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On a logarithmic criterion for uniform polynomial stability

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Abstract. The paper presents a logarithmic criterion for uniform polynomial stability of evolution operators in Banach spaces. As applications, another four characterizations of uniform polynomial stability are obtained.

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1 Introduction

The classical notion of (uniform) exponential stability, essentially introduced by Perron in [13] plays an important role in a large part of the theory of dynamical systems. We refer to the books [1], [2], [5], [6] and [12]. However, the exponential stability is a strong requirement and hence is of considerable interest to look for another growth rates that are not necessarily exponentially, in particular pollynomial growth rates. This approach is present in the work of Barreira and Valls [3], Bento and Silva [4], Hai [8], Megan, Ceauşu and Rămneanţu [9], [10], [11], [12], [14], [15]. The main objective of this paper is to give a new characterization for uniform polynomial stability of evolution operators in Banach spaces. As applications of this result we obtain another four characterizations for this property.

2 Definitions and notations

Let X be a real or complex Banach space, X^* the dual space of X and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms on X, on X^* and on $\mathcal{B}(X)$ will be denoted by $\|.\|$. The identity operator on X is denoted by I. We also denote by

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}, \qquad \Delta_1 = \{(t,s) \in \Delta : s \geq 1\}$$

and

$$T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}, \qquad T_1 = \{(t,s,t_0) \in T : t_0 \geq 1\}.$$

Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator on X (i.e. $U(t,t) = I$ for every $t \geq 0$ and $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$.)

Definition 2.1. An evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be *strongly measurable* if for all $(s,x) \in \mathbb{R}_+ \times X$, the mapping $t \mapsto \|U(t,s)x\|$ is measurable on $[s,\infty)$.

Definition 2.2. An evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be **-measurable* if for all $(s,x^*) \in \mathbb{R}_+ \times X^*$, the mapping $t \mapsto \|U(t,s)^*x^*\|$ is measurable on $[0,t)$.

Definition 2.3. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly polynomially stable (u.p.s.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$(t+1)^\nu \|U(t,t_0)x_0\| \leq N(s+1)^\nu \|U(s,t_0)x_0\|$$

for all $(t,s,t_0,x_0) \in T \times X$.

Remark 2.1. The following assertions are equivalent:

(i) U is u.p.s.

(ii) there are $N \geq 1$ and $\nu > 0$ such that

$$(t+1)^\nu \|U(t,s)x\| \leq N(s+1)^\nu \|x\|$$

for all $(t,s,x) \in \Delta \times X$.

(iii) there are $N \geq 1$ and $\nu > 0$ such that

$$(t+1)^\nu \|U(t,s)x\| \leq N(s+1)^\nu \|x\|$$

for all $(t,s,x) \in \Delta_1 \times X$.

(iv) there are $N \geq 1$ and $\nu > 0$ such that

$$(t+1)^{\nu}\|U(t,t_0)x_0\| \leq N(s+1)^{\nu}\|U(s,t_0)x_0\|$$

for all $(t,s,t_0,x_0) \in T_1 \times X$.

Definition 2.4. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly stable (u.s.)* if there exists $N \geq 1$ such that

$$\|U(t,t_0)x_0\| \leq N\|U(s,t_0)x_0\|$$

for all $(t,s,t_0,x_0) \in T \times X$.

Remark 2.2. The following assertions are equivalent

(i) U is u.s.

(ii) $\sup\limits_{(t,s)\in\Delta} \|U(t,s)\| < \infty$.

(iii) $\sup\limits_{(t,s)\in\Delta_1} \|U(t,s)\| < \infty$.

(iv) there exists $N \geq 1$ with

$$\|U(t,t_0)x_0\| \leq N\|U(s,t_0)x_0\||$$

for all $(t,s,t_0,x_0) \in T_1 \times X$.

Definition 2.5. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has a uniform polynomial growth (u.p.g.)* if there are $M \geq 1$ and $\omega > 0$ such that

$$(s+1)^{\omega}\|U(t,t_0)x_0\| \leq M(t+1)^{\omega}\|U(s,t_0)x_0\|$$

for all $(t,s,t_0,x_0) \in T \times X$.

Remark 2.3. The following conditions are equivalent

(i) U has u.p.g.

(ii) there are $M \geq 1$ and $\omega > 0$ with

$$(s+1)^{\omega}\|U(t,s)x\| \leq M(t+1)^{\omega}\|x\|$$

for all $(t,s,x) \in \Delta \times X$.

(iii) there are $M \geq 1$ and $\omega > 0$ with

$$(s+1)^{\omega}\|U(t,s)x\| \leq M(t+1)^{\omega}\|x\|$$

for all $(t,s,x) \in \Delta_1 \times X$.

