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ON A LINEAR AGE-DEPENDENT POPULATION DIFFUSION MODEL*

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1. Introduction. We discuss here a model described by Gurtin [3] for age-dependent populations with diffusion in a bounded set Ω of $\mathbb{R}^N$.

In this linear theory the age-space structure is studied through the population distribution $u(t, a, x)$ where t is a time, a age $(0 < a < A)$ and x spatial position. The evolution of u is governed by the equation (balance law):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u - \Delta \int_0^A k(a, \alpha)u(t, \alpha, x)\, d\alpha = 0 \tag{1}$$

where $\mu = \mu(t, a, x)$ is the death-modulus and $\nabla \int_0^A k(a, \alpha)u(t, \alpha, x)\, d\alpha$ is the flux of population by spatial diffusion. Here Δ is the Laplacian and ∇ is the gradient in $\mathbb{R}^N$. The reader is referred to Hoppensteadt [6] for equations of this form, but without diffusion.

We assume that the birth process is given by the birth law:

$$u(t, 0, x) = \int_0^A \beta(t, a, x)u(t, a, x)\, da \tag{2}$$

(β is the birth modulus), that there is no diffusion through the boundary $\partial\Omega$ of Ω, that is:

$$\frac{\partial}{\partial \eta} \int_0^A k(a, \alpha)u(t, \alpha, x)\, d\alpha = 0 \tag{3}$$

where $\partial/\partial\eta$ is the normal derivative, and that the initial population is known:

$$u(0, a, x) = u_0(a, x). \tag{4}$$

A is the maximum life expectancy of the species.

The initial boundary value problem (1)–(4) will be referred to as problem (I), namely (subscripts indicate partial differentiation):

$$\begin{aligned}
&u_t + u_a + \mu u - \Delta \int_0^A k(a, \alpha)u(t, \alpha, x)\, d\alpha = 0, && t > 0,\ 0 < a < A,\ x \in \Omega,\\
&u(0, a, x) = u_0(a, x), && 0 < a < A,\ x \in \Omega,\\
&u(t, 0, x) = \int_0^A \beta(t, a, x)u(t, a, x)\, da, && t > 0,\ x \in \Omega,\\
&\frac{\partial}{\partial \eta}\int_0^A k(a, \alpha)u(t, \alpha, x)\, d\alpha = 0, && t > 0,\ 0 < a < A,\ x \in \Omega.
\end{aligned} \tag{I}$$

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In the particular case when the kernel k is independent of the variable α, the equation (1) and the boundary condition (3) can be expressed in a simpler form and problem (I) becomes problem (II), that is:

$$\begin{aligned}
&u_t + u_a + \mu u - k(a)\,\Delta \int_0^A u(t, \alpha, x)\, d\alpha = 0,\ t > 0, && 0 < a < A,\ x \in \Omega,\\
&u(0, a, x) = u_0(a, x), && 0 < a < A,\ x \in \Omega,\\
&u(t, 0, x) = \int_0^A \beta(t, a, x) u(t, a, x)\, da, && t > 0,\ x \in \Omega,\\
&\frac{\partial}{\partial \eta} \int_0^A u(t, a, x)\, da = 0, && t > 0,\ x \in \partial\Omega.
\end{aligned} \tag{II}$$

Some results concerning problem (I) are given in di Blasio and Lamberti [1]. The method emphasized here (initiated in Langlais [8]) is quite different. Under suitable assumptions this method provides existence and uniqueness in problem (II) (which is the first model derived in [3]), and is helpful for the nonlinear model investigated in Garroni and Langlais [2]. On the other hand, the hypotheses needed to solve problem (II) when μ is not bounded appear again in the model with nonlinear diffusion studied by Langlais [9], and we expect them to be useful in the nonlinear diffusion model described in Gurtin and MacCamy [4, 5] (investigated for constant μ by MacCamy [11]). In these two linear models the solution can become negative in a finite time (see [4, 9]). This paper is a first step towards nonlinear models.

2. Notation and basic assumptions. T and A are positive and finite real numbers; Ω is a bounded open set of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The variables t, a and x lie respectively in $(0, T)$, $(0, A)$ and Ω. The set $(0, T) \times (0, A)$ is denoted $\mathcal{O}$ and Q is the product $\mathcal{O} \times \Omega$.

Let μ be a real-valued function on Q satisfying:

$$\begin{aligned}
&\mu \text{ is continuous on } [0, T] \times [0, A) \times \bar{\Omega};\\
&\mu(t, a, x) \geq 0 \text{ on } Q;\\
&\nabla\mu \text{ is bounded on } Q.
\end{aligned} \tag{$(\mu)^1$}$$

μ is not and will not be assumed to be bounded near $a = A$ (see further remarks).

We suppose that β is a real-valued function on Q such that:

$$\begin{aligned}
&\beta \text{ is bounded on } Q;\\
&\text{there exists a contant } C_1 \text{ such that}\\
&\int_0^A [\beta^2 + |\nabla\beta|^2](t, a, x)\, da \leq C_1 \text{ in } (0, T) \times \Omega.
\end{aligned} \tag{$(\beta)^1$}$$

The kernel k is a real measurable function defined in $(0, A) \times (0, A)$; in problem (I) we shall suppose:

$$\begin{aligned}
&\text{there exists a constant } C_2 \text{ such that for each } \varphi \in L^2(0, A):\\
&\int_0^A \left[\int_0^A k(a, \alpha)\varphi(\alpha)\, d\alpha\right]^2 d\alpha \leq C_2^2 \int_0^A \varphi^2(a)\, da:
\end{aligned} \tag{$(\mathrm{k})^1$}$$

for each $\varphi \in L^2(0, A)$ we have:

$$\int_0^A \int_0^A k(a, \alpha)\varphi(\alpha)\varphi(a)\, da\, d\alpha \geq 0.$$

The first part of $(\mathrm{k})^1$ means that the map:

$$\varphi \in L^2(0, A) \to \int_0^A k(a, \alpha)\varphi(\alpha)\, d\alpha \in L^2(0, A)$$

is continuous. The last part of $(\mathrm{k})^1$ is a monotonicity-type condition used in [1].

