginal saving propensity of the agents, the second corresponds to the money transaction, and the last contains the effects of an open economy describing the market returns. As we will show, the main consequence of the natural hypothesis to discard trades that produce debts, is that the total amount of money in the system is increasing. We refer the reader to Section 5 for numerical results in this direction.

This binary interaction model can be also related to some recent Lotka-Volterra type models [4]. A similar trade rule where  $\eta$  and  $\eta_*$  have the same positive constant value has been considered recently in [23]. A collection of various types of binary interactions recently considered in the literature can be found in [20].

In a closed economical system it is assumed that the total amount of money is conserved ( $\eta, \eta_* \equiv 0$ ). This conservation law is reminiscent of analogous conservations which take place in kinetic theory. In such a situation, thanks to the bounds on the transaction coefficient equations (1) correspond to a granular gas like model (or to a traffic flow

model[15]) where the stationary state is a Dirac delta centered in the average wealth (usually referred to as synchronized traffic state in traffic flow modelling). This behavior is a consequence of the fact that, in a single trade, the difference of wealths is diminishing, with  $|w' - w'_*| = (1 - 2\gamma)|w - w_*|$ . Thus all agents will end up in the market with exactly the same amount of money.

Let  $f(w, t)$  denote the distribution of money  $w \in \mathbb{R}_+$  at time  $t \geq 0$ . By standard methods of kinetic theory, it is easy to show that the time evolution of  $f$  is driven by the following integro-differential equation of Boltzmann type,

$$\frac{\partial f}{\partial t} = \int_{\mathbb{R}^2} \int_0^\infty \left( \beta_{(w', w'_*) \rightarrow (w, w_*)} \frac{1}{J} f(w') f(w'_*) - \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*) \right) dw_* d\eta d\eta_* \quad (2)$$

where  $(w', w'_*)$  are the pre-trading money that generates the couple  $(w, w_*)$  after the interaction. In (2)  $J$  is jacobian of the transformation of  $(w, w_*)$  into  $(w', w'_*)$ ,

$$J = (1 - \gamma + \eta)(1 - \gamma + \eta_*) - \gamma^2,$$

while the kernel  $\beta$  is related to the details of the binary interaction.

As usual in classical kinetic theory of rarefied gases, the interaction integral on right-hand side of (2) represents the instantaneous variation of the distribution of money, due to the balance between gain and loss of wealth in binary trades. The presence of the Jacobian  $J$ , guarantees that equation (2) preserves the mass (total number of agents), for any choice of the rate function  $\beta$ . We shall restrict here to a transition rate of the form

$$\beta_{(w, w_*) \rightarrow (w', w'_*)} = \Theta(\eta) \Theta(\eta_*) \Psi(w' \geq 0) \Psi(w'_* \geq 0),$$

where  $\Psi(A)$  is the indicator function of the set  $A$ , and  $\Theta(\cdot)$  is a symmetric probability density with zero mean and variance  $\sigma^2$ . The rate function  $\beta_{(w, w_*) \rightarrow (w', w'_*)}$  characterizes the effects of the open economy through the distribution of the random variables  $\eta$  and  $\eta_*$  and takes into account the hypothesis that no-debts are allowed. The above equation can be included in a more general settings where the trade rule has a more complex structure including, for example, risk, taxes and subsidies as described in [20].

We remark that, for general probability density  $\Theta(\cdot)$ , the rate function  $\beta$  depends on the wealth variables  $(w, w_*)$  through the indicator functions  $\Psi$ . This is analogous to what happens in the classical Boltzmann equation [5, 8], where the rate function depends on the relative velocity. A simplified situation occurs when the random variables take values on the set  $-(1 - \gamma), 1 - \gamma$ . In this case in fact, both  $w' \geq 0$  and  $w'_* \geq 0$ , and the kernel  $\beta$  does not depend on the wealth variables  $(w, w_*)$ . In this case the kinetic equation (2) is the corresponding of the classical Boltzmann equation for Maxwell molecules [2], which presents several mathematical simplifications. In all cases however, methods borrowed from kinetic theory of rarefied gas can be used to study the evolution of the function  $f$ .

### 3 Main properties of the kinetic equation

We will start our analysis by introducing some notations and by discussing the main properties of the kinetic equation (2). Our goal here will be both to describe the time

evolution of the average wealth, and to obtain upper bounds for the evolution of higher moments. The problem of the determination of a steady state for the equation is addressed in the next section. As we shall see, the reasonable and simple hypothesis of the constraint on post-collisional trade (the debt is not allowed) brings to the conclusion that in the general case the average wealth increases at least exponentially in time.

Let  $Q(f, f)$  denote the interaction integral,

$$Q(f, f)(w) = \int_{\mathbb{R}^2} \int_0^\infty (\beta_{(w', w'_*) \rightarrow (w, w_*)} \frac{1}{J} f(w') f(w'_*) - \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*)) dw_* d\eta d\eta_*. \quad (3)$$

Let  $\mathcal{M}_0$  the space of all probability measures in  $\mathbb{R}_+$  and by

$$\mathcal{M}_p = \left\{ \Theta \in \mathcal{M}_0 : \int_{\mathbb{R}_+} |\eta|^p \Theta(\eta) d\eta < +\infty, p \geq 0 \right\}, \quad (4)$$

the space of all Borel probability measures of finite momentum of order  $p$ , equipped with the topology of the weak convergence of the measures.

Let  $\mathcal{F}_p(\mathbb{R}_+)$ ,  $p > 1$  be the class of all real functions on  $\mathbb{R}_+$  such that  $g(0) = g'(0) = 0$ , and  $g^{(m)}(v)$  is Hölder continuous of order  $\delta$ ,

$$\|g^{(m)}\|_\delta = \sup_{v \neq w} \frac{|g^{(m)}(v) - g^{(m)}(w)|}{|v - w|^\delta} < \infty, \quad (5)$$

the integer  $m$  and the number  $0 < \delta \leq 1$  are such that  $m + \delta = p$ , and  $g^{(m)}$  denotes the  $m$ -th derivative of  $g$ .