(iv) there exist $M \geq 1$ and $\omega > 0$ with

$$(s+1)^{\omega}\|U(t,t_0)x_0\| \leq M(t+1)^{\omega}\|U(s,t_0)x_0\||$$

for all $(t,s,t_0,x_0) \in T_1 \times X$.

Remark 2.4. It is obvious that

$$u.p.s. \Rightarrow u.s. \Rightarrow u.p.g.$$

3 Uniform polynomial stability

The main result of this paper is

Theorem 3.1. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform polynomial growth. Then, U is uniformly polynomially stable if and only if*

$$S \stackrel{d}{=} \sup_{(t,s)\in\Delta_1} \|U(t,s)\| \ln \frac{t+1}{s+1} < \infty$$

for all $(t,s) \in \Delta_1$.

Proof. Necessity. If U is u.p.s. then there exist $N \geq 1$ and $\nu > 0$ such that $\|U(t,s)\| \leq N \left(\frac{t+1}{s+1}\right)^{-\nu}$. Then, we have that

$$S < N \left(\frac{t+1}{s+1}\right)^{-\nu} \ln \frac{t+1}{s+1} = f\left(\frac{t+1}{s+1}\right),$$

where $f(u) = \frac{\ln u}{u^{\nu}}$. It is easy to see that $S \leq f(e^{\frac{1}{\nu}}) < \infty$.

Sufficiency. **Step 1.** Firstly, we consider $n \stackrel{d}{=} \left[\frac{\ln\left(\frac{t}{s}\right)}{4S}\right]$, with $(t,s) \in \Delta_1$.

Then, it is easy to observe that the following inequalities hold:

(1) $se^{4nS} \leq t < se^{4(n+1)S}$

(2) $\left(\frac{t+1}{s+1}\right)^{\frac{\ln 2}{4S}} \leq 2^{n+1}$

for all $(t, s) \in \Delta_1$.

Step 2. We prove that

$$\|U(se^{4S}, s)\| \leq \frac{1}{2}, \ \forall s \geq 1.$$

Indeed, from $\dfrac{1+s^{4S}}{1+s} \geq e^{2S}$, it results that

$$\|U(se^{4S}, s)\| \leq \frac{S}{\ln \dfrac{1+se^{4S}}{1+s}} \leq \frac{S}{\ln e^{2S}} = \frac{1}{2}.$$

Step 3. We have that

$$\|U(se^{4nS}, s)\| \leq \frac{1}{2^n}, \forall s \geq 1.$$

Indeed,

$$\|U(se^{4nS}, s)\| = \|U(se^{4nS}, se^{4(n-1)S})\| \cdot \|U(se^{4(n-1)S}, se^{4(n-2)S})\| \cdot \ldots \cdot \|U(se^{4S}, s)\|.$$

Using the previous step, we obtain that

$$\|U(se^{4nS}, s)\| \leq \frac{1}{2^n}.$$

Finally, we prove that U is u.p.s. Using the evolution property, the first and the third step, we have

$$\|U(t, s)\| \leq \|U(t, se^{4nS})\| \cdot \|U(se^{4nS}, s)\| \leq M \left(\frac{1+t}{1+se^{4nS}}\right)^{\omega} \cdot \frac{1}{2^n} \leq N \left(\frac{t+1}{s+1}\right)^{-\nu},$$

where $N = 2Me^{4\omega S} > 1$ and $\nu = \dfrac{\ln 2}{4S} > 0$.

□

In what follows, we will give a Barbashin [1] characterization for the uniform polynomial stability.

Theorem 3.2. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator *-strongly measurable with uniform polynomial growth. Then U is uniformly polynomially stable if and only if there exists $B > 1$ such that*

$$\int_0^t \frac{\|U(t,s)^*x^*\|}{s+1} ds \leq B\|x^*\|$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$.

Proof. Necessity. If U is u.p.s. then

$$\int_0^t \frac{\|U(t,s)^*x^*\|}{s+1} ds \leq \frac{N\|x^*\|}{(t+1)^\nu} \int_0^t (s+1)^{\nu-1} ds \leq B\|x^*\|,$$

where $B = \dfrac{N}{\nu} + 1$.
Sufficiency. If $t \geq 2s+1$ then

$$\begin{aligned} |\langle x^*, U(t,s)x\rangle| &= \frac{1}{s+1} \int_s^{2s+1} |\langle U(t,\tau)^*x^*, U(\tau,s)x\rangle| d\tau \leq \\ &\leq 2M \int_s^{2s+1} \left(\frac{\tau+1}{s+1}\right)^\omega \|x\| \frac{\|U(t,\tau)^*x^*\|}{\tau+1} d\tau \leq \\ &\leq 2^{\omega+1}\|x\|M \int_0^t \frac{\|U(t,\tau)^*x^*\|}{\tau+1} d\tau \leq 2^{\omega+1}MB\|x\|\|x^*\| \end{aligned}$$

