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On a Leibniz type formula for fractional derivatives

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Abstract. We prove that L^2-norm of fractional derivatives of product of two functions can be estimated by L^2 and L^1-norms of derivatives of the functions themselves.

1. Introduction

Fractional derivatives are becoming more and more important tool in modeling different physical phenomena. For instance, in mathematical finance, option pricing models based on jump processes (cf. [9]) give rise to linear partial differential equations with fractional terms. They can also be found in dislocation dynamics, hydrodynamics and molecular biology [10], semiconductors devices and explosives [14]. Many situations in viscoelasticity have been described with the help of fractional calculus [3, 11].

They seem to appear rather naturally in transport phenomena in surroundings with fractal dimension greater than one [16]. Indeed, let us consider a flow in a porous strip containing pockets (denoted by A on Figure 1). If we denote by u a concentration of the considered liquid in the strip, by u_A the concentration in the pocket, and assume that the transport is driven merely by the convection, then u and u_A satisfy the conservation laws:

$$\partial_t u + \partial_x f(u) = p_A(u - u_A), \tag{1}$$
$$\partial_t u_A = p_A(u_A - u), \tag{2}$$

for an appropriate flux function f, and the constant p_A which represent a probability that an elementary quantity of liquid will enter or leave pocket.

Remark that we can explicitly solve (2), i.e. $u_A = f_A \star u$, for appropriate kernel f_A. After substitution of such obtained u_A in (1), we obtain a single equation to solve.

If we have many pockets as in Figure 2 (which actually means that fractal dimension of the ”coast” is not one; e.g. [4]), on the right-hand side of equation (1) we will have the sum of the form

$$\sum_{k=1}^{n} p_{A_k}(u_{A_k} - u), \tag{3}$$

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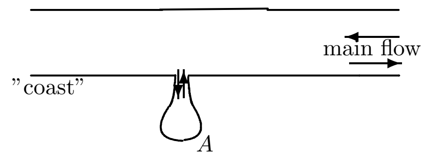

Figure 1: Flow along an irregular coast.

in which each term corresponds to a pocket. If we denote $p_{A_k} = p(A_k)$, we see that (3) can be considered as an integral sum for a function p. Appropriate form of the function p will lead to the fractional derivative of the unknown function u since one of equivalent definitions of fractional derivatives is given via appropriate integrals (see e.g. [5]). A concrete situation of such a model can be found in [7].

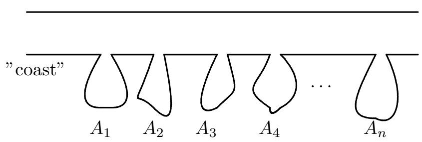

Figure 2: Example of a coast with many pockets denoted by A_i, $i = 1, \dots, n$. Paramount examples are e.g. Adriatic or Norwegian coast where our pockets are actually bays or fjords.

Practical importance of fractional derivatives also led to many theoretical studies (e.g. [1, 8, 13, 15]). As expected, it was noticed that if a fractional partial differential equation contains nonlinear terms [2], then a solution to such equation is non-unique. To gain a unique physically relevant solution, one needs to introduce appropriate physical conditions. They are usually called entropy admissibility conditions [12], and they were adapted for fractional equations in [1]. In the mentioned work, equations were linear with respect to its part which is under fractional derivatives. On the other hand, if an equation contains nonlinear terms under a fractional derivative (see e.g. [6, 15]), then we are not able to derive appropriate admissibility conditions. The reason for this lies in a fact that the Leibnitz product rule does not hold for fractional derivatives. In the current contribution, we make a step forward toward a satisfactory generalization of the product rule.

2. Definition of the fractional derivative and the Leibnitz type rule

There are many definitions of fractional derivatives. In the paper, we shall work with the one given via the Fourier transform since it seems the easiest to handle with from the viewpoint of the current contribution.