In the particular case where the kernel k does not depend on α we shall simply need:

$$k \in L^2(0, A); \qquad k(a) \geq 0 \quad \text{in} \quad (0, A). \tag{(k)^2}$$

Now if U is any open set of either $\mathbb{R}$ or $\mathbb{R}^2$ and if H is any of the Sobolev spaces of order 1 or 2 $H^1(\Omega)$ and $H^2(\Omega)$, then $L^2(U, H)$ is the Hilbert space of measurable and square integral function $v\colon U \to H$ (see Lions and Magenes [10]).

3. Statement of results. The first result concerns problem (I), that is the initial boundary value problem (1)–(4).

THEOREM 1. Let (μ, β, k) satisfy $(\mu)^1$, $(\beta)^1$, $(\mathrm{k})^1$ and

$$\Delta\mu \in L^\infty(Q), \qquad \partial\mu/\partial\eta = 0 \text{ in } \mathcal{O} \times \partial\Omega; \tag{$(\mu)^2$}$$

$$\int_0^A |\Delta\beta|^2(t, a, x)\, da \leq C_1 \text{ in } (0, T) \times \Omega, \qquad \frac{\partial\beta}{\partial\eta} = 0 \text{ in } \mathcal{O} \times \partial\Omega. \tag{$(\beta)^2$}$$

Then given any u_0 in $L^2(0, A; H^2(\Omega))$ such that $\partial u_0/\partial\eta = 0$ in $(0, A) \times \partial\Omega$ there exists a unique u belonging to $L^2(\mathcal{O}; H^2(\Omega))$, a solution to problem (I) and verifying $\partial u/\partial\eta = 0$ in $\mathcal{O} \times \partial\Omega$.

It is worth while to notice that the initial boundary conditions (2–4) make sense provided the solution belongs to $L^2(\mathcal{O}; H^2(\Omega))$.

As the solution u lies in $L^2(\mathcal{O}; H^2(\Omega))$, the properties of k ensure that $\int_0^A k(a, \alpha)u(t, \alpha, x)\, d\alpha$ lies in $L^2(\mathcal{O}; H^2(\Omega))$; hence the Neumann boundary condition (3) makes sense and is fulfilled because u satisfies the homogeneous boundary Neumann condition. Moreover, from Eq. (1) we deduce that $u_t + u_a + \mu \cdot u$ belongs to $L^2(Q)$. Now take any A_0, $0 < A_0 < A$, and set $\mathcal{O}_0 = (0, T) \times (0, A_0)$, $Q_0 = \mathcal{O}_0 \times \Omega$; then μ is bounded on Q_0 and $u_t + u_a$ lies in $L^2(Q_0)$. Thus bearing in mind that u is in $L^2(Q_0)$, initial conditions (2) and (4) make sense.

When the data u_0 is not smooth enough or when the conditions $(\mu)^2$ and $(\beta)^2$ are not fulfilled we can prove the existence and uniqueness of a suitable weak solution. Hence Theorem 1 can be considered as a regularity theorem. But it is more interesting to view it as a basic result from which we can derive particular cases.

Let us assume first that:

$$k(a, \alpha) = h(a)h(\alpha), \qquad h \in L^2(0, A).$$

This assumption is stronger that $(\mathrm{k})^1$. The Newmann boundary condition (3) becomes:

$$\frac{\partial}{\partial\eta} \int_0^A h(\alpha)u(t, \alpha, x)\, d\alpha = 0 \text{ in } (0, T) \times \partial\Omega. \tag{5}$$

Condition $(\mathrm{k})^3$ allows us to remove $(\mu)^2$ and $(\beta)^2$ in Theorem 1.

THEOREM 2. Suppose that $(\mu)^1$, $(\beta)^1$, $(k)^3$ are satisfied; then given any u_0 in $L^2(0, A; H^1(\Omega))$ there exists a unique u in $L^2(\mathcal{O}; H^1(\Omega))$, a solution to problem (I) and such that:

$$\int_0^A h(\alpha)u(t, \alpha, x)\, d\alpha \in L^2(0. T; H^2(\Omega)). \tag{6}$$

We investigate now the particular case when the kernel k is independent of α, that is, problem (II). Assume first that k lies in $C^1([0, A])$ and that:

$$\text{there exists a real constant } k_0 \text{ such that } k(a) \geq k_0 > 0 \text{ in } (0, A) \tag{k^4}$$

If we let $u(t, a, x) = k(a)v(t, a, x)$ in Q, then, at least formally, v is a solution of the equations:

$$\begin{aligned}
&k(v_t + v_a) + (\mu k + k_a)v - k\,\Delta \int_0^A k(\alpha)v(t, \alpha, x)\, d\alpha = 0 \text{ in } Q,\\
&v(0, a, x) = [k(a)]^{-1}u_0(a, x) \text{ in } (0, A) \times \Omega\\
&v(t, 0, x) = [k(0)]^{-1} \int_0^A \beta(t, a, x)k(a)v(t, a, x)\, da \text{ on } (0, T) \times \Omega,\\
&\frac{\partial}{\partial \eta} \int_0^A k(a)v(t, a, x)\, da = 0 \text{ on } \mathcal{O} \times \partial\Omega.
\end{aligned} \tag{III}$$

This boundary-value problem with $(k)^4$ and the problem (I) with $(k)^3$ have the same qualitatives properties. We can prove that when $(\mu)^1$, $(\beta)^1$, $(k)^2$ and $(k)^4$ are satisfied for any u_0 given in $L^2(0, A; H^1(\Omega))$ there exists a unique v verifying (6)—see Theorem 2—a solution to problem (III). From v we get u, a solution to problem (II).

