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On a logarithmic Hartree equation

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Abstract: We study the existence of radially symmetric solutions for a nonlinear planar Schrödinger-Poisson system in presence of a superlinear reaction term which doesn't satisfy the Ambrosetti-Rabinowitz condition. The system is re-written as a nonlinear Hartree equation with a logarithmic convolution term, and the existence of a positive and a negative solution is established via critical point theory.

Keywords: planar Schrödinger-Poisson system, logarithmic Hartree equation, Hardy-Littlewood-Sobolev inequality, superlinear source

MSC: 35J50, 35Q40

1 Introduction

In recent past years many papers have been devoted to finding solutions of Schrödinger-Poisson systems of the form

$$\begin{cases} i\psi_t - \Delta\psi + E(x)\psi + \gamma w\psi = f(x,t), & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ \Delta w = \psi^2, & \text{in } \mathbb{R}^N, \end{cases} \tag{SP}$$

and often the main objects were standing wave solutions, i.e. solutions of the form

$$\psi(x,t) = e^{-i\omega t}u(x), \quad \omega \in \mathbb{R},$$

so that (SP) reads

$$\begin{cases} -\Delta u + b(x)u + \gamma wu = \tilde{f}(x,t), & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ \Delta w = u^2, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $b(x) = E(x) + \omega$ and $\tilde{f}(x,t) = e^{i\omega t}f(x,t)$.

Due to the numerous several Physical applications, the most studied case is $N = 3$ (or $N \geq 3$). On the other hand, the 1–dimensional case was considered in [1] when $f = 0$, and the existence of a unique ground state was established by decreasing symmetric rearrangements tools. The 2–dimensional case when $f = 0$ was first approached in [2], Section 6, only from a numerical point of view, while the first rigorous existence result was given in [3] by using a shooting method for ODEs.

Moving down to lower dimensions, in particular to $N = 2$, introduces several complications, the first important one is that in this case *the Coulomb potential is not positive*.

In [4] the authors studied the eigenvalue problem for the Schrödinger operator in a bounded domain of $\mathbb{R}^3$, with electromagnetic field **E-H** that is not assigned; in this case the unknowns are the wave function $\psi(x,t)$ and the gauge potentials $\mathbf{A}(x,t)$, $\phi(x,t)$ related to **E-H**. In particular, they considered the problem in which $\mathbf{A}$ and ϕ do not depend on the time and $\psi(x,t) = e^{-i\omega t}u(x)$, with $\omega \in \mathbb{R}$ and u real function. With these

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consideration they assume $\mathbf{A} = 0$, and thus the system reduces to

$$\begin{cases} -\Delta u - \phi u = \omega u, & \text{in } \Omega \subset \mathbb{R}^3, \\ \Delta \phi = 4\pi u^2, & \text{in } \Omega \subset \mathbb{R}^3. \end{cases}$$

Under this hypotheses, they proved the existence of a sequence of solution in a bounded domain of $\mathbb{R}^3$.

Later on, in [5] the authors considered the problem

$$\begin{cases} -\Delta u - \phi u - \omega u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \Delta \phi = 4\pi u^2, & \text{in } \mathbb{R}^3 \end{cases} \tag{1.2}$$

with $p \in \mathbb{R}^+$ and they proved the existence of radially symmetric solutions in $\mathbb{R}^3$, for $p \in [4, 6)$, while in [6] they showed the nonexistence of solutions for $p \in (0, 2]$ or $p \in [6, \infty)$. After that, system (1.2) has been object of an intensive study, where generalizations of several type where considered; we refer to [7–14] for other references and improvements on this subject.

All these works have been done in the whole of $\mathbb{R}^3$, while the two dimensional case has remained for a long time a quite open field of study. Indeed, a theoretical approach in dimension 2 is harder than in higher dimensions due to the lack of positivity of the Coulomb interaction term: precisely, the Coulomb potential is neither bounded from above nor from below.

However, in 2008 Stubbe [15] bypassed this problem giving a suitable variational framework for the problem

$$-\Delta u + au - \frac{1}{2\pi}\left[\ln \frac{1}{|x|} \star |u|^2\right] u = 0 \tag{1.3}$$

and proving the existence of ground states, which is a positive spherically symmetric strictly decreasing function, by solving an appropriate minimization problem for the energy functional associated to the system (see also [16]).