It results that $\|U(t,s)\| \leq 2^{\omega+1}BM$, for all $t \geq 2s+1$.
If $t \in [s, 2s+1)$ then

$$\|U(t,s)x\| \leq M\left(\frac{t+1}{s+1}\right)^\omega \leq M2^\omega \leq 2^{\omega+1}BM.$$

Finally, we obtain that U is uniformly stable. Now, we prove that U is uniformly polynomially stable. We observe that

$$\begin{aligned} \|U(t,s)^*x^*\| \ln \frac{t+1}{s+1} &= \int_s^t \frac{\|U(t,s)^*x^*\|}{\tau+1} d\tau \leq \int_s^t \|U(\tau,s)^*\| \frac{\|U(t,\tau)^*x^*\|}{\tau+1} d\tau \leq \\ &\leq M_1 \int_s^t \frac{\|U(t,\tau)^*x^*\|}{\tau+1} d\tau \leq M_1 B\|x^*\|. \end{aligned}$$

So, we obtained $\|U(t,s)\| \ln \dfrac{t+1}{s+1} \le BM_1$, and from Theorem (3.1) we have that U is u.p.s. $\square$

In what follows, we will present a characterization of Datko type [7] of the uniform polynomial stability concept.

Theorem 3.3. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform polynomial growth. Then U is uniformly polynomially stable if and only if there exists $D > 1$ with*

$$\int_s^\infty \frac{\|U(\tau,t_0)x_0\|}{\tau+1} d\tau \le D\|U(s,t_0)x_0\|$$

for all $(s,t_0,x_0) \in \Delta \times X$.

Proof. Necessity. If U u.p.s. then there exist $N > 1$ and $\nu > 0$ with

$$\int_s^\infty \frac{\|U(\tau,t_0)x_0\|}{\tau+1} d\tau \le N \int_s^\infty \left(\frac{s+1}{\tau+1}\right)^\nu \frac{\|U(\tau,t_0)x_0\|}{\tau+1} d\tau =$$

$$= \frac{N}{\nu}\|U(s,t_0)x_0\| \le D\|U(s,t_0)x_0\|.$$

where $D = 1 + \dfrac{N}{\nu}$.

Sufficiency. Step 1. We show that U is uniformly stable. If $t \ge 2s+1$ then

$$\|U(t,t_0)x_0\| = \frac{2}{t+1}\int_{\frac{t-1}{2}}^t \|U(t,t_0)x_0\| d\tau \le 2\int_{\frac{t-1}{2}}^t \frac{\|U(t,t_0)x_0\|}{\tau+1} d\tau \le$$

$$\le 2M \int_{\frac{t-1}{2}}^t \left(\frac{t+1}{\tau+1}\right)^\omega \frac{\|U(\tau,t_0)x_0\|}{\tau+1} d\tau \le M2^{\omega+1}\int_s^\infty \frac{\|U(\tau,t_0)x_0\|}{\tau+1} d\tau \le$$

$$\le DM2^{\omega+1}\|U(s,t_0)x_0\|$$

So,

$$\|U(t,t_0)x_0\| \le DM2^{\omega+1}\|U(s,t_0)x_0\|, \forall t \ge 2s+1, \forall s \ge t_0 \ge 0, \forall x_0 \in X.$$

If $t \in [s, 2s+1)$ then $1 \le \dfrac{t+1}{s+1} \le 2$. We obtain

$$\|U(t,t_0)x_0\| \le M\left(\frac{t+1}{s+1}\right)^\omega \|U(s,t_0)x_0\| \le DM2^{\omega+1}\|U(s,t_0)x_0\|.$$

So,

$$\|U(t,t_0)x_0\| \le M_1\|U(s,t_0)x_0\|$$

for all $(t,s,t_0,x_0) \in T \times X$, where $M_1 = DM2^{\omega+1}$.
Step 2. We prove that U is u.p.s. using Theorem (3.1).

$$\|U(t,t_0)x_0\| \ln\frac{t+1}{s+1} = \int_s^t \frac{\|U(t,t_0)x_0\|}{\tau+1}d\tau \le M_1\int_s^t \frac{\|U(\tau,t_0)x_0\|}{\tau+1}d\tau \le$$
$$\le DM_1\|U(s,t_0)x_0\|.$$

For $t_0 = s$ and from Theorem (3.1) we obtain the conclusion. □

The next theorem is a characterization which uses Lyapunov functions for the uniform polynomial stability of an evolution operator.

