The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2108–2113

ON A LIMIT DISTRIBUTION OF HIGH LEVEL CROSSINGS OF A STATIONARY GAUSSIAN PROCESS[1]

By Clifford Qualls

University of New Mexico

1. Introduction. Let $\{\xi(t), -\infty < t < \infty\}$ be a real stationary Gaussian process with zero mean function and having continuous sample paths with probability one. Denote the covariance function by r (taking $r(0) = 1$ for convenience), and the corresponding spectral distribution function by F. Let μ be the expected number of upcrossings of the level u by $\xi(t)$ in a t-interval of length 1.

H. Cramér [4] in a recent and important paper proved that for a certain class of these processes the number of upcrossings of a level tending to infinity during a t-interval of length T behaves asymptotically like a Poisson process provided the time T is measured in units of $1/\mu$. Cramér's requirements determining this class of processes are:

$$(1') \qquad r^{(iv)}(0) \text{ exists, or equivalently, } \int_{-\infty}^{\infty} \lambda^4 dF(\lambda) < \infty,$$

and

$$(2) \qquad r(t) = O(t^{-\alpha}) \text{ as } t \to \infty \text{ for some } \alpha > 0$$

In this paper we shall show that Cramér's result applies to a significantly wider class of stationary Gaussian processes by replacing his condition $(1')$ by a condition only slightly stronger than the existence of $r''(0)$:

$$(1) \qquad \lambda_2 = -r''(0) \quad \text{exists and} \quad \int_0^{\delta} (\lambda_2 + r''(t))/t\, dt < \infty \quad \text{for some } \delta > 0,$$

$$\text{or equivalently,} \qquad \int_0^{\infty} \log(1 + \lambda)\lambda^2\, dF(\lambda) < \infty.$$

(The equivalence of the two statements in condition (1) was proved in [9], but now essentially the same proof can be found in a recent paper by R. P. Boas [3], Theorem 3.) To be more precise, the result proved in this paper is the following limit theorem.

Theorem 1.1. *Suppose the process $\xi(t)$ satisfies conditions* (1) *and* (2). *Let $N(a_i, b_i)$ be the number of upcrossings of the level u by $\xi(t)$ in the t-interval (a_i, b_i). The t-intervals $(a_1, b_1), \cdots, (a_j, b_j)$ are disjoint and depend on the level u in that $b_i - a_i = \tau_i/\mu$, $i = 1, \cdots, j$, where $\tau_1, \cdots, \tau_j$ are fixed positive numbers. Then for every j-tuple of non-negative integers $k_1, \cdots, k_j$,*

$$\lim_{u\to\infty} P\{N(a_i, b_i) = k_i,\ i = 1, \cdots, j\} = \prod_{i=1}^{j} e^{-\tau_i}\tau_i^{k_i}/k_i!.$$

Received 21 July 1967.

[1] This paper is based on part of the author's doctoral dissertation completed June 1967 at the University of California at Riverside under the direction of Professor Howard G. Tucker.

Cramér proved his theorem in 5 lemmas. (See [5], pp. 258 ff, for a more recent version of Cramér's proof.) It is the second lemma, which deals with the asymptotic behavior of the second factorial moment of the number of upcrossings of a high level, that is decisive in the proof and that can be significantly improved. The generalization of Lemma 2 is given in section 2, and the difference between Cramér's proof and the one given here will be discussed at the end of that section. The reader is referred to Cramér's work for the proofs of the remaining lemmas, which apply without change.

We note here that a recent result of Yu. K. Belayev [2] overlaps part of the result of this paper. A brief discussion of Belayev's work is included in Section 3.

2. Generalization of Cramér's Lemma 2. In this section Cramér's Lemma 2 of [4] is shown to be valid under our weaker conditions (1) and (2). Lemma 2 was first proved by Volkonskii and Rozanov [10], and, in fact, Cramér's paper [4] is based on their work.