Denote by $\mathcal{F}$ and $\overline{\mathcal{F}}$ the Fourier transform and the inverse Fourier transform, respectively, and denote by $\mathcal{M}_d$ the space of functions $\phi : \mathbf{R}^d \to \mathbf{C}$ which satisfy $(1+|x|^2)^{-k/2}\phi \in L^1(\mathbf{R}^d)$ for some $k \in \mathbf{N}_0$. Next, we give the definition of the fractional order derivatives.

Definition 2.1. Let $\phi \in \mathcal{M}_d$ It is said that ϕ has r-th partial fractional derivative of order $\kappa \in \mathbf{R}$ in $L^\infty(\mathbf{R}^d)$ if $\bar{\mathcal{F}}(\xi_r^\kappa \mathcal{F}(\phi)) \in L^\infty(\mathbf{R}^d)$ for every $r \in \{1, ..., d\}$ ($\xi = (\xi_1, \dots, \xi_d) \in \mathbf{R}^d$) and we write

$$D^\kappa_{x_r}\phi(x) = \bar{\mathcal{F}}((i\xi_r)^\kappa \mathcal{F}(\phi))(x), \;\; x \in \mathbf{R}^d, \; r = 1, ..., d,$$

where $i^\kappa := e^{\frac{i\pi\kappa}{2}}$.

Next two statements will provide an L^2-estimate for the fractional derivative of a product of functions. Such estimates can be derived from existing generalization of the Leibnitz rule (e.g. [17]), but the latter holds only for analytic functions. In the sequel, we have much weaker demands on the involved functions.

Theorem 2.2. *Let $v \in C_c^\infty(\mathbf{R}^d)$, $u \in L^2 \cap L^1(\mathbf{R}^d)$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, and $k \in \mathbf{N}$ such that $\alpha - k > 0$ and $\alpha - k - 1 < 0$. Assume that $D^{\alpha-m}_{x_i} u \in L^2 \cap L^1(\mathbf{R}^d)$ for every $i \in \{1, \dots, d\}$ and $m = 0, 1, \dots, k+1$. Then there exists $C > 0$ such that for every $M > 0$ there holds*

$$\begin{aligned} \|D^\alpha_{x_i}(uv)\|_2^2 \leq C\Big(&\|\sum_{m=0}^{k} D^m_{x_i} v D^{\alpha-m}_{x_i} u\|_2^2 + \|D^{\alpha-k}_{x_i} u\|_2^2\, \|\xi_i^k \hat{v}\|_1^2 \\ &+ M^{d+\alpha-k-1}\|u\|_1^2\, \|D^{k+1}_{x_i} v\|_2^2 + M^{2(\alpha-k-1)}\|u\|_2^2\, \|\xi_i^{k+1}\hat{v}\|_1^2\Big), \end{aligned} \tag{4}$$

where (here and in the sequel) $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbf{R}^d)}$.

Remark 2.3. If $\alpha \in \mathbf{N}$ then there exists $C > 0$ depending on d and α such that

$$\|D^\alpha_{x_i}(uv)\|^2_{L^2(\mathbf{R}^d)} \leq C \sum_{m=0}^{k} \|D^m_{x_i} v D^{\alpha-m}_{x_i} u\|^2_{L^2(\mathbf{R}^d)} \tag{5}$$

The latter directly follows from the classical Leibnitz rule for the derivative of products.

Proof. We have

$$\begin{aligned} \mathcal{F}(D^\alpha_{x_j}(uv)) &= (i\xi_j)^\alpha \mathcal{F}(uv) = (i\xi_j)^\alpha \mathcal{F}(u) \star \mathcal{F}(v) = \int_\eta (i\xi_j)^\alpha \hat{u}(\xi - \eta)\hat{v}(\eta)d\eta \\ &= i^\alpha \int_\eta \left(\xi_j^\alpha - (\xi_j - \eta_j)^\alpha\right) \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta + i^\alpha \int_\eta (\xi_j - \eta_j)^\alpha \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta \\ &= i^\alpha \int_\eta \left(\xi_j^\alpha - (\xi_j - \eta_j)^\alpha\right) \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta + \mathcal{F}(v D^\alpha_{x_j}(u)). \end{aligned} \tag{6}$$