In some recent works, a local nonlinear terms of the form $b|u|^{p-2}u$, $p > 2$ has been added; this kind of nonlinearity are frequently used in Schrödinger equations to model the interaction among particles, like recalled above (see [4]). Thus in [17], they studied a Schrödinger-Poisson system of the type

$$-\Delta u + a(x)u - \frac{1}{2\pi}\left[\ln \frac{1}{|x|} \star |u|^2\right] u = b|u|^{p-2}u \quad \text{in } \mathbb{R}^2,$$

with $b \geq 0$, $p \geq 2$ and $a \in L^\infty(\mathbb{R}^2)$ and they proved that if $p \geq 4$ then the problem has a sequence of solution pairs $\pm u_n$ such that $I(u_n) \to \infty$ as $n \to \infty$.

In this work we are concerned with the integro-differential equation

$$-\Delta u + au - \frac{1}{2\pi}\left[\ln \frac{1}{|x|} \star |u|^2\right] u = f(x, u) \quad \text{in } \mathbb{R}^2, \tag{P}$$

where $a > 0$ and f is a superlinear function. We refer to (P) as the logarithmic Choquard equation. Note that, if compared with (SP), we have chosen $y = -1$; since y represents the charge of the particle that we are studying, it means that we are considering electrons.

In order to generalize and include the previous cases, on the reaction term $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ we assume that it is a Carathéodory function having superlinear growth and not verifying the Ambrosetti-Rabinowitz condition, from now on **(*AR*)**.

Our main result has the following flavour (see Theorem 3.1 for the precise statement):

Theorem 1.1. *Under suitable hypotheses on f problem (P) has two nontrivial constant sign solutions.*

This work is organized as follows. In Section 2 we recall some useful definitions and results that we shall use, we set up an appropriate variational framework and define the energy functional associated to the problem. Moreover, we give an extended results of the estimates given in the Strauss theorem.

In Section 3 we prove the well-posedness and the regularity of our functional and we give a Lemma that plays a fundamental role in the proof of the Cerami condition. Finally, we give the proof of the main existence theorem.

2 Background and Variational Framework

We provide a suitable variational framework for studying (P): indeed, the associated functional is not well defined on the natural Sobolev space $H^1(\mathbb{R}^2)$, and so we need some adjustments taken from [15], see also [17].

We first recall an important result for L^p-spaces.

Theorem 2.1 (Hardy-Littlewood-Sobolev's inequality, [18]). *Let $p, q > 1$ and $0 < \lambda < N$ with $\frac{1}{p} + \frac{\lambda}{N} + \frac{1}{q} = 2$. Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, \lambda, p)$, independent of f and g, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)g(y)|}{|x-y|^\lambda} dxdy \le C(N, \lambda, p)\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}. \tag{2.1}$$

The sharp constant satisfies

$$C(N, \lambda, p) \le \frac{N}{(N-\lambda)} \left(\frac{|S^{N-1}|}{N}\right)^{\frac{\lambda}{N}} \frac{1}{pq} \left(\left(\frac{\lambda/N}{1-\frac{1}{p}}\right)^{\frac{\lambda}{N}} + \left(\frac{\lambda/N}{1-\frac{1}{q}}\right)^{\frac{\lambda}{N}} \right).$$

If $p = q = \frac{2N}{2N-\lambda}$, then

$$C(N, \lambda, p) = C(N, \lambda) = \pi^{\frac{\lambda}{2}} \frac{\Gamma(N/2 - \lambda/2)}{\Gamma(N-\lambda/2)} \left\{ \frac{\Gamma(N/2)}{\Gamma(N)} \right\}^{-1+\lambda/N}.$$

In this case there is equality in (2.1) *if and only if $g \equiv cf$ with c constant and*

$$f(x) = A\left(\gamma^2 + |x - x_0|^2\right)^{-(2N-\lambda)/2}$$

for some $A \in \mathbb{R}$, $0 \ne \gamma \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$. Here $|S^{N-1}|$ denotes the area of the unit sphere in $\mathbb{R}^N$.

First of all, we endow $H^1(\mathbb{R}^2)$ with the scalar product (recall that $a > 0$ is a constant)

$$(u|v) = \int_{\mathbb{R}^2} (Du \cdot Dv + auv)\, dx, \quad \text{for } u, v \in H^1(\mathbb{R}^2),$$

and we introduce the space

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u(x)|^2 \ln(1+|x|)dx < \infty \right\}$$

with the norm defined by

$$\|u\|_X^2 = \int_{\mathbb{R}^2} \left[|Du|^2 + |u|^2 \left(a + \ln(1+|x|)\right) \right] dx.$$