Remark 1. Up to now we did not suppose μ to be bounded at $a = A$ (see $(\mu)^1$). Actually if μ is rapidly increasing at $a = A$ then any of the solutions whose existence has been previously established vanishes at $a = A$. More precisely, the conditions $u(t, A, x) = 0$ in $(0, T) \times \Omega$ and the two conditions:

$$\begin{aligned}
&0 < t < A, \qquad x \in \Omega \lim_{a \to A} \int_0^t \mu(\tau, a - t + \tau, x)\, dt = +\infty\\
&A < t < T, \qquad x \in \Omega \lim_{a \to A} \int \mu(t - a + \alpha, \alpha, x)\, d\alpha = +\infty
\end{aligned} \tag{μ^{3}}$$

are equivalent (see Langhaar [7], [2] and [8]: this property is independent of the diffusion term in (1)).

Unfortunately, we have not been able to solve Problem (II) when $(\mu)^3$ is fulfilled (except when $(k)^4$ is true) without additional hypotheses on u, k and u_0.

THEOREM 3. Let $(\mu)^1$, $(\beta)^1$, $(k)^2$ be satisfied. Let u_0 be in $L^2((0, A) \times \Omega)$and verify $\int_0^A u_0(a, x)\, da \in H^1(\Omega)$. Assume either:

there exist two constants λ_1, λ_2 such that

$$\begin{aligned}
&\mu^2 - \mu_t - \mu_a \geq \lambda_1\mu + \lambda_2 \text{ on } Q; && \text{(i)}\\
&\mu(0, a, x)u_0 \in L^2((0, A) \times \Omega); && \text{(ii)}
\end{aligned} \tag{μ^{4}}$$

there exists a constant M such that

$$\int_0^A \mu^2(t, a, x)k^2(a)\, da \leq M \text{ on } (0, T) \times \Omega; \qquad \text{(iii)}$$

or:

$$\text{there exists a constant } m \text{ such that } \int_0^A \mu(t, a, x)\, da \leq m \text{ in } (0, T) \times \Omega. \qquad (\mu)^5$$

Then there exists a unique u in $L^2(Q)$ satisfying (6)—$\mu u \in L^2(Q)$ when $(\mu)^4$ is true—a solution to problem (II).

Remark 2. When $(\mu)^3$ is satisfied and when k is a constant then $(\mu)^4$(iii) is not fulfilled; nevertheless, to conclude we merely apply Theorem 2. More generally, $(\mathrm{k})^4$, $(\mu)^3$, and $(\mu)^4$(iii), are not consistent, but we have previously dealt with $(\mathrm{k})^4$.

The conditions $(\mu)^3$ and $(\mu)^4$(i) are realized when:

$$\mu(a) = q(A - a)^{-p}, \quad q > 0 \text{ and } p > 1 \text{ or } q \geq 1 \text{ and } p = 1,$$

$$\mu(a) = q \exp\ + 1/A - a, \quad q > 0.$$

4. Proofs. We first discuss a preliminary result from which Theorem 1 is proved, for bounded μ, using a fixed-point method. Then we turn to the general case. Theorem 3 is proved along the same lines, but we shall point out the differences in the first and last steps. The proof of Theorem 2 is omitted.

Now let u be a solution to problem (I) or problem (II) and set:

$$u(t, a, x) = e^{\lambda t} v(t, a, x) \text{ in } Q,\ \lambda \text{ constant};$$

then v is the solution to problem (I) or problem (II) with μ replaced by $\mu + \lambda$. This change of unknown function will be done throughout this section, for suitable positive values of λ, and v will be simply denoted u.

4-1. *Preliminary results.* We investigate the folowing initial-boundary value problem:

$$\begin{aligned} &u_t + u_a + \lambda u - \Delta \int_0^A k(\alpha, a)u(t, \alpha, x)\, d\alpha = f \text{ in } Q,\\ &u(0, a, x) = u_0(a, x) \text{ in } (0, A) \times \Omega,\\ &u(t, 0, x) = b(t, x) \text{ in } (0, T) \times \Omega,\\ &\frac{\partial}{\partial \eta} \int_0^A k(a, \alpha)u(t, \alpha, x)\, da = 0 \text{ in } \mathcal{O} \times \partial\Omega. \end{aligned} \qquad \text{(IV)}$$

Here λ is a positive constant. In order to get the existence of a solution we introduce the eigenfunctions of the Neumann problem in Ω:

$$-\Delta w_j = \nu_j w_j \text{ in } \Omega, \qquad \partial w_j/\partial \eta = 0 \text{ in } \partial\Omega,$$

$$\int_\Omega w_j^2 = 1, \qquad \int_\Omega w_j \cdot w_i\, dx = 0 \qquad j \neq i.$$

Expressing the data on the form:

$$f(t, a, x) = \sum_j f_j(t, a) w_j(x),$$
$$b(t, x) = \sum_j b_j(t) w_j(x), \qquad u_0(a, x) = \sum_j u_{0,j}(a) w_j(x), \tag{7}$$

it is appealing to seek as a formal solution to problem (III) the series

$$u(t, a, x) = \sum_j u_j(t, a) w_j(x).$$

It turns out that for suitable data (f, b, u_0) this gives a solution.