In the rest of the paper we will assume that the symmetric probability density  $\Theta(\eta)$  which characterizes the transition rate belongs to  $\mathcal{M}_{2+\alpha}$ , for some  $\alpha > 0$ . Moreover, to simplify computations, we assume that this density is obtained from a given random variable  $Y$  with zero mean and unit variance, that belongs to  $\mathcal{M}_{2+\alpha}$ . Thus,  $\Theta$  of variance  $\sigma^2$  is the density of  $\sigma Y$ . By this assumption, we can easily obtain the dependence on  $\sigma$  of the moments of  $\Theta$ . In fact, for any  $p > 2$  such that the  $p$ -th moment of  $Y$  exists,

$$\int_{\mathbb{R}} |\eta|^p \Theta(\eta) d\eta = E(|\sigma Y|^p) = \sigma^p E(|Y|^p).$$

By a weak solution of the initial value problem for equation (2), corresponding to the initial probability density  $f_0(w) \in \mathcal{M}_p$ ,  $p > 1$  we shall mean any probability density  $f \in C^1(\mathbb{R}_+, \mathcal{M}_p)$  satisfying the weak form of the equation

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}_+} \phi(w) f(w, t) dw &= (Q(f, f), \phi) = \\ \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*) (\phi(w') - \phi(w)) dw_* dw d\eta d\eta_*, \end{aligned} \quad (6)$$

for  $t > 0$  and all  $\phi \in \mathcal{F}_p(\mathbb{R}_+)$ , and such that for all  $\phi \in \mathcal{F}_p(\mathbb{R}_+)$

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}_+} \phi(w) f(w, t) dw = \int_{\mathbb{R}_+} \phi(w) f_0(w) dw. \quad (7)$$

The form (6) is easier to handle, and it is the starting point to reckon the evolution of macroscopic quantities (moments). By symmetry reasons, we can alternatively use the symmetric form

$$\begin{aligned} \frac{d}{dt} \int_0^\infty f(w) \phi(w) dw &= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*) \\ &\quad (\phi(w') + \phi(w'_*) - \phi(w) - \phi(w_*)) dw_* dw d\eta d\eta_*. \end{aligned} \quad (8)$$

Existence of a weak solution to the initial value problem for equation (2) can be easily obtained by using methods first applied to the Boltzmann equation [5]. The main difference here is that the classical Boltzmann  $H$ -theorem, which prevents formation of concentration, does not hold.

From (6) (or equivalently from (8)) conservation of the total number of agents is obtained for  $\phi(w) = 1$ , which represents the only conservation property satisfied by the system. The choice  $\phi(w) = w$  is of particular interest since it gives the time evolution of the average wealth. We have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}_+} w f(w, t) dw &= \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*) (\gamma(w_* - w) + \eta w) dw_* dw d\eta d\eta_* = \\ &\quad \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*) \gamma(w_* - w) dw_* dw d\eta d\eta_* + \\ &\quad \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*) \eta w dw_* dw d\eta d\eta_* = \\ &\quad \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \eta \Theta(\eta) \Theta(\eta_*) \Psi(w' \geq 0) \Psi(w'_* \geq 0) w f(w) f(w_*) dw_* dw d\eta d\eta_*. \end{aligned} \quad (9)$$

The previous equality follows from the fact that, by symmetry,

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*) \gamma(w_* - w) dw_* dw d\eta d\eta_* = 0.$$

Let us study into details the last integral in formula (9). By definition 1,

$$\Psi(w' \geq 0) = \Psi\left(\eta \geq -(1 - \gamma) - \gamma \frac{w_*}{w}\right), \quad \Psi(w'_* \geq 0) = \Psi\left(\eta_* \geq -(1 - \gamma) - \gamma \frac{w}{w_*}\right).$$

Since  $\Theta(\cdot)$  is a symmetric probability density, we have the bounds

$$\frac{1}{2} \leq \int_{\mathbb{R}} \Theta(\eta_*) \Psi\left(\eta_* \geq -(1 - \gamma) - \gamma \frac{w}{w_*}\right) d\eta_* \leq 1. \quad (10)$$

For the same reasons,

$$\int_{\mathbb{R}} \eta \Theta(\eta) \Psi\left(\eta \geq -(1 - \gamma) - \gamma \frac{w_*}{w}\right) d\eta = \int_{\mathbb{R}} \eta \Theta(\eta) \Psi\left(\eta \geq 1 - \gamma + \gamma \frac{w_*}{w}\right) d\eta > 0. \quad (11)$$

Bounds (10) and (11) show that the average wealth is increasing, unless  $w' \geq 0$  and  $w'_* \geq 0$  for all exchanges (in such case the average wealth would be conserved). This property enlightens the importance of the choice of the probability density function  $\Theta(\eta)$  in the model. If  $\Theta(\eta)$  has unbounded support, an accurate analysis of (9) permits to extract information on the rate of growth of the average wealth. To this aim, let us remark that

$$\Psi\left(\eta \geq 1 - \gamma + \gamma \frac{w_*}{w}\right) \leq \Psi(\eta \geq 1 - \gamma).$$

Hence, if the random variable  $X \in \mathcal{M}_{2+\alpha}$ , for some  $\alpha > 0$

$$\int_{\mathbb{R}} \eta \Theta(\eta) \Psi\left(\eta \geq 1 - \gamma + \gamma \frac{w_*}{w}\right) d\eta \leq \int_{\mathbb{R}} \eta \Theta(\eta) \Psi(\eta \geq 1 - \gamma) d\eta \leq \frac{E(X^{2+\alpha})}{(1 - \gamma)^{1+\alpha}}. \quad (12)$$