Before stating and proving Lemma 2, we ne ' a result and the notation contained in Cramér's Lemma 1. Let $\xi(t)$ satisfy t e condition that $\lambda_2 = -r''(0) < \infty$. Let $N(T)$ be the number of upcrossings of the level u by $\xi(t)$ during $(0, T)$. Suppose $T = \tau/\mu$, where $\mu = EN(1) = \lambda_2^{\frac{1}{2}} \exp\{-u^2/2\}/2\pi$ and τ is a fixed positive number. Note that the expression for $EN(1)$ is Rice's formula, which was shown to hold in this present context by both K. Itô [7] and N. D. Ylvisaker [11]. Let $M = [T\mu^{-\beta}]$, $0 < \beta < 1$, where [ ] denotes the greatest integer function, and let $q = T/M$. Note that $q \sim \mu^{\beta}$ as $u \to \infty$. Let $\xi_q(t)$ be the process that is equal to $\xi(t)$ at times which are integral multiples of q and is the linear interpolation of $\xi(t)$ between these times. The symbol N_q will denote the number of upcrossings by the process ξ_q. Cramér proved the following as the main part of his Lemma 1:

Lemma 2.1. *Let $\xi(t)$ be as described above ($r''(0)$ exists). Then $EN_q(q) = q\mu + o(q\mu)$, as $u \to \infty$.*

In the original paper [4], Cramér proved this result assuming $r^{(iv)}(0)$ exists, but in [5], p. 260, it has been proved in the form given above. An alternate proof of Lemma 2.1 based on the estimation technique used by Itô [7] is given in [9].

Let $m_1 = [\mu^{-1}]$ and $t_1 = m_1 q$. Note $m_1 \sim \mu^{-1}$ and $t_1 \sim \mu^{\beta-1}$ as $u \to \infty$. With this notation, we state the following generalization of Cramér's Lemma 2.

Theorem 2.2 *Let $\xi(t)$ satisfy*

(1) $\lambda_2 = -r''(0)$ *exists and* $\int_0^{\delta} (\lambda_2 + r''(t))/t \, dt < \infty$ *for some* $\delta > 0$, *and*

(2″) $r(t) \to 0$ *as* $t \to \infty$.

Then

$$\lim_{u\to\infty} E\{N_q(t_1)[N_q(t_1) - 1]\}/EN_q(t_1) = 0.$$

Proof. Note that by stationarity $EN_q(t_1) = m_1 q\mu + o(m_1 q\mu) \sim q$ as $u \to \infty$ and also that $N_q(t_1) < N(t_1)$, so that it suffices to show

$$\lim_{u\to\infty} E\{N(t_1)[N(t_1) - 1]\}/q = 0.$$

Under the conditions that $r''(0)$ exists and that the spectral distribution F has a continuous component, Ylvisaker [12] (improving the work of Cramér and

Leadbetter [6]) obtained an integral expression for $E\{N(t_1)[N(t_1) - 1]\}$ finite or not. See also [5], p. 204. Note condition (2) of Theorem 1.1 implies condition (2″) which in turn implies that F has a continuous component. After performing the reduction in [5], p. 262, and another reduction which uses the fact that $r(s_1, s_2) = E\xi(s_1)\xi(s_2)$ depends only on $t = |s_1 - s_2|$, the integral expression becomes:

$$E\{N(t_1)[N(t_1) - 1]\} = 2\int_0^{t_1} (t_1 - t) \exp\{-u^2/(1 + r)\}A(t)B(t, u)\, dt,$$

where

$$A(t) = [\lambda_2(1 - r^2) - (r')^2]/2\pi(1 - r^2)^{3/2} > 0,$$

$$B(t, u) = \int_0^\infty \int_0^\infty z_1 z_2 \varphi(z_1, z_2)\, dz_1\, dz_2,$$

φ is the density of a bivariate Gaussian distribution with mean vector $(-h, h)$ and covariance matrix $\Sigma = \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$,

$$h = r'u(1 + r)^{-1}[(1 - r^2)/(\lambda_2(1 - r^2) - (r')^2)]^{\frac{1}{2}},$$

and

$$\rho = [r''(1 - r^2) + r(r')^2]/[\lambda_2(1 - r^2) - (r')^2].$$

Here r, r', and r'' are all functions of t. It is a result of these reductions that $|\rho| < 1$ for $t \neq 0$. We then have

$$(1/q)E\{N(t_1)[N(t_1) - 1]\} \leqq 2(t_1/q)\int_0^{t_1} \exp\{-u^2/(1 + r)\}A(t)B(t, u)\, dt.$$