Let $n \in \mathbf{N}$. By the Taylor formula,

$$\begin{aligned} \xi_j^\alpha - (\xi_j - \eta_j)^\alpha =& \alpha(\xi_j - \eta_j)^{\alpha-1}\eta_j + \alpha(\alpha-1)(\xi_j-\eta_j)^{\alpha-2}\frac{\eta_j^2}{2} + \dots \\ &+ \frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}\tilde{\xi}^{\alpha-n}\eta_j^n, \end{aligned} \tag{7}$$

where $\tilde{\xi}$ belongs to the interval with the endpoints ξ_j and $\xi_j - \eta_j$.

From (6) and (7), we conclude

$$\begin{aligned} \mathcal{F}(D^\alpha_{x_j}(uv)) = \mathcal{F}(vD^\alpha_{x_j}(u)) &+ \sum_{m=1}^{n-1} \frac{\alpha(\alpha-1)\dots(\alpha-m)}{m!}\mathcal{F}(D^{\alpha-m}_{x_j}(u) D^m_{x_j}(v)) \\ &+ i^\alpha \int_\eta \frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}\tilde{\xi}^{\alpha-n}\eta_j^n \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta. \end{aligned} \tag{8}$$

Consider the L^2-norm of the expression $D^\alpha_{x_j}(uv)$. The Plancherel formula and (8) with $n = k$, imply that there exists $C_1 > 0$ such that

$$\|D^\alpha_{x_j}(uv)\|^2_{L^2(\mathbf{R}^d)} = \int_\xi |\xi_j^\alpha \mathcal{F}(uv)(\xi)|^2 d\xi \tag{9}$$

$$\leq C_1\Big(\|\sum_{m=0}^{k-1} D^{\alpha-m}_{x_j}(u) D^m_{x_j}(v)\|_{L^2(\mathbf{R}^d)} + \int_\xi |\int_\eta \tilde{\xi}^{\alpha-k}\eta_j^k \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta|^2 d\xi\Big).$$

Let us consider the last term in (9). We have

$$\int_\xi |\int_\eta \tilde{\xi}^{\alpha-k}\eta_j^k \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta|^2 d\xi \tag{10}$$

$$= \int_\xi \Big|\Big(\int_{|\xi_j-\eta_j|>|\xi_j|} + \int_{|\xi_j-\eta_j|<|\xi_j|}\Big) \tilde{\xi}^{\alpha-k}\eta_j^k \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta|^2 d\xi$$

$$\leq 2\int_\xi |\int_{|\xi_j-\eta_j|>|\xi_j|} \tilde{\xi}^{\alpha-k}\eta_j^k \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta|^2 d\xi$$

$$+ 2\int_\xi |\int_{|\xi_j-\eta_j|<|\xi_j|} \tilde{\xi}^{\alpha-k}\eta_j^k \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta|^2 d\xi.$$

For the first summand on the right hand side of (10) we use the Young inequality for the convolution and $|\tilde{\xi}|^{\alpha-k} < |\xi_j - \eta_j|^{\alpha-k}$. This implies

$$\int_\xi |\int_{|\xi_j-\eta_j|>|\tilde{\xi}|>|\xi_j|} \tilde{\xi}^{\alpha-k}\eta_j^k \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta|^2 d\xi \tag{11}$$

$$\leq \int_\xi \Big| \int_\eta |\xi_j-\eta_j|^{\alpha-k} \cdot |\eta_j|^k \cdot |\hat{u}(\xi-\eta)| \cdot |\hat{v}(\eta)| d\eta\Big|^2 d\xi$$

$$= \int_\xi \Big(|\xi_j^{\alpha-k}\hat{u}| \star |\xi_j^k \hat{v}|\Big)^2 d\xi \leq \|D^{\alpha-k}_{x_j}u\|_2^2 \|\xi_j^k \hat{v}\|_1^2.$$

For the second summand on the right hand side of (10), we first notice that

$$\tilde{\xi}^{\alpha-k}\eta_j^k = (\xi_j - \eta_j)^{\alpha-k}\eta_j^k + \frac{\alpha-k-1}{k+1}\bar{\xi}^{\alpha-k-1}\eta_j^{k+1}, \tag{12}$$

for some $\bar{\xi}$ belonging to the interval with the endpoints $\xi_j - \eta_j$ and ξ_j. The latter relation follows by subtracting (8) when $n = k$ from (8) with $n = k+1$.