Thanks to (10) and (12), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}_+} w f(w, t) dw \leq \frac{E(X^{2+\alpha})}{(1 - \gamma)^{1+\alpha}} \int_{\mathbb{R}_+} w f(w) dw. \quad (13)$$

Moreover, by (10)

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \eta \Theta(\eta) \Theta(\eta_*) \Psi(w' \geq 0) \Psi(w'_* \geq 0) w f(w) f(w_*) dw_* dw d\eta d\eta_* \geq \\ & \frac{1}{2} \int_{\mathbb{R}_+^2} w f(w) f(w_*) dw_* dw \int_{\mathbb{R}} \eta \Theta(\eta) \Psi(\eta > 1 - \gamma + \gamma(w_*/w)) d\eta \geq \\ & \frac{1}{2} \int_{\mathbb{R}_+^2} \Psi(w_* < w) w f(w) f(w_*) dw_* dw \int_{\mathbb{R}} \eta \Theta(\eta) \Psi(\eta > 1 - \gamma + \gamma(w_*/w)) d\eta. \end{aligned} \quad (14)$$

Last, on the set  $\{w_* < w\}$ ,  $\Psi(\eta > 1 - \gamma + \gamma(w_*/w)) \geq \Psi(\eta > 1)$ . Moreover

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \Psi(w_* < w) w f(w) f(w_*) dw_* dw = \int_{\mathbb{R}_+^2} \Psi(w < w_*) w_* f(w) f(w_*) dw_* dw = \\ & \frac{1}{2} \int_{\mathbb{R}_+^2} [\Psi(w_* < w) w + \Psi(w < w_*) w_*] f(w) f(w_*) dw_* dw \geq \\ & \frac{1}{2} \int_{\mathbb{R}_+^2} [\Psi(w_* < w) w + \Psi(w < w_*) w] f(w) f(w_*) dw_* dw = \frac{1}{2} \int_{\mathbb{R}_+} w f(w) dw. \end{aligned} \quad (15)$$

Grouping together all the previous inequalities we conclude that the average wealth satisfies the inequalities

$$\frac{d}{dt} \int_{\mathbb{R}_+} w f(w, t) dw \geq \frac{1}{4} \int_{\mathbb{R}} \eta \Theta(\eta) \Psi(\eta > 1) d\eta \int_{\mathbb{R}_+} w f(w, t) dw. \quad (16)$$

If  $X$  is a random variable of density  $\Theta(\eta)$ , taking values on a interval  $(-a, a)$ , with  $a > 1$ ,

$$\int_{\mathbb{R}} \eta \Theta(\eta) \Psi(\eta > 1) d\eta = A > 0, \quad (17)$$

and (16) implies that the average wealth in the model increases at least exponentially with rate  $A/4$ .

Once more, we remark that, if the kernel  $\beta$  does not depend on the wealth variables, the kinetic model preserves the average wealth.

Similar bounds for moments of order higher than 2 can be obtained in a similar way. We will consider here only upper bounds, which will be used later on.

Let us fix  $\phi(w) = w^p$  for some  $p > 2$ . Using the same trick as in (9) we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}_+} w^p f(w, t) dw &= \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \beta_{(w, w_*) \rightarrow (w', w'_*)} f(w) f(w_*) (|w'|^p - w^p) dw_* dw d\eta d\eta_* = \\
&\int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) f(w) f(w_*) (|w'|^p - w^p) dw_* dw d\eta d\eta_* + \\
&- \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) (1 - \Psi(w' \geq 0) \Psi(w'_* \geq 0)) (|w'|^p - w^p) f(w) f(w_*) dw_* dw d\eta d\eta_* \leq \\
&\int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) f(w) f(w_*) (|w'|^p - w^p) dw_* dw d\eta d\eta_* + \\
&+ \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) (1 - \Psi(w' \geq 0) \Psi(w'_* \geq 0)) w^p f(w) f(w_*) dw_* dw d\eta d\eta_*. \quad (18)
\end{aligned}$$

Using again Markov inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) (1 - \Psi(w' \geq 0) \Psi(w'_* \geq 0)) w^p f(w) f(w_*) dw_* dw d\eta d\eta_* \leq \\
&\int_{\mathbb{R}^2} \Theta(\eta) \Theta(\eta_*) (\Psi(1 - \gamma + \eta < 0) + \Psi(1 - \gamma + \eta_* < 0)) d\eta d\eta_* \int_{\mathbb{R}_+} w^p f(w) dw \leq \\
&2 \frac{\sigma^{2+\alpha}}{(1 - \gamma)^{2+\alpha}} \int_{\mathbb{R}_+} w^p f(w) dw. \quad (19)
\end{aligned}$$

Moreover, we can write

$$|w'|^p = w^p + pw^{p-1}(w' - w) + \frac{1}{2}p(p-1)|\tilde{w}|^{p-2}(w' - w)^2,$$

where, for some  $0 \leq \theta \leq 1$

$$\tilde{w} = \theta w' + (1 - \theta)w.$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) f(w) f(w_*) (|w'|^p - w^p) dw_* dw d\eta d\eta_* = \\
&\int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) f(w) f(w_*) (pw^{p-1}(w' - w) + \frac{1}{2}p(p-1)|\tilde{w}|^{p-2}(w' - w)^2) dw_* dw d\eta d\eta_* \leq
\end{aligned}$$



$$\frac{1}{2}p(p-1) \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta)\Theta(\eta_*)f(w)f(w_*)|\tilde{w}|^{p-2}(w'-w)^2 dw_* dw d\eta d\eta_*. \quad (20)$$