THEOREM 4. Let k satisfy $(k)^1$ and let (f, b, u_0) be such that:

$$\begin{aligned} &f \in L^2(\mathcal{O}; H^2(\Omega)), && \partial f/\partial\eta = 0 \text{ in } \mathcal{O} \times \partial\Omega, \\ &b \in L^2(0, T; H^2(\Omega)), && \partial b/\partial\eta = 0 \text{ in } (0, T) \times \partial\Omega, \\ &u_0 \in L^2(0, T; H^2(\Omega)), && \partial u_0/\partial\eta = 0 \text{ in } (0, A) \times \partial\Omega. \end{aligned} \tag{8}$$

Then there exists a unique solution u in $L^2(\mathcal{O}; H^2(\Omega))$ to problem (IV). Moreover, this solution satisfies the boundary condition:

$$\partial u/\partial\eta = 0 \text{ on } \mathcal{O} \times \partial\Omega$$

and the a priori estimates:

$$\lambda \int_Q [u^2 + |\nabla u|^2]\, dt\, da\, dx \le \frac{1}{2} \int_{(0, A)\times\Omega} [u_0^2 + |\nabla u_0|^2]\, da\, dx$$
$$+ \frac{1}{2} \int_{(0, T)\times\Omega} [b^2 + |\nabla b|^2\, dt\, dx + \int_Q [f \cdot u + \nabla f \cdot \nabla u]\, dt\, da\, dx, \tag{9}$$

$$\lambda \int_Q |\Delta u|^2\, dt\, da\, dx \le \frac{1}{2} \int_{(0, A)\times\Omega} |\Delta u_0|^2\, da\, dx$$
$$+ \frac{1}{2} \int_{(0, T)\times\Omega} |\Delta b|^2\, da\, dx + \int_Q \Delta f \cdot \Delta u\, dt\, da\, dx. \tag{10}$$

Proof. The existence of a solution is obtained by the method of separation of variables outlined above. The solution satisfies the estimates (9) and (10).

Now let u^1 and u^2 be solutions of (IV). The difference $u = u^1 - u^2$ is solution of the same problem (IV) with $f = 0$, $b = 0$ and $u_0 = 0$. Multiplying equation $(IV)^1$ by u and integrating over Q yields:

$$\int_O (u_t + u_a) u\, dt\, da\, dx + \lambda \int_Q u^2\, dt\, da\, dx - \int_Q \Delta \int_0^A k(a, \alpha) u(t, a, x)\, d\alpha \cdot u\, dt\, da\, dx = 0 \tag{11}$$

Integrating by parts the last term and using $(k)^1$ we have:

$$\int_Q \nabla \int_{\substack{0\\V}}^A k(a, \alpha) u(t, \alpha, x)\, d\alpha \cdot \nabla u(t, a, x)\, dt\, da\, dx$$
$$= \int_{(0, T)\times\Omega} \left[\int_0^A \int_0^A k(a, \alpha) \nabla u(t, \alpha, x) \cdot \nabla u(t, a, x)\, d\alpha\ \ da \right] dt\, dx \ge 0$$

u belonging to $L^2(\mathcal{O}; H^2(\Omega))$. From equation (IV)[1] we get $u_t + u_a \in L^2(Q)$. We can integrate by parts the first term in (11); this gives:

$$\int_Q (u_t + u_a)u\, dt\, da\, dx = \frac{1}{2}\left[\int_{(0,A)\times\Omega} u^2(T, a, x)\, da\, dx + \int_{(0,T)\times\Omega} u^2(t, A, x)\, dt\, dx\right] \geq 0,$$

because u vanishes on $t = 0$ and $a = 0$. It is easy now to prove that $u = 0$; from (11) we derive $\int_Q u^2(t, a, x)\, dt\, da\, dx \leq 0$.

4.2 *Proof of Theorem* 1 *for bounded* μ. We assume that μ is bounded. Let λ be a positive constant large enough compared with C_i (see the hypotheses on β), with the norms of μ and $\Delta\mu$ in $L^\infty(Q)$ and with the norms of $\nabla\mu$ in $[L^\infty(Q)]^N$. We deal with $v = e^{-\lambda t}u$ still denoted u.

We want to prove Theorem 1 using a fixed-point method and (IV), (9), (10). Let E be the Hilbert space:

$$F = \{u \in L^2(\mathcal{O}; H^2(\Omega)),\ \partial u/\partial\eta = 0 \text{ in } \mathcal{O} \times \partial\Omega\}.$$

Assuming that the hypotheses of theorem 1 (μ bounded) are satisfied, for any w in E there exists a unique $u = Sw$ in E, a solution to

$$u_t + u_a + \lambda u - \Delta \int_0^A k(a, \alpha)u(t, \alpha, x)\, d\alpha = -\mu w \text{ in } Q,$$

$$u(0, a, x) = u_0(a, x) \text{ in } (0, A) \times \Omega,$$

$$u(t, 0, x) = \int_0^A \beta(t, a, x)w(t, a, x)\, da \text{ in } (0, T) \times \Omega,$$

$$\frac{\partial}{\partial\eta}\int_0^A k(a, \alpha)u(t, \alpha, x)\, d\alpha = 0 \text{ in } \mathcal{O} \times \partial\Omega.$$

Thus we define a map S: $E \to E$. Its fixed points are the solutions to problem (I) in E. From (9) and (10) we deduce that S is continuous.

Let w^1 (resp. w^2) be in E and set $u^1 = Sw^1$ (resp. $u^2 = Sw^2$). The function $u = u^1 - u^2$ is a solution to problem (IV) with:

$$f = -\mu w \text{ in } Q, \qquad w = w^1 - w^2 \text{ in } Q.$$

$$b(t, x) = \int_0^A \beta(t, a, x)w(t, a, x)\, da \text{ in } (0, T) \times \Omega, \tag{12}$$

$$u_0(a, x) = 0 \text{ in } (0, A) \times \Omega.$$

Before employing the estimates (9) (10), we need to bound the right-hand side of (9), (10) when f, b and u_0 are given by (12).