Denote by

$$\chi(\xi) = \begin{cases} 1, & |\xi| < M \\ 0, & |\xi| \geq M \end{cases}, \quad \xi \in \mathbf{R}^d.$$

We will use below that $\alpha - k - 1 < 0$ and $|\bar{\xi}| > |\xi_j - \eta_j|$ imply $|\bar{\xi}|^{\alpha-k-1} < |\xi_j - \eta_j|^{\alpha-k-1}$. We have

$$\begin{aligned}
&\int_\xi |\int_{|\xi_j-\eta_j|<\tilde{\xi}<|\xi_j|} \tilde{\xi}^{\alpha-k}\eta_j^k \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta|^2 d\xi \qquad (13)\\
&= \int_\xi \Big| \int_{|\xi_j-\eta_j|<\hat{\xi}<|\xi_j|} \left((\xi_j-\eta_j)^{\alpha-k}\eta_j^k + \frac{\alpha-k-1}{k+1}\bar{\xi}_j^{\alpha-k-1}\eta_j^{k+1}\right)\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta\Big|^2 d\xi\\
&\leq \int_\xi \Big| \int_\eta \left(|\xi_j-\eta_j|^{\alpha-k}|\eta_j|^k + \frac{|\alpha-k-1|}{k+1}|\xi_j-\eta_j|^{\alpha-k-1}|\eta_j|^{k+1}\right)|\hat{u}(\xi-\eta)\hat{v}(\eta)|d\eta\Big|^2 d\xi\\
&\leq 2\left(\int_\xi \left(|\xi_j^{\alpha-k}\hat{u}| \star |\xi_j^k\hat{v}|\right)^2 d\xi + \int_\xi \left(|\xi_j^{\alpha-k-1}\hat{u}| \star |\xi_j^{k+1}\hat{v}|\right)^2 d\xi\right)\\
&\leq 4\Big(\int_\xi \left(|\xi_j^{\alpha-k}\hat{u}| \star |\xi_j^k\hat{v}|\right)^2 d\xi + \int_\xi \left(|\xi_j^{\alpha-k-1}\chi\hat{u}| \star |\xi_j^{k+1}\hat{v}|\right)^2 d\xi\\
&\qquad\qquad + \int_\xi \left(|\xi_j^{\alpha-k-1}(1-\chi)\hat{u}| \star |\xi_j^{k+1}\hat{v}|\right)^2 d\xi\Big)\\
&= 4\Big(\|D_{x_j}^{\alpha-k}u\|_2^2\,\|{\xi_j}^k\hat{v}\|_1^2 + \|{\xi_j}^{\alpha-k-1}\chi\hat{u}\|_1^2\,\|D_{x_j}^{k+1}v\|_2^2 + \|{\xi_j}^{\alpha-k-1}(1-\chi)\hat{u}\|_2^2\,\|\xi_j^{k+1}\hat{v}\|_1^2\Big)
\end{aligned}$$

Finally, we want to estimate $\|{\xi_j}^{\alpha-k-1}\chi\hat{u}\|_1$ and $\|{\xi_j}^{\alpha-k-1}(1-\chi)\hat{u}\|_2$. The use of Hausdorff-Young inequality $\|\hat{u}\|_\infty \leq \|u\|_1$ implies that there exists $C_2 > 0$ depending on d such that

$$\|{\xi_j}^{\alpha-k-1}\chi\hat{u}\|_1 = \int_{\|\xi\|<M} {\xi_j}^{\alpha-k-1}\hat{u}d\xi \leq C_2 M^{d+\alpha-k-1}\|u\|_1. \tag{14}$$