In fact,

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta)\Theta(\eta_*)f(w)f(w_*)w^{p-1}(w'-w) dw_* dw d\eta d\eta_* = \\ & \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta)\Theta(\eta_*)f(w)f(w_*)w^{p-1}(\gamma(w_*-w) + \eta w) dw_* dw d\eta d\eta_* = \\ & \gamma \int_{\mathbb{R}_+^2} f(w)f(w_*)w^{p-1}(w_*-w) dw_* dw \leq 0. \end{aligned} \quad (21)$$

The last bound follows by Hölder inequality since,

$$\begin{aligned} & \int_{\mathbb{R}_+^2} f(w)f(w_*)w^{p-1}w_* dw_* dw = \int_{\mathbb{R}_+} f(w)w^{p-1} dw \int_{\mathbb{R}_+} f(w_*)w_* dw_* \leq \\ & \left( \int_{\mathbb{R}_+} f(w)w^p dw \right)^{(p-1)/p} \left( \int_{\mathbb{R}_+} f(w_*)w_*^p dw_* \right)^{1/p} = \int_{\mathbb{R}_+} f(w)w^p dw. \end{aligned}$$

Finally, from

$$|\tilde{w}|^{p-2} = |(1-\theta\gamma)w + \theta\gamma w_* + \theta\eta w|^{p-2} \leq c_p [w^{p-2} + \gamma^{p-2}w_*^{p-2} + |\eta|^{p-2}w^{p-2}],$$

we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta)\Theta(\eta_*)f(w)f(w_*)|\tilde{w}|^{p-2}(w'-w)^2 dw_* dw d\eta d\eta_* \leq \\ & c_p \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta)\Theta(\eta_*)f(w)f(w_*) [w^{p-2} + \gamma^{p-2}w_*^{p-2} + |\eta|^{p-2}w^{p-2}] \cdot \\ & \cdot [\gamma^2(w_*-w)^2 + \eta^2w^2 + 2\eta w(w_*-w)] dw_* dw d\eta d\eta_* \leq \\ & c_p A_p(\sigma, \gamma) \int_{\mathbb{R}_+} w^p f(w) dw, \end{aligned} \quad (22)$$

where

$$A_p(\sigma, \gamma) = \sigma^2(1 + \gamma^{p-2}) + \sigma^p + 2\gamma^2 + 2\gamma^p + 2\gamma^2\sigma^{p-2}. \quad (23)$$

Grouping all estimates together, we finally obtain the bound for the moments

$$\frac{d}{dt} \int_{\mathbb{R}_+} w^p f(w, t) dw \leq \left( \frac{1}{2}p(p-1)c_p A_p(\sigma, \gamma) + 2 \frac{\sigma^{2+\alpha}}{(1-\gamma)^{2+\alpha}} \right) \int_{\mathbb{R}_+} w^p f(w) dw. \quad (24)$$

we proved

**Theorem 3.1** *Let the probability density  $f_0 \in \mathcal{M}_p$ , where  $p = 2 + \delta$  for some  $\delta > 0$ , and let the symmetric random variable  $Y$  which characterizes the kernel have a density  $\Theta$  in  $\mathcal{M}_{2+\alpha}$ , with  $\alpha > \delta$ . Then, if  $\Theta$  has unbounded support, the average wealth is increasing with time at least exponentially*

$$\int_{\mathbb{R}_+} w f(w, t) dw \geq \exp \left\{ \frac{A}{4} t \right\} \int_{\mathbb{R}_+} w f_0(w) dw, \quad (25)$$

where the constant  $A$  is given by (17). Moreover, the average wealth does not increase more than exponentially in time

$$\int_{\mathbb{R}_+} w f(w, t) dw \leq \exp \left\{ \frac{\sigma^{2+\alpha}}{(1-\gamma)^{1+\alpha}} t \right\} \int_{\mathbb{R}_+} w f_0(w) dw. \quad (26)$$

Similarly, higher order moments does not increase more than exponentially, and the bound

$$\int_{\mathbb{R}_+} w^q f(w, t) dw \exp \left\{ \left( \frac{1}{2} p(p-1) c_p A_p(\sigma, \gamma) + 2 \frac{\sigma^{2+\alpha}}{(1-\gamma)^{2+\alpha}} \right) t \right\} \int_{\mathbb{R}_+} w^q f_0(w) dw, \quad (27)$$

holds for  $q \leq p$ .

## 4 The continuous trading limit

The previous analysis shows that in general it is quite difficult both to study in details the evolution of the wealth function, and to describe the asymptotic behavior. For a general kernel one has in addition to take into account the exponential growth of the average wealth. In this case, the only way to get information on the properties of the solution for large time relies in the scaling of the solution with respect to its average wealth. This corresponds to study the behavior of  $\tilde{f}(w, t) = m(t) f(m(t)w, t)$  where  $m(t)$  is the (variable in time) average amount of money in the system. Note that the density  $\tilde{f}$  is normalized to have the first moment equal to one. We present in Section 5 various numerical tests which show in particular the exponential growth of the average wealth, and the formation of heavy tails. As is usual in kinetic theory, however, particular asymptotics of the equation result in simplified models (generally of Fokker-Planck type), for which it is relatively easier to find steady states, and to prove their stability. In order to give a physical basis to these asymptotics, it is relevant to discuss in some detail the interaction rule (1). To skip inessential difficulties, that as we will see later on do not change the structure of the limit equation, we suppose here that the random variables take values on the set  $(-(1-\gamma), 1-\gamma)$ . As remarked in the previous section, in this case both  $w' \geq 0$  and  $w'_* \geq 0$ , and the kernel  $\beta$  does not depend on the wealth variables  $(w, w_*)$ . Let us denote by  $E(X)$  the mathematical expectation of the random variable  $X$ . Then the following properties follow from (1)

$$E[w' + w'_*] = w + w_*, \quad E[w' - w'_*] = (1 - 2\gamma)(w - w_*). \quad (28)$$

The first equality in (28) describes the property of mean conservation of wealth. The second refers to the tendency of the trade to decrease (in mean) the distance between

wealths after the interaction. This tendency is a universal consequence of the rule (1), in that it holds whatever distribution one assigns to  $\Theta$ , namely to the random variable which accounts for the effects of the market returns in an open economy. This universality is false for the first equality, which in general has to be substituted by the inequality

$$E[w' + w'_*] \geq w + w_*. \quad (29)$$

Hence, in general the effects of the market returns in an open economy account for an increasing of the microscopic mean wealth.