First, the properties of β—that is, $(\beta)^1$, $(\beta)^2$—give:

$$\begin{aligned}\int_{(0,T)\times\Omega}[b^2 + |\nabla b|^2]\, dt\, dx &\leq 3C_1 \int_Q [w^2 + |\nabla w|^2]\, dt\, da\, dx,\\ \int_{(0,T)\times\Omega} |\Delta b|^2\, dt\, dx &\leq 4C_1 \int_Q [w^2 + |\nabla w|^2 + |\Delta w|^2]\, dt\, da\, dx.\end{aligned} \tag{13}$$

Here we used the Holder inequality. The same calculation for μ leads to:

$$\int_Q [f\cdot u + \nabla f\cdot \nabla u]\, dt\, da\, dx \le C_3\left[\int_Q (u^2 + |\nabla u|^2)\, dt\, da\, dx\right]^{1/2} \cdot\left[\int_Q (w^2 + |\nabla w|^2)\, dt\, da\, dx\right]^{1/2}, \tag{14}$$

$$\int_Q \Delta f\cdot \Delta u\, dt\, da\, dx \le C_4\left[\int_Q |\Delta u|^2\, dt\, da\, dx\right]^{1/2}\left[\int_Q (w^2 + |\nabla w|^2 + |\Delta w|^2)\, dt\, da\, dx\right]^{1/2}$$

where:

$$\begin{aligned} C_3 &= 2|\mu|_{L^\infty(Q)} + |\nabla\mu|_{L^\infty(Q)}, \\ C_4 &= |\mu|_{L^\infty(Q)} + |\Delta\mu|_{L^\infty(Q)} + 2|\nabla\mu|_{L^\infty(Q)}. \end{aligned} \tag{15}$$

If we substitute (13) and (14) into the estimate (9), using the Schwarz inequality we obtain

$$\left(\lambda - \frac{C_3}{2}\right)\int_Q [u^2 + |\nabla u|^2]\, dt\, da\, dx \le \frac{3C_1 + C_3}{2}\int_Q [w^2 + |\nabla w|^2]\, dt\, da\, dx. \tag{16}$$

If we use (13), (14), (10) and the Schwarz inequality we have:

$$\left(\lambda - \frac{C_4}{2}\right)\int_Q |\Delta u|^2\, dt\, da\, dx \le \frac{4C_1 + C_4}{2}\int_Q [w^2 + |\nabla w|^2 + |\Delta w|^2]\, dt\, da\, dx. \tag{17}$$

Remembering that λ was chosen very large compared with (C_1, C_3, C_4), we easily derive from (16), (17) that there exists a constant $K = K(\lambda, C_1, C_3, C_4)$ that satisfies $0 < K < 1$ and such that:

$$\int_Q [u^2 + |\nabla u|^2 + |\Delta u|^2]\, dt\, da\, dx \le K\int_Q [w^2 + |\nabla w|^2 + |\Delta w|^2]\, dt\, da\, dx.$$

But $w = w^1 - w^2$ and $u = u^1 - u^2 = Sw^1 - Sw^2$. Therefore we proved that S has a unique fixed point. Hence for bounded μ the problem (1–4) has a unique solution belonging to the space E.

4.3 *Proof of Theorem* 1: *Existence.* In the general case μ is not bounded near $a = A$. Let λ be a positive constant very large compared with C_1 and the norms of $\nabla\mu$ and $\Delta\mu$ in $L^\infty(Q)$. Once more we deal with $e^{-\lambda t}u$, still denoted u. There exists a sequence $(\mu_n)_n$ such that:

$$\begin{aligned} &\mu_n \in L^\infty(Q), \\ &|\nabla\mu_n|_{L^\infty(Q)} + |\Delta\mu_n|_{L^\infty(Q)} \le C_5 \text{ independent of } n, \\ &\mu_n(t, a, x) = \mu(t, a, x) \text{ in } (0, T)\times(0, A - 1/n)\times\Omega. \end{aligned} \tag{18}$$

For each n there exists a unique u_n belonging to E, a solution of the problem (I) with μ changed into $\mu_n + \lambda$. But u_n is the solution to problem (IV) with:

$$f = -\mu_n u_n \text{ in } Q, \qquad b(t, x) = \int_0^A \beta(t, a, x)u_n(t, a, x)\, da \text{ in } (0, T)\times\Omega.$$

Again we use the estimates (9), (10) and we need some preliminary calculations. It is obvious that the inequalities (13) are valid with u_n instead of w. Now we have:

$$\int_Q f \cdot u_n + \nabla f \cdot \nabla u_n \, dt\, da\, dx = -\int_Q \mu_n [u_n^2 + |\nabla u_n|^2] \, dt\, da\, dx - \int_Q u_n \cdot \nabla \mu_n \cdot \nabla u_n \, dt\, da\, dx.$$

hence from (18) and Schwarz and Holder inequalities we obtain:

$$\int_Q [f \cdot u_n + \nabla f \cdot \nabla u_n] \, dt\, da\, dx$$

$$\leq -\int_Q \mu_n [u_n^2 + |\nabla u_n|^2] \, dt\, da\, dx + \frac{C_5}{2} \int_Q [u_n^2 + |\nabla u_n|^2] \, dt\, da\, dx.$$

Along the same lines we transform $\int_Q \Delta f \cdot \Delta u_n \, dt\, da\, dx$ into:

$$-\int_Q \mu_n \cdot |\Delta u_n|^2 \, dt\, da\, dx - \int_Q [2\nabla \mu_n \cdot \Delta u_n \nabla u_n + \Delta \mu_n \cdot u_n \Delta u_n] \, dt\, da\, dx;$$

therefore:

$$\int_Q \Delta f \cdot \Delta u_n \, dt\, da\, dx$$

$$\leq -\int_Q \mu_n |\Delta u_n|^2 \, dt\, da\, dx + C_6 \int_Q [u_n^2 + |\nabla u_n|^2 + |\Delta u_n|^2] \, dt\, da\, dx,$$

where C_6 is a constant depending only on C_5.