Since $t_1/q = m_1 \sim 1/\mu$ as $u \to \infty$, we may replace the expression on the right by

$$J = C \cdot \int_0^{t_1} \exp\{-\tfrac{1}{2}u^2(1 - r)(1 + r)^{-1}\}A(t)B(t, u)\, dt, \qquad C = 4\pi/\lambda_2^{\frac{1}{2}}.$$

It suffices to show that $\lim_{u\to\infty} J = 0$. The following lemma will be needed to complete the proof.

LEMMA 2.3. *Under the conditions that $r''(0)$ exists and $r(t) \to 0$ as $t \to \infty$, we have*

(i) *$B(t, u)$ is a decreasing function of $|h|$,*

(ii) *$B(t, u)$ is a decreasing function of ρ, and*

(iii) $0 \leqq B(t, u) \leqq \frac{1}{2}$.

PROOF. If $\varphi(z_1, z_2)$ is a bivariate normal density with mean vector (μ_1, μ_2) and covariance matrix $\Lambda = (\lambda_{ij})$, we can use the inversion formula for the characteristic function of φ to see that $\partial\varphi/\partial\mu_i = -\partial\varphi/\partial z_i$ and $\partial\varphi/\partial\lambda_{ij} = \partial^2\varphi/\partial z_i \partial z_j$, $i \neq j$. Consequently

$$\begin{aligned}
\partial B(t, u)/\partial h &= \int_0^\infty \int_0^\infty z_1 z_2 (\partial\varphi(z_1, z_2)/\partial h)\, dz_1\, dz_2 \\
&= \int_0^\infty \int_0^\infty z_1 z_2 \{\partial\varphi/\partial z_1 - \partial\varphi/\partial z_2\}\, dz_1\, dz_2 \quad (\text{since } \mu_1 = -h \text{ and } \mu_2 = h) \\
&= \int_0^\infty \int_0^\infty (z_1 - z_2)\varphi(z_1, z_2)\, dz_1\, dz_2 \qquad (\text{integrating by parts}) \\
&= \exp\{-h^2/(1 + \rho)\} \int_0^\infty \int_0^\infty (z_1 - z_2)\varphi_0(z_1, z_2) \\
&\quad \cdot \exp\{-h(z_1 - z_2)/(1 + \rho)\}\, dz_1\, dz_2
\end{aligned}$$

where φ_0 is the density of $N(\mathbf{0}, \Sigma)$. By symmetry,

$$(\partial B/\partial h) \mid_{h=0} = \int_0^\infty \int_0^\infty (z_1 - z_2)\varphi_0(z_1, z_2)\, dz_1\, dz_2 = 0.$$

Now consider $\exp\{-h(z_1 - z_2)/(1 + \rho)\} = w(z_1 - z_2)$ to be a weighting factor in the integrand of $\partial B/\partial h$. For $h > 0$, $w(z_1 - z_2) > 1$ if and only if $(z_1 - z_2) < 0$, and therefore $\partial B/\partial h < 0$. Similarly for $h < 0$, $\partial B/\partial h > 0$ which establishes (i).

To prove (ii), it suffices to consider $\partial B/\partial \rho$ and use the fact that $\partial\varphi/\partial\rho = -\partial^2\varphi/\partial z_1 \partial z_2$.

Now by (i) and (ii) we have

$$0 \leqq B(t, u) \leqq B(t, 0) \leqq \lim_{\rho\to-1} B(t, 0).$$

In this limiting case, the bivariate Gaussian distribution becomes singular with the mass being distributed along the line $z_1 = z_2$ and has marginal densities $\varphi(z_1)$ and $\varphi(z_2)$ both of which are $N(0, 1)$. Therefore

$$\lim_{\rho\to-1} B(t, 0) = \int_0^\infty z_1^2\varphi(z_1)\, dz_1 = \tfrac{1}{2}.$$

This completes the proof of the lemma.