The second property in (28) is analogous to the similar one that holds in a collision between molecules in a granular gas. There the quantity  $e = 2\gamma$  is called "coefficient of restitution", and describes the peculiar fact that energy is dissipated [25]. If we want to consider the situation in which most of the trades corresponds to a very small exchange of money ( $\gamma \rightarrow 0$ ), and at the same time we want to maintain both properties (28) at a macroscopic level, we have to pretend that

$$\int_{\mathbb{R}_+^2} (w + w_*) f(w) f(w_*) dw dw_* = 2 \int_{\mathbb{R}_+} w f(w) dw = 2m(t) \quad (30)$$

remains constant, while

$$\int_{\mathbb{R}_+^2} (w - w_*)^2 f(w) f(w_*) dw dw_* = A_f(t) \quad (31)$$

is varying with time, and decays to zero when the market returns are not present (i.e.  $\sigma = 0$ ).

When the kernel  $\beta$  does not depend on the wealth variables, (9) implies that  $m(t) = m_0$ . Moreover, explicit computations show that  $A_f(t)$  varies with law

$$\frac{dA_f(t)}{dt} = -4 \left[ \gamma(1 - \gamma) - \frac{\sigma^2}{2} \right] A_f(t) + 2\sigma^2 m^2. \quad (32)$$

Hence, when  $\sigma = 0$ ,  $A_f(t)$  decays exponentially to zero. It is now evident that (30) is satisfied for any value of  $\gamma$  and  $\sigma$ , while (31) loses its meaning as  $\gamma$  and  $\sigma$  tend to zero. Of course, one can rewrite (32) as

$$\frac{dA_f(t)}{dt} = -4\gamma \left[ \left( 1 - \gamma - \frac{\sigma^2}{2\gamma} \right) A_f(t) - \frac{\sigma^2}{2\gamma} m^2 \right]. \quad (33)$$

Hence, if we set

$$\tau = \gamma t, \quad g(w, \tau) = f(w, t), \quad (34)$$

which implies  $f_0(w) = g_0(w)$ , we obtain

$$\frac{dA_g(\tau)}{d\tau} = -4 \left( 1 - \gamma - \frac{\sigma^2}{2\gamma} \right) A_g(\tau) + 2 \frac{\sigma^2}{\gamma} m^2. \quad (35)$$

Letting now both  $\gamma \rightarrow 0$  and  $\sigma \rightarrow 0$  in such a way that  $\sigma^2/\gamma = \lambda$ , (35) becomes in the limit

$$\frac{dA_g(\tau)}{d\tau} = -(4 - 2\lambda) A_g(\tau) + 2\lambda m^2. \quad (36)$$

This formal argument shows that the value of the ratio  $\sigma^2/\gamma$  is of paramount importance to get asymptotics which maintain memory of the microscopic interactions. In particular, while for  $\lambda < 2$   $A_g(\tau)$  converges to the finite value  $\bar{A}_g = \lambda m^2/(2 - \lambda)$ ,  $A_g(\tau)$  diverges to infinity as time goes to infinity when  $\lambda \geq 2$ .

In the remaining of this section, we shall present a rigorous derivation of a Fokker-Planck model from the Boltzmann equation for the wealth function  $g(w, \tau)$ , when both  $\gamma \rightarrow 0$  and  $\sigma \rightarrow 0$  in such a way that  $\sigma^2/\gamma \rightarrow \lambda$ . This derivation, which is similar to the quasi-elastic limit of granular gases, is of major relevance for the study of the asymptotic equilibrium states of the kinetic model. First, we show how the Fokker-Planck equation comes out for the simpler case of a kernel which does not depend on the wealth variables. Second, we extend the result to a general kernel.

The scaled density  $g(v, \tau) = f(v, t)$  satisfies the weak form

$$\frac{d}{d\tau} \int_0^\infty g\phi dw = \frac{1}{\gamma} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta)\Theta(\eta_*)g(w)g(w_*)(\phi(w') - \phi(w))dw_*dw d\eta d\eta_*. \quad (37)$$

Given  $0 < \delta \leq \alpha$ , let us set  $\phi \in \mathcal{F}_{2+\delta}(\mathbb{R}_+)$ .

By (1),

$$w' - w = \gamma(w_* - w) + \eta w.$$

Then, if we use a second order Taylor expansion of  $\phi$  around  $w$

$$\phi(w') - \phi(w) = (\gamma(w_* - w) + \eta w)\phi'(w) + \frac{1}{2}(\gamma(w_* - w) + \eta w)^2\phi''(\tilde{w}),$$

where, for some  $0 \leq \theta \leq 1$

$$\tilde{w} = \theta w' + (1 - \theta)w.$$

Inserting this expansion in the collision operator, we get

$$\begin{aligned} \frac{d}{d\tau} \int_0^\infty g\phi dw &= \frac{1}{\gamma} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta)\Theta(\eta_*)[(\gamma(w_* - w) + \eta w)\phi'(w) + \\ &+ \frac{1}{2}(\gamma(w_* - w) + \eta w)^2\phi''(\tilde{w})]g(w_*)g(w)dw_*dw d\eta d\eta_* + R(\gamma, \sigma), \end{aligned} \quad (38)$$

where

$$\begin{aligned} R(\gamma, \sigma) &= \frac{1}{2\gamma} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta)\Theta(\eta_*)(\gamma(w_* - w) + \eta w)^2 \cdot \\ &\cdot (\phi''(\tilde{w}) - \phi''(w))g(w_*)g(w)dw_*dw d\eta d\eta_* \end{aligned}$$