Substituting these results in (9) and (10) yields, respectively:

$$\int_Q \left[\lambda + \mu_n - \frac{3C_1}{2} - \frac{C_5}{2}\right] \left([u_n^2 + |\nabla u_n|^2] \, dt\, da\, dx \leq \frac{1}{2} \int_{(0, A) \times \Omega} [u_0^2 + |\nabla u_0|^2] \, da\, dx. \quad (19)$$

$$\int_Q [\lambda + \mu_n] |\Delta u_n|^2 \, dt\, da\, dx - C_6 \int_Q [|u_n|^2 + |\nabla u_n|^2 + |\Delta u_n|^2] \, dt\, da\, dx$$

$$\leq \frac{1}{2} \int_{(0, A) \times \Omega} |\Delta u_0| \, da\, dx. \quad (20)$$

But λ has been chosen large. So from (19) and (20) we can deduce that the sequence (u_n) is bounded in the $L^2(\mathcal{O}; H^2(\Omega))$ norm. The clue to getting this estimate is that μ is non-negative. This allows us to remove the hypothesis "μ bounded."

Hence there exists u in $L^2(\mathcal{O}; H^2(\Omega)$ and a sub-sequence $(u_{n_p})_p$ (which we simply denote (u_p)) such that:

$$u_p \xrightarrow[p \to \infty]{} u \text{ weakly in } L^2(\mathcal{O}; H^2(\Omega)).$$

Each u_p satisfies the homogeneous Neumann boundary condition; it follows that u satisfies the same homogeneous Neumann boundary condition.

Now $(\Delta u_n)_n$ is bounded in $L^2(Q)$; from the equation

$$\frac{\partial u_p}{\partial t} + \frac{\partial u_p}{\partial a} + \mu_p u_p = \Delta \int_0^A k(a, \alpha) u_p(t, \alpha, x)\, d\alpha \text{ in } Q, \tag{21}$$

we conclude that the sequence $(\partial u_p/\partial t + \partial u_p/\partial a + \mu_p u_p)$ is bounded in the $L^2(Q)$-norm. For a subsequence still denoted with the p-indices we have:

$$\frac{\partial u_p}{\partial t} + \frac{\partial u_p}{\partial a} + \mu_p u_p \xrightarrow[p\to\infty]{} h \text{ weakly in } L^2(Q).$$

But the choice of the sequence (μ_n)—see $(18)^3$—ensures that:

$$\frac{\partial u_p}{\partial t} + \frac{\partial u_p}{\partial a} + \mu_p u_p \xrightarrow[p\to\infty]{} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u \text{ in } \mathscr{D}'(Q),$$

that is, in the distributional sense in Q. Thus $h = u_t + u_a + \mu u$ and, letting $p \to +\infty$ in (21), we obtain that u satisfies Eq. (1).

We may notice that weakly in $L^2((0, T) \times \Omega))$ we have:

$$u_p(t, 0, x) = \int_0^A \beta(t, a, x) u_p(t, a, x)\, da \xrightarrow[p\to\infty]{} \int_0^A \beta(t, a, x) u(t, a, x)\, da. \tag{22}$$

Let A_0 be such that $0 < A_0 < A$ and define $\mathcal{O}_0 = (0, T) \times (0, A_0)$; for $n \geq n(A_0)$ $\mu_n(t, a, x) = \mu(t, a, x)$ in Q_0. Hence we have $\mu \in L^\infty(Q_0)$ and from Eq. (21) we get $(\partial u_p/\partial t + \partial u_p/\partial a)_p$ bounded in $L^2(Q_0)$. Thus:

$$u_p \xrightarrow[p\to\infty]{} u \text{ weakly in } L^2(Q_0),$$

$$\frac{\partial u_p}{\partial t} + \frac{\partial u_p}{\partial a} \longrightarrow \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} \text{ weakly in } L^2(Q_0),$$

and by continuity it follows that:

$$u_p(0, a, x) \xrightarrow[p\to\infty]{} u(0, a, x) \text{ weakly in } L^2((0, A_0) \times \Omega),$$

$$u_p(t, 0, x) \xrightarrow[p\to\infty]{} u(t, 0, x) \text{ weakly in } L^2(0, T) \times \Omega.$$

From (22) we deduce that u satisfies the initial condition (2)). For each u_p verifying (4) on $(0, A) \times \Omega$ we have:

$$u(0, a, x) = u_0(a, x) \text{ in } (0, A_0) \times \Omega;$$

this equality being true for any A_0 lying in $(0, A)$ is true in the open set $(0, A) \times \Omega$. So the initial condition (4) is fulfilled.

We can also derive that $\mu^{1/2} u$ belongs to $L^2(\mathcal{O}; H^2(\Omega))$.

4.4 *Proof of Theorem* 1: *Uniqueness.* Let λ be large enough with respect to C_1; we perform the change of unknown function $u(t, a, x) = e^{\lambda t} v(t, a, x)$ and we deal with v which we simply denote u.

Let u be in $L^2(\mathcal{O}; H^2(\Omega))$, a solution to problem (I), and satisfy the homogeneous Neumann condition on $\mathcal{O} \times \partial\Omega$, and let $u(0, a, x) = 0$ in $(0, A) \times \Omega$.