Theorem 3.4. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform polynomial growth. Then U is uniformly polynomially stable if and only if there are $D > 1$ and $L : \Delta \times X \to \mathbb{R}_+$ with the properties*

(i) $L(s,t_0,x_0) \le D\|U(s,t_0)x_0\|, \ \forall(s,t_0,x_0) \in \Delta \times X$

(ii) $L(t,t_0,x_0) = L(s,t_0,x_0) - \int_s^t \frac{\|U(\tau,t_0)x_0\|}{\tau+1}d\tau, \ \forall(t,s,t_0,x_0) \in T \times X.$

Proof. Necessity. If U is u.p.s. then by Theorem (3.3) the function $L : \Delta \times X \to \mathbb{R}_+$ defined by

$$L(s,t_0,x_0) = \int_s^\infty \frac{\|U(\tau,t_0)x_0\|}{\tau+1}d\tau$$

satisfies the conditions (i) and (ii).
Sufficiency. If there exists a function $L : \Delta \times X \to \mathbb{R}_+$ with the properties (i) and (ii) then

$$\int_s^t \frac{\|U(\tau,t_0)x_0\|}{\tau+1}d\tau \le L(s,t_0,x_0) \le D\|U(s,t_0)x_0\|$$

for all $(t,s,t_0,x_0) \in T \times X$. For $t \to \infty$ we obtain

$$\int_s^\infty \frac{\|U(\tau,t_0)x_0\|}{\tau+1}d\tau \le D\|U(s,t_0)x_0\|$$

for all $(s,t_0,x_0) \in \Delta \times X$. By Theorem (3.3) it results that U is u.p.s. □

Finally, we present a new proof of a result due to Hai [8].

Theorem 3.5. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform polynomial growth. Then U is uniformly polynomially stable if and only if there exists $r > 1$ such that*

$$S_1 \stackrel{d}{=} \sup_{s\geq 1} \|U(rs, s)\| < 1. \tag{3.1}$$

Proof. Necessity. If U is u.p.s. then $r = e^{4S}$, where S is given by Theorem (3.1) and $\dfrac{1+rs}{1+s} > e^{2S}$. Thus

$$S_1 = \sup_{s\geq 1} \|U(rs, s)\| \leq \frac{S}{\ln \dfrac{1+rs}{1+s}} \leq \frac{1}{2}.$$

Sufficiency. If $(t, s) \in \Delta_1$ then there exists a natural number n such that

$$sr^n \leq t < sr^{n+1}$$

$\Big(n$ is the integer part of $\dfrac{\ln \frac{t}{s}}{\ln r}$, where r is given by (3.1)$\Big)$. Then

$$\frac{sr^n + 1}{s+1} \leq \frac{t+1}{s+1} \leq \frac{sr^{n+1} + 1}{s+1} < r^{n+1}$$

and hence

$$\ln \frac{t+1}{s+1} < (n+1) \ln r \tag{3.2}$$

From here and u.p.g. property of U it results that there are $M > 1$ and $\omega > 0$ such that

$$\|U(t, s)\| \leq \|U(t, sr^n)\| \|U(sr^n, s)\| \leq M \left(\frac{t+1}{sr^n+1}\right)^{\omega} \|U(sr^n, sr^{n-1})\| \|U(sr^{n-1}, s)\| \leq$$
$$\leq MS_1 \left(\frac{sr^{n+1}+1}{sr^n+1}\right)^{\omega} \|U(sr^{n-1}, sr^{n-2})\| \|U(sr^{n-2}, s)\| \leq$$
$$\leq MS_1^2 (r+1)^{\omega} \|U(sr^{n-2}, s)\| \leq MS_1^n (r+1)^{\omega}$$

for all $(t, s) \in \Delta_1$. Using (3.2) we obtain

$$\|U(t, s)\| \ln \frac{t+1}{s+1} \leq MS_1^n (r+1)^{\omega} (n+1) \ln r \leq MN(r+1)^{\omega} \ln r,$$

for all $(t, s) \in \Delta_1$, where $N = \sup\limits_{n\in\mathbb{N}} (n+1) S_1^n < \infty$. Finally, it results that

$$S \stackrel{d}{=} \sup_{(t,s)\in\Delta_1} \|U(t, s)\| \ln \frac{t+1}{s+1} \leq MN(r+1)^{\omega} \ln r < \infty$$

By Theorem (3.1) it follows that U is u.p.s. □

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