Since  $\phi \in \mathcal{F}_{2+\delta}(\mathbb{R}_+)$ , and  $|\tilde{w} - w| = \theta|w' - w|$

$$|\phi''(\tilde{w}) - \phi''(w)| \leq \|\phi''\|_\delta |\tilde{w} - w|^\delta \leq \|\phi''\|_\delta |w' - w|^\delta. \quad (39)$$

Hence

$$|R(\gamma, \sigma)| \leq \frac{\|\phi''\|_\delta}{2\gamma} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) \cdot |\gamma(w_* - w) + \eta w|^{2+\delta} g(w_*) g(w) dw_* dw d\eta d\eta_*$$

By virtue of the inequality

$$|\gamma(w_* - w) + \eta w|^{2+\delta} \leq 4^{1+\delta} \left( |\gamma w_*|^{2+\delta} + |\gamma w|^{2+\delta} + |\eta w|^{2+\delta} \right),$$

we finally obtain the bound

$$|R(\gamma, \sigma)| \leq 2^{1+2\delta} \|\phi''\|_\delta \left( 2\gamma^{1+\delta} + \frac{1}{\gamma} \int_{\mathbb{R}} |\eta|^{2+\delta} \Theta(\eta) d\eta \right) \int_{\mathbb{R}_+} w^{2+\delta} g(w) dw \quad (40)$$

Since  $\Theta$  is a probability density with zero mean and  $\lambda\gamma$  variance, and  $\Theta$  belongs to  $\mathcal{M}_{2+\alpha}$ , for  $\alpha > \delta$ ,

$$\int_{\mathbb{R}} |\eta|^{2+\delta} \Theta(\eta) d\eta = E \left( \left| \sqrt{\lambda\gamma} Y \right|^{2+\delta} \right) = (\lambda\gamma)^{1+\delta/2} E \left( |Y|^{2+\delta} \right),$$

and  $E \left( |Y|^{2+\delta} \right)$  is bounded. Using this equality into (40) one shows that  $R(\gamma, \sigma)$  converges to zero as  $\gamma \rightarrow 0$ , if  $\int_{\mathbb{R}_+} w^{2+\delta} g(w, \tau)$  remains bounded at any fixed time  $\tau > 0$ , provided the same bound holds at time  $\tau = 0$ . By virtue of bound (24), taking  $\sigma^2 = \lambda\gamma$ , we obtain

$$\begin{aligned} & \frac{d}{d\tau} \int_{\mathbb{R}_+} w^{2+\delta} g(w, \tau) dw \leq \\ & \frac{1}{\gamma} \left( \frac{1}{2} p(p-1) c_{2+\delta} A_{2+\delta}(\sqrt{\lambda\gamma}, \gamma) + 2 \frac{(\lambda\gamma)^{1+\alpha/2}}{(1-\gamma)^{2+\alpha}} \right) \int_{\mathbb{R}_+} w^p g(w, \tau) dw. \end{aligned} \quad (41)$$

Since the lower order term in  $A_{2+\delta}(\sqrt{\lambda\gamma}, \gamma)$  is  $\lambda\gamma(1-\gamma)^\delta$ , it follows that the boundedness of the moment holds independently of the value of  $\gamma$ . Therefore, at any fixed time  $\tau$

$$|R(\gamma, \sigma)|(\tau) \leq 2^{1+2\delta} \|\phi''\|_\delta \left( 2\gamma^{1+\delta} + \lambda^{2+\delta} \gamma^{\delta/2} E \left( |Y|^{2+\delta} \right) \right) C\tau \int_{\mathbb{R}_+} w^{2+\delta} g_0(w) dw \quad (42)$$

Hence, the remainder  $R(\gamma, \sigma)$  converges to zero as both  $\gamma$  and  $\sigma$  converge to zero, in such a way that  $\sigma^2 = \lambda\gamma$ . Within the same scaling,

$$\begin{aligned} & \lim_{\gamma \rightarrow 0} \frac{1}{\gamma} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) [(\gamma(w_* - w) + \eta w) \phi'(w) + \\ & + \frac{1}{2} (\gamma(w_* - w) + \eta w)^2 \phi''(w)] g(w_*) g(w) dw_* dw d\eta d\eta_* = \\ & \int_{\mathbb{R}_+} \left[ (m - w) \phi'(w) + \frac{\lambda}{2} w^2 \phi''(w) \right] g(w) dw \end{aligned} \quad (43)$$

The right-hand side of (43) is nothing but the weak form of the Fokker-Planck equation

$$\frac{\partial g}{\partial \tau} = \frac{\lambda}{2} \frac{\partial^2}{\partial w^2} (w^2 g) + \frac{\partial}{\partial w} ((w - m)g). \quad (44)$$

The limit Fokker-Planck equation can be rewritten as

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial w} \left[ \left( \left(1 + \frac{\lambda}{2}\right)w - m \right) g + \frac{\lambda}{2} w \frac{\partial}{\partial w} (wg) \right]. \quad (45)$$