For any A_0 such that $0 < A_0 < A$ we let $Q_0 = (0, T) \times (0, A_0) \times \Omega$. Multiplying Eq. (1) by u and integrating over Q_0 yields:

$$\int_{Q_0} (u_t + u_a + \mu u + \lambda u) u \, dt \, da \, dx - \int_{Q_0} \left[\Delta \int_0^A k(a, \alpha) u(t, \alpha, x) \, d\alpha \right] u \, dt \, da \, dx = 0. \quad (23)$$

We already know that $u_t + u_a$ lie in $L^2(Q_0)$ because $\mu \in L^\infty(Q_0)$. So we can integrate by parts the first term in the left-hand side of (23); we obtain, using (2) and $u(0, a, x) = 0$:

$$\frac{1}{2} \int_{(0, A_0) \times \Omega} u^2(T, a, x) \, da \, dx + \frac{1}{2} \int_{(0, A_0) \times \Omega} u^2(t, A_0, x) \, dt \, dx + \int_{Q_0} (\lambda + \mu) u^2 \, dt \, da \, dx - \frac{1}{2} \int_{(0, T) \times \Omega} \left[\int_0^A \beta \cdot u \, da \right]^2 dt \, dx.$$

Moreover, we can integrate by parts the second integral in the left-hand side of (23). These two calculations lead to the estimate:

$$\lambda \int_{Q_0} u^2(t, a, x) \, dt \, da \, dx + \int_{(0, T) \times \Omega} \left[\nabla \int_0^A k(a, \alpha) u(t, \alpha, x) \, d\alpha \cdot \nabla \int_0^{A_0} u(t, a, x) \, da \right] dt \, dx \le \frac{1}{2} \int_{(0, T) \times \Omega} \left[\int_0^A \beta u \, da \right]^2 dt \, dx.$$

So if we let $A_0 \to A$, using the properties of the kernel k we get

$$\lambda \int_Q u^2 \, dt \, da \, dx \le \frac{1}{2} \int_{(0, T) \times \Omega} \left[\int_0^A \beta u \, da \right]^2 dt \, dx.$$

From the condition $(\beta)^1$ we derive:

$$\left(\lambda - \frac{C_1}{2} \right) \int_Q u^2 \, dt \, da \, dx \le 0 \quad \text{and} \quad u = 0 \text{ in } Q.$$

4.5 *Sketch of the proof of Theorem* 3. We must modify the first step, because (9), (10) are not necessarily fulfilled when we change $(\mathrm{k})^1$ into $(\mathrm{k})^2$.

When $(\mathrm{k})^2$ is satisfied the analogue to problem (IV) is:

$$\left.\begin{aligned} & u_t + u_a + \lambda u - k(a) \, \Delta \int_0^A u(t, a, x) \, da = f \text{ in } Q, \\ & u(0. a, x) = u_0(a, x) \text{ in } (0, A) \times \Omega, \\ & u(t, 0, x) = b(t, x) \text{ in } (0, T) \times \Omega, \\ & \frac{\partial}{\partial \eta} \int_0^A u(t, a, x) \, da = 0 \text{ in } \mathcal{O} \times \partial\Omega. \end{aligned}\right\} \quad \text{(V)}$$

Let $Q_T = (0, T) \times \Omega$. Assuming:

$$f \in L^2(Q), \qquad u_0 \in L^2((0, A) \times \Omega),$$
$$\int_0^A u_0(a, x)\, da \in H^1(\Omega), \qquad b \in L^2((0, A) \times (\Omega)), \tag{24}$$

then any u in $L^2(Q)$ verifying (6), that is:

$$\int_0^A u(t, a, x)\, da = P(t, x) \in L^2(0, T; H^2(\Omega)), \tag{6}$$

and a solution to problem (V) is such that:

$$\begin{aligned} P_t - K\, \Delta P + \lambda P &= \int_0^A f(t, a, x)\, da + b(t, x) - u(t, A, x) \text{ in } Q_T, \\ P(0, x) &= \int_0^A u_0(a, x)\, da = p_0(x) \text{ in } \Omega, \\ \partial P/\partial \eta &= 0 \text{ in } (0, T) \times \partial\Omega, \end{aligned} \tag{VI}$$

where we have used $k = \int_0^A k(a)\, da$. This is obtained by integrating Eq. (V)[1] with respect to the variable a. The hypotheses on (f, b, u_0) and (u, P) are consistent. Now u is the solution to:

$$\begin{aligned} u(0, a, x) &= u_0(a, x) \text{ in } (0, A) \times \Omega, \\ u(t, 0, x) &= b(t, x) \text{ in } (0, T) \times \Omega, \\ u_t + u_a + \lambda u &= k(a)\, \Delta P + f \text{ in } Q. \end{aligned} \tag{VII}$$

If u belongs to $L^2(Q)$ and satisfies (6) and if (f, b, u_0) satisfies (24), a priori bounds can be derived from Eq. (VI) and (VII).

For the parabolic equation (VI) we have:

$$\frac{1}{2}\int_\Omega |\nabla P(T, x)|^2\, dx + \int_{Q_T} \left[\lambda |\nabla P|^2 + K |\Delta P|^2\right] dt\, dx$$
$$= \frac{1}{2}\int_\Omega |\nabla p_0|^2\, dx + \int_{Q_T} \left[\int_0^A f(t, a, x)\, da + b - u(t, A, x)\right][-\Delta P]\, dt\, dx. \tag{25}$$

As for the first-order equation (VII), integrating by parts yields:

$$\frac{1}{2}\int_Q u^2(t, A, x)\, dt\, dx + \lambda \int_Q u^2\, dt\, da\, dx \le \int_Q [f - k(a)\, \Delta P] u\, dt\, da\, dx$$
$$+ \frac{1}{2}\int_{QT} b^2(t, x)\, dt\, dx + \frac{1}{2}\int_{(0,\, A) \times \Omega} u_0^2\, da\, dx. \tag{26}$$

Employing the method of separation of variables and (25), (26) we prove for problem (V) the analogue to Theorem 4. Via a fixed-point method we get Theorem 3 for bounded μ. It suffices to use a suitable linear combination of (25) and (26).