Moreover, we have

$$\|{\xi_j}^{\alpha-k-1}(1-\chi)\hat{u}\|_2^2 = \int_{\|\xi\|>M} {\xi_j}^{2(\alpha-k-1)}|\hat{u}|^2 d\xi \leq M^{2(\alpha-k-1)}\|\hat{u}\|_2^2 = M^{2(\alpha-k-1)}\|u\|_2^2. \tag{15}$$

Inserting (15) and (14) in (13), and then such obtained (13) together with (11) and (10) in (9), we conclude that (4) holds. □

The following theorem is used to estimate the last term on the right-hand side of (4).

Theorem 2.4. *Let $u \in L^2 \cap L^1(\mathbf{R}^d)$, $\beta > 0$ and $\gamma \in [0,1)$. Assume $D_{\xi_j}^{\frac{d+\beta}{2}} u \in L^2(\mathbf{R}^d)$, $j = 1, \dots d$. Then, there exists $C > 0$ such that for any $M > 0$ and $k = 1, \dots, d$,*

$$\|\xi_k^{-\gamma}\hat{u}\|_{L^1(\mathbf{R}^d)} \leq C\left(M^{-\gamma-\beta/2}\sum_{j=1}^d \|D_{x_j}^{\frac{d+\beta}{2}}u\|_{L^2} + M^{d-\gamma}\|u\|_{L^1(\mathbf{R}^d)}\right). \tag{16}$$

Proof. We have

$$\|\xi_k^{-\gamma}\hat{u}\|_{L^1(\mathbf{R}^d)} = \int_{\mathbf{R}^d} |\xi_k^{-\gamma}\hat{u}(\xi)|d\xi = \left(\int_{|\xi|\geq M} + \int_{|\xi|\leq M}\right)|\xi_k^{-\gamma}\hat{u}(\xi)|d\xi. \tag{17}$$

Since $\|\hat{u}\|_{L^\infty(\mathbf{R}^d)} \leq \|u\|_{L^1(\mathbf{R}^d)}$, it follows that there exists $C_1 > 0$ such that

$$\int_{|\xi|\leq M} |\xi_k^{-\gamma}\hat{u}(\xi)|d\xi \leq C_1 M^{d-\gamma}\|u\|_{L^1(\mathbf{R}^d)}. \tag{18}$$

Further on, with suitable $C_2 > 0$,

$$\int_{|\xi|\geq M} |\xi_k^{-\gamma} \hat{u}(\xi)| d\xi = \int_{|\xi|\geq M} \xi_k^{-\gamma} |\prod_{j=1}^{d} \xi_j^{-\frac{1}{2}-\frac{\beta}{2d}}| \cdot \prod_{j=1}^{d} |\xi_j^{\frac{1}{2}+\frac{\beta}{2d}} \hat{u}(\xi)| d\xi \tag{19}$$

$$\leq \frac{1}{M^\gamma} \left(\int_{|\xi|\geq M} |\prod_{j=1}^{d} \xi_j^{-1-\frac{\beta}{d}}| d\xi \right)^{1/2} \left(\int_{|\xi|\geq M} \prod_{j=1}^{d} |\xi_j^{\frac{1}{2}+\frac{\beta}{2d}} \hat{u}(\xi)|^2 d\xi \right)^{1/2}$$

$$\leq C_2 M^{-\gamma-\beta/2} \sum_{j=1}^{d} \left(\int_{\mathbf{R}^d} |\xi_j^{\frac{d}{2}+\frac{\beta}{2}} \hat{u}(\xi)|^2 d\xi \right)^{1/2} = C_2 M^{-\gamma-\beta/2} \sum_{j=1}^{d} \|D_{x_j}^{\frac{d+\beta}{2}} u\|_{L^2(\mathbf{R}^d)}.$$

Collecting (19), (18), and (17), we reach to (16) and conclude the lemma. □

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