The general case of a wealth-variables depending rate function can be easily obtained from the previous computations. With respect to formula (38) we have to consider two more terms in the remainder. The first one comes out from the possibility to have negative wealth variables as outcome of the interaction. This term reads

$$R_1(\gamma, \sigma) = -\frac{1}{\gamma} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+^2} \Theta(\eta) \Theta(\eta_*) \cdot (1 - \Psi(w' \geq 0) \Psi(w'_* \geq 0)) (\phi(w') - \phi(w)) g(w) g(w_*) dw_* dw d\eta d\eta_*. \quad (46)$$

The integrand in (46) coincides with the integrand in the right-hand side of (37), multiplied by the factor  $1 - \Psi(w' \geq 0) \Psi(w'_* \geq 0)$ . Proceeding as in (19) we obtain

$$\int_{\mathbb{R}} \eta^2 \Theta(\eta) d\eta = \sigma^2, \\ \int_{\{|\eta| > 1-\gamma\}} \eta^2 \Theta(\eta) d\eta \leq \frac{1}{(1-\gamma)^\alpha} \int_{\{|\eta| > 1-\gamma\}} \eta^{2+\alpha} \Theta(\eta) d\eta \leq \sigma^{2+\alpha} E(Y^{2+\alpha}).$$

Thus, we can use the same expansion following formula (37) to conclude that  $R_1(\gamma, \sigma)$  converges to zero as  $\gamma \rightarrow 0$ .

The second remainder we have to take into account comes out from the fact that in general the average wealth  $m(t)$  is not constant in time. Thus we have the additional term

$$R_2(\gamma, \sigma) = -\frac{1}{\gamma} \int_{\mathbb{R}^2} \Theta(\eta) \Theta(\eta_*) \left( \int_{\mathbb{R}_+} w_* g(w_*, \tau) dw_* - m \right) g(w, \tau) \phi'(w) dw_* dw d\eta d\eta_*. \quad (47)$$

On the other hand, thanks to (26)

$$\left| \int_{\mathbb{R}_+} w_* g(w_*, \tau) dw_* - m \right| \leq \left[ \exp \left\{ \frac{\sigma^{2+\alpha}}{(1-\gamma)^{1+\alpha}} \tau \right\} - 1 \right] m,$$

which implies the convergence to zero of the remainder as  $\gamma \rightarrow 0$ . Hence we proved

**Theorem 4.1** *Let the probability density  $f_0 \in \mathcal{M}_p$ , where  $p = 2 + \delta$  for some  $\delta > 0$ , and let the symmetric random variable  $Y$  which characterizes the kernel have a density in  $\mathcal{M}_{2+\alpha}$ , with  $\alpha > \delta$ . Then, as  $\gamma \rightarrow 0$ ,  $\sigma \rightarrow 0$  in such a way that  $\sigma^2 = \lambda\gamma$  the weak solution to the Boltzmann equation for the scaled density  $g_\gamma(v, \tau) = f(v, t)$ , with  $\tau = \gamma t$  converges, up to extraction of a subsequence, to a probability density  $g(w, \tau)$ . This density is a weak solution of the Fokker-Planck equation (44), and it is such that the mean wealth is conserved.*

The stationary state of the Fokker-Planck equation can be directly computed and, by assuming for simplicity

$$m = \int_{\mathbb{R}_+} g_\infty(w) w dw = 1,$$

it can be written as

$$g_\infty(w) = \frac{(\mu - 1)^\mu}{\Gamma(\mu)} \frac{\exp\left(-\frac{\mu-1}{w}\right)}{w^{1+\mu}} \quad (48)$$

where

$$\mu = 1 + \frac{2}{\lambda} > 1.$$

Therefore the stationary distribution exhibits a Pareto power law tail for large  $w$ 's.

Note that this equation is essentially the same Fokker-Planck equation derived from a Lotka-Volterra interaction in [4, 24, 17].

## 5 Numerical results

In this section, we shall present some numerical test and we shall compare the stationary results obtained by using Monte Carlo simulation of the kinetic model with the stationary state of the Fokker-Planck model. We start from a situation where all individuals shares the same money. At each iteration, we choose randomly two individual. Then, the parameter  $\eta$  is chosen accordingly to a normal centered law (one for each individual) and the trade is performed if it is admissible i.e. when the new money of each individual remain positive. We report the results for the density  $\tilde{f}(w, t) = m(t)f(m(t)w, t)$  where  $m(t)$  is the (variable in time) average amount of money in the system. Note that the density  $\tilde{f}$  is normalized to have the first moment equal to one.

We use  $N = 2000$  individuals and perform several iterations until a stationary state is reached. The distribution is then averaged over the next 250 iterations.

We plot the distribution function for different values of  $(\gamma, \sigma^2) = \{(0.1, 0.2), (0.01, 0.02)\}$ .

The values for  $\gamma$  and  $\sigma^2$  are such that we have a fixed  $\lambda = 2$  corresponding to a coefficient  $\mu = 2$  which is a realistic value for the income distribution of capitalistic economies [4, 17]. As prescribed from our theoretical analysis we observe that the equilibrium distribution converges toward the function  $f_\infty$  as both  $\gamma$  and  $\sigma^2$  go to 0, with  $\sigma^2/\gamma = 2$  (see Figures 1 and 2).

The corresponding behaviors of the average amount of money obtained with  $N = 10000$  agents are given in Figure 3. Note that, as expected, asymptotically the exponential growth vanishes and the model preserves the average amount of money.