This lemma makes some things obvious. We note that the integrand of J is positive and is a decreasing function of u. We also note that $E\{N(t_1)[N(t_1) - 1]\}$ is dominated above by the corresponding second factorial moment for the zero level $u = 0$. Under hypothesis (1) in Theorem 2.2 Leadbetter and Cryer [8] show that this second moment for $u = 0$ is finite. Therefore, under hypothesis (1), one can see that $J < \infty$ by considering an interval twice the length of t_1.

By the Lesbesgue dominated convergence theorem, we have, for any fixed $k > 0$,

$$\lim_{u\to\infty} \int_0^k \exp\{-\tfrac{1}{2}u^2(1 - r)/(1 + r)\}A(t)B(t, u)\, dt = 0.$$

Now consider

$$\begin{aligned} J_k &= C\int_k^{t_1} \exp\{-\tfrac{1}{2}u^2(1 - r)/(1 + r)\}A(t)B(t, u)\, dt \\ &= C\int_{k/t_1}^1 \exp\{-\tfrac{1}{2}u^2(1 - r(st_1))/(1 + r(st_1))\}A(st_1)B(st_1, u)t_1\, ds. \end{aligned}$$

By Lemma 2.3, $B(t, u)$ is bounded. In addition, $A(t)$ is bounded for all $t \geqq k > 0$, since $A(t)$ is continuous and by hypothesis ($2''$) of Theorem 2.2

$$0 < A(t) < \lambda_2(1 - r^2)/2\pi(1 - r^2)^{3/2} \to \lambda_2/2\pi \quad \text{as} \quad t \to \infty.$$

Therefore

$$0 \leqq J_k \leqq C_1\int_{k/t_1}^1 \exp\{-\tfrac{1}{2}u^2(1 - r(st_1))/(1 + r(st_1))\}t_1\, ds.$$

But $t_1 \sim \mu^{\beta-1} = K\exp\{-(u^2/2)(\beta - 1)\}$ as $u \to \infty$. So this last integral can be replaced by

$$I_k = C_2\int_{k/t_1}^1 \exp\{-(u^2/2)(\beta - 2r(st_1)/(1 + r(st_1)))\}\, ds,$$

and it follows that

$$\begin{aligned} I_k &\leqq C_2\int_{k/t_1}^1 \exp\{-(u^2/2)(\beta - \epsilon)\}\, ds \\ &\leqq C_2 \exp\{-(u^2/2)(\beta - \epsilon)\} \to 0 \quad \text{as} \quad u \to \infty, \end{aligned}$$

since $2r(t)/(1 + r(t)) < \epsilon < \beta$ for all $t > k$ and k sufficiently large.

This completes the proof of Theorem 2.2. It differs from Cramér's proof in that the existence of $r^{(iv)}(0)$ is used there to show $A(t)$ is bounded for $0 \leqq t < \infty$, but here we allow $A(t)$ to be unbounded as $t \downarrow 0$.

3. Discussion. Note that the full strength of condition (2) in Theorem 1.1 was not used in Section 2. It is used to conclude the proof of Theorem 1.1. For the proof of Theorem 2.2 all that was really necessary is

$$(1'') \quad E\{N(T)[N(T) - 1]\} < \infty \quad \text{and} \quad (2'') \quad r(t) \to 0 \quad \text{as} \quad t \to \infty.$$

In fact we can replace condition (1) in Theorem 1.1 of this paper by condition $(1'')$ if we wish. There are no known necessary and sufficient conditions on the covariance function or spectral distribution giving $E\{N(T)[N(T) - 1]\} < \infty$. However condition (1) of Theorem 1.1 cannot be weakened too much as is indicated by a counterexample due to Cramér and Leadbetter [6]. In fact our condition (1) is slightly better (by actual example) than Hunt's condition which guarantees continuity of the sample derivatives of $\xi(t)$; namely,

$$\int_0^\infty [\log(1 + \lambda)]^a \lambda^2 \, dF(\lambda) < \infty$$

for some $a > 1$. It is an open question whether continuity of the sample derivatives is either a necessary or a sufficient condition for $E\{N(T)[N(T) - 1]\}$ to be finite.