Now we want to turn to the general case, namely, the case when μ is not bounded. μ can be approximated by a sequence $(\mu_n)_n$ satisfying (18); for each integer there is a unique u_n in $L^2(Q)$ verifying (6) and solution to problem (II); that is, u_n and $P_n(t, x) = \int_0^A u(t, a, x)\, da$ are solutions to (VI) and (VII) with:

$$f = -\mu_n u_n \text{ in } Q, \qquad b = \int_0^A \beta u_n \, da \text{ in } (0, T) \times \Omega. \tag{27}$$

From (26) we can get an estimate on $(\mu_n^{1/2} \cdot u_n)_n$ in $L^2(Q)$ but not on the term $-\int_0^A \mu_n \cdot u_n \, da$ that appears on the right-hand side of (25) when f is given by (27). However, when the condition $(\mu)^5$ is satisfied we have:

$$\left| \int_0^A \mu_n \cdot u_n \, da \right| \le \int_0^A \mu_n^{1/2} \mu_n^{1/2} |u_n| \, da \le m^{1/2} \cdot \left(\int_0^A \mu_n u_n^2 \, da \right)^{1/2}.$$

This is enough to obtain that (u_n) and $(\mu_n^{1/2} \cdot u_n)$ (resp. P_n) are bounded in $L^2(Q)$ (resp. $L^2(0, T; H^2(\Omega))$) and to prove Theorem 3 by letting $n \to \infty$.

When $(\mu)^5$ is not fulfilled we shall derive from $(\mu)^4$ that $(\mu_n u_n)$ is bounded in the $L^2(Q)$-norm. If we multiply Eq. (VII) that u_n satisfies by $\mu_n^2 u_n$ and if we integrate over Q we obtain:

$$\int_Q \left[\mu_n^2 + \lambda \mu_n - \frac{\partial \mu_n}{\partial t} - \frac{\partial \mu_n}{\partial a} \right] \mu_n u_n^2 \, dt \, da \, dx$$
$$\le \frac{1}{2} \int_{(0, A) \times \Omega} \left[\mu_n^2(0, a, x) u_0^2(a, x) \, da \, dx + \frac{1}{2} \int_{Q_T} \mu_n^2(t, 0, x) \left[\int_0^A \beta u_n \, da \right]^2 dt \, dx \right.$$
$$- \int_Q [\mu_n k \, \Delta P_n] \cdot [\mu_n u_n] \, dt \, da \, dx.$$

We can choose the sequence (μ_n) so that each μ_n satisfies $(\mu)^4$ uniformly with respect to n. Thus:

$$(\lambda + \lambda_1) \int_Q \mu_n^2 u_n^2 \, dt \, da \, dx + \lambda_2 \int_Q \mu_n u_n^2 \, dt \, da \, dx$$
$$\le C_7 + C_8 \int_Q u_n^2 \, dt \, da \, dx + M^{1/2} \cdot \left[\int_{Q_T} |\Delta P_n|^2 \, dt \, dx \right]^{1/2} \left[\int_Q \mu_n^2 u_n^2 \, dt \, da \, dx \right]^{1/2} \tag{28}$$

where C_7 and C_8 are independent of n. From (28), (25) and (26) are satisfied by u_n and P_n when (f, b) is given by (27) and the inequality

$$\left| \int_0^A \mu_n u_n \, da \right| \le A^{1/2} \left(\int_0^A \mu_n^2 u_n^2 \, da \right)^{1/2}.$$

We derive that if λ has been chosen large enough:

$$\int_Q [u_n^2 + \mu_n^2 u_n^2] \, dt \, da \, dx + \int_{Q_T} |\Delta P_n|^2 \, dt \, dx \le C_9 \text{ independent of } n.$$

So letting $n \to \infty$ we conclude.

Remark 3. An alternative method is to approximate the kernel k in such a way that the condition $(k)^4$ is satisfied (see Sec. 3).

Bibliography

[1]. G. diBlasio and L. Lamberti, *An initial boundary value problem for age population diffusion*, SIAM J. Appl. Math. **35**, 530–615 (1978)

[2] M. G. Garroni and M. Langlais, *Age-dependent population diffusion with external constraints*, J. Math. Biol., to appear

[3] M. E. Gurtin, *A system of equations for age-dependent population diffusion*, J. Theor. Biol. **40** 389–392 (1973)

[4] M. E. Gurtin and R. C. MacCamy, *On the diffusion of biological population*, Math. Biosci. **38**, 35–49 (1977)

[5] M. E. Gurtin and R. C. MacCamy, *Some simple models for nonlinear age-dependent population dynamics*, Math. Biosci. **43**, 199–211 (1979)

[6] F. Hoppensteadt, *Mathematical theories of populations: demographics, genetics, and epidemics*, Soc. Ind. Appl. Math., Philadelphia (1975)

[7] H. L. Langhaar, *General population theory in age-time continuum*, J. Franklin Inst. **293**, 199–214 (1972)

[8] M. Langlais, *Sur un problème de dynamique de population*, Tech. Rep. A. A. I., University of Bordeaux 1 (1980)

[9] M. Langlais, Thesis, University of Bordeaux 1 (1981)

[10] J. L. Lions and E. Magènes, *Problèmes aux limites homogènes et applications*, tôme 1, Dunod, Paris (1968)

[11] R. C. MacCamy, *A population model with nonlinear diffusion*, J. Diff. Eqs. **39**, 52–72 (1981)