## 6 Conclusions

We introduced and discussed a nonlinear kinetic model for a simple market economy, which is based on binary exchanges of money and speculative trading. We showed that at suitably large times, in presence of a large number of trades in which agents exchange

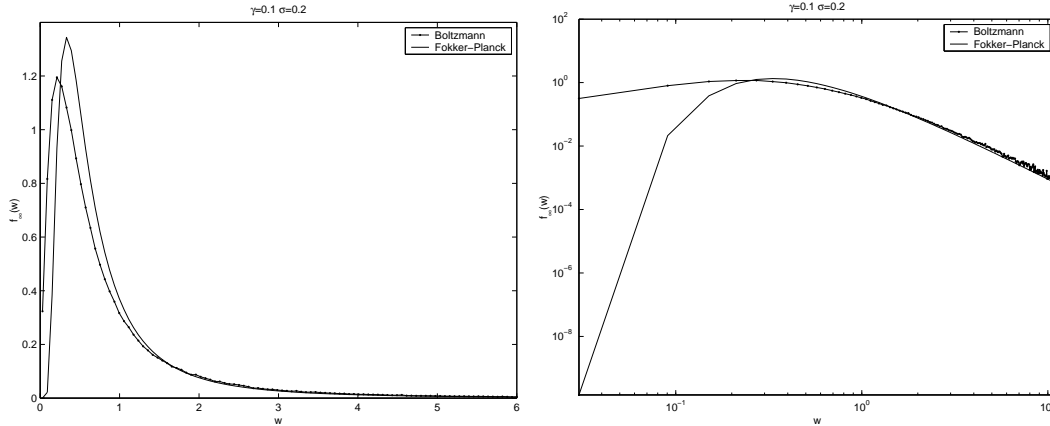


Figure 1: Asymptotic behavior of the Fokker-Planck model and the Boltzmann model for  $\mu = 2.0$ ,  $\gamma = 0.1$  and  $\sigma = 0.2$ . Figure on the right is in loglog-scale.

a small amount of money, the nonlinear kinetic equation is well-approximated by a linear Fokker-Planck type equation, which admits a stationary steady state with Pareto tails. Convergence towards a similar steady state is shown numerically for the solution of the kinetic model after a suitable normalization which guarantees the conservation of the mean wealth. Our analysis enlightens both the range of validity of the Fokker-Planck equation (44), and why in the continuous trading limit the mean wealth, which is increasing for the kinetic model, remains constant. The present derivation takes advantages of the deep similarities between the trade rule (1) and a molecular dissipative collision. However, it is expected that the asymptotic analysis discussed in this paper is not restricted to the trade rule (1), so that different microscopic interactions could be treated as well. Finally we remark that the formation of power law tails for large times in the present kinetic model is in accord with the overpopulated energy tails for the Boltzmann equation for a dissipative granular Maxwellian gas [3, 1, 9, 10]. Pushing further these analogies may help to clarify many aspects of large-time behavior of market economies and financial markets.

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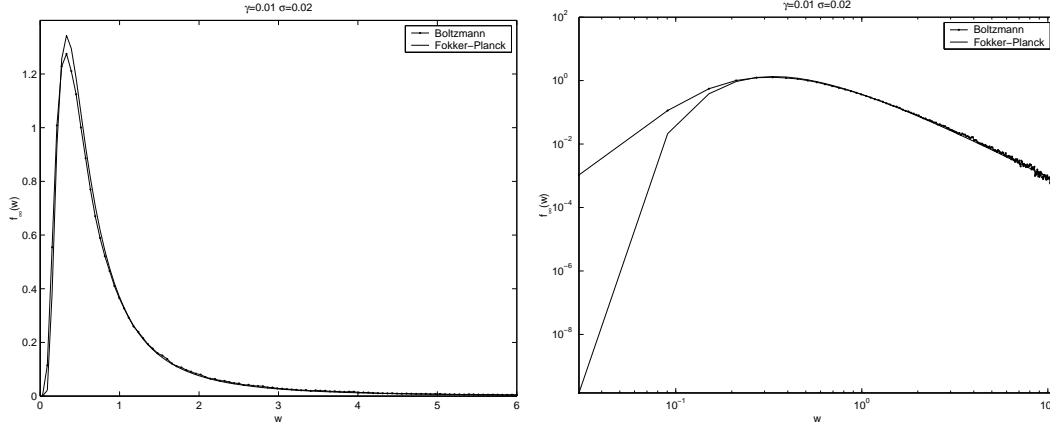


Figure 2: Asymptotic behavior of the Fokker-Planck model and the Boltzmann model for  $\mu = 2.0$ ,  $\gamma = 0.01$  and  $\sigma = 0.02$ . Figure on the right is in loglog-scale.

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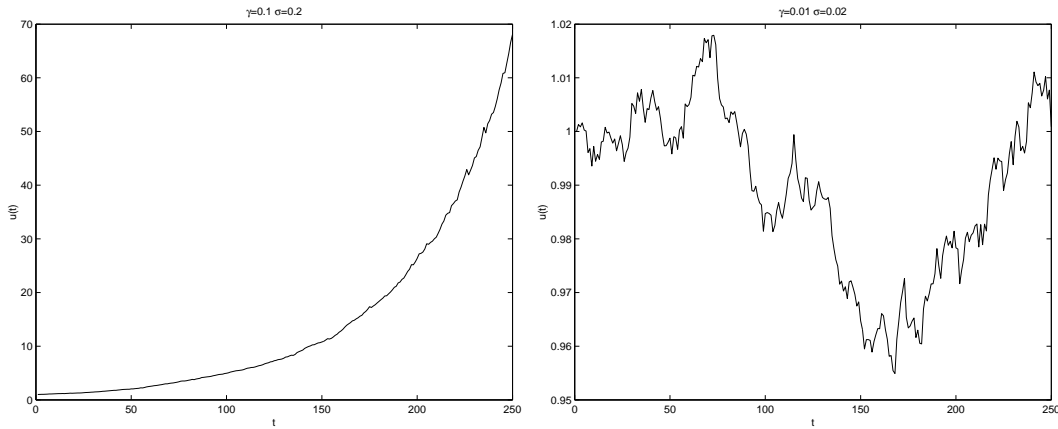


Figure 3: Growth of the average amount of money in the market corresponding to the data of the previous figure.

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