It is interesting to note that condition (1) can be stated in terms of the covariance function r instead of its second derivative. Given $\lambda_2 = -r''(0) < \infty$, the following two conditions are equivalent to condition (1).

$$(A) \qquad \int_0^{\delta_1} (r(t) - 1 + \tfrac{1}{2}\lambda_2 t^2)/t^3 \, dt < \infty, \text{ for some } \delta_1 > 0,$$

$$(B) \qquad \int_0^{\delta_2} (\lambda_2 t + r'(t))/t^2 \, dt < \infty, \text{ for some } \delta_2 > 0.$$

For the proof that $A \Leftrightarrow B$, write the numerator of the integrand of A as an integral of $(\lambda_2 t + r'(t))$. Fubini's theorem and a change of variable then gives the result. Similarly, $B \Leftrightarrow$ condition (1).

Recently Belayev [2] obtained a theorem similar to Theorem 1.1 with conditions (1) and (2) replaced by

(1^*) $r(t)$ is twice differentiable and $|r''(t) - r''(0)| \leqq C/|\ln|t||^{1+\epsilon}$ as $t \to 0$ for some $C, \epsilon > 0$, and

$$(2^*) \qquad r(t) = o(1/\ln t), \; r'(t) = o(1/(\ln t)^{\frac{1}{2}}) \text{ as } t \to \infty.$$

Since condition (1*) is equivalent to Hunt's condition (Belayev [1]), our condition (1) is better than (1*). However condition (2*) appears to be better than (2). This author was unable to determine the exact relationship between conditions (2*) and (2). It is possible to obtain the combined theorem using conditions (1) and (2*).

4. Acknowledgment. The author wishes to acknowledge a great debt to Professor Howard G. Tucker. Professor Tucker brought the author's attention to the problem treated here. His technical advice and encouragement are greatly appreciated. The author also appreciates his former colleagues at the California State College at Fullerton for their patience and encouragement.

REFERENCES

[1] Belayev, Yu. K. (1961). Continuity and Hölder's conditions for sample functions of stationary Gaussian processes. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **2** 23–33. Univ. of California Press.

[2] Belayev, Yu. K. (1967). On the number of intersections of a level by a Gaussian stochastic process, II. *Teor. Veroyatnost. i. Primenen.* **12** 444–457 (in Russian).

[3] Boas, R. P. (1967). Lipschitz behavior and integrability of characteristic functions. *Ann. Math. Statist.* **38** 32–36.

[4] Cramér, Harald (1966). On the intersections between the trajectories of a normal stationary stochastic process and a high level. *Arkiv För Matematik* **6** 337–349.

[5] Cramér, Harald and Leadbetter, M. R. (1967). *Stationary and Related Stochastic Processes.* Wiley, New York.

[6] Cramér, Harald and Leadbetter, M. R. (1965). The moments of the number of crossings of a level by a stationary normal process. *Ann. Math. Statist.* **36** 1656–1663.

[7] Itô, Kiyosi (1964). The expected number of zeros of continuous stationary Gaussian processes. *J. Math. Kyoto Univ.* **3** 207–216.

[8] Leadbetter, M. R. and Cryer, J. D. (1965). The variance of the number of zeros of a stationary normal process. *Bull. Amer. Math. Soc.* **71** 561–563.

[9] Qualls, C. R. (1967). On the limit distributions of high level crossings of a stationary Gaussian process. University of California at Riverside. Doctoral dissertation, June 1967.

[10] Volkonskii, V. A. and Rozanov, Yu. A. (1961). Some limit theorems for random functions, II. *Theor. Prob. Appl.* **6** 187–198 (English translation).

[11] Ylvisaker, N. Donald (1965). The expected number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* **36** 1043–1046.

[12] Ylvisaker, N. Donald (1966). On a theorem of Cramér and Leadbetter. *Ann. Math. Statist.* **37** 682–685.