Then, we define the symmetric bilinear forms

$$B_1(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)u(x)v(y)dxdy,$$

$$B_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|}\right) u(x)v(y)dxdy,$$

and

$$B(u, v) = B_1(u, v) - B_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)\, u(x)v(y)dxdy,$$

since for all $r > 0$ we have

$$\ln r = \ln(1+r) - \ln\left(1+\frac{1}{r}\right). \tag{2.2}$$

The definitions above are restricted to measurable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that the corresponding double integral is well defined in the Lebesgue sense. Finally, for any measurable functions $u : \mathbb{R}^2 \to \mathbb{R}$ we consider the seminorm in X

$$|u|_\star^2 = \int_{\mathbb{R}^2} \ln(1+|x|)u^2(x)dx.$$

We note that, since

$$\ln(1+|x-y|) \le \ln(1+|x|+|y|) \le \ln(1+|x|) + \ln(1+|y|), \tag{2.3}$$

we have by the Schwarz inequality

$$\begin{aligned}|B_1(uv, wz)| &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[\ln(1+|x|) + \ln(1+|y|)\right] |u(x)v(x)||w(y)z(y)|dxdy \\ &\le |u|_\star|v|_\star\|w\|_{L^2(\mathbb{R}^2)}\|z\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}|w|_\star|z|_\star \end{aligned}\tag{2.4}$$

for $u, v, w, z \in L^2(\mathbb{R}^2)$. Next, since $0 \le \ln(1+r) \le r$ for all $r > 0$, we have by Theorem 2.1

$$|B_2(u,v)| \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|}u(x)v(y)dxdy \le C\|u\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}, \tag{2.5}$$

for $u, v \in L^{\frac{4}{3}}(\mathbb{R}^2)$, for some constant $C > 0$. In particular, from (2.4) we have

$$B_1(u^2, u^2) \le 2|u|_\star^2\|u\|_{L^2(\mathbb{R}^2)}^2 \tag{2.6}$$

for all $u \in L^2(\mathbb{R}^2)$ and from (2.5) we have

$$B_2(u^2, u^2) \le C\|u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}^4 \tag{2.7}$$

for all $u \in L^{\frac{8}{3}}(\mathbb{R}^2)$.

The energy functional $I : X \to \mathbb{R}$ associated to (P) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|Du|^2 + a|u|^2\right)dx - \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\frac{1}{|x-y|}|u(x)|^2|u(y)|^2dxdy - \int_{\mathbb{R}^2}F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,s)ds$ and the Gâteaux derivative of I along $v \in X$ is

$$\begin{aligned}I'(u)(v) = &\int_{\mathbb{R}^2}(Du\cdot Dv + auv)\,dx \\ &- \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\frac{1}{|x-y|}|u(x)|^2u(y)v(y)dxdy - \int_{\mathbb{R}^2}f(x,u)vdx.\end{aligned}$$

Definition 1. We say that $u \in X$ is a *weak solution* of (P) if

$$I'(u)(v) = 0 \quad \text{for all } v \in X,$$

thus if u is a critical point for I.

Of course, these consideration are only formal, since, without any assumption on f, we cannot differentiate I. In Section 3 we will give some sufficient conditions for I to be of class C^1 in X, while for the moment we continue with formal computations.

We have the following results, the second statement being new, as far as we know, and extending Strauss' Radial Lemma [19] to the space X and its N– dimensional version. Indeed, though later on we shall use only the compact embedding in dimension $N = 2$ presented below, we can prove an asymptotic result which is valid in any space dimension. Since we believe that this property is of independent interest, we present our result in the general case. For this, let us introduce the sets

$$\mathcal{X} = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^2 \ln(1+|x|)dx < \infty \right\}$$

and

$$\mathcal{X}_r = \left\{ u \in \mathcal{X} : u(x) = u(|x|) \right\}.$$

Proposition 2.1. *The following properties hold true:*

- *X is compactly embedded in $L^s(\mathbb{R}^2)$, for all $s \in [2, \infty)$.*
- *There exists $c \in \mathbb{R}$ such that for all $u \in \mathcal{X}_r$*

$$|u(x)| \le c \frac{\|u\|_{\mathcal{X}_r}}{x^{\frac{N-1}{2}} \sqrt[4]{\ln(1+x)}}.$$

Proof. The compact embedding for $N = 2$ is an application of the Riesz criterion (see [20, Theorem XIII.66]). Indeed, if S is a bounded subset of X, then S is bounded in $L^q(\mathbb{R}^2)$ for any $q \in [2, \infty)$, as well. Moreover, for any $R > 0$ and any $u \in S$ we have

$$\int_{\{|x|>R\}} |u|^p dx \le \|u\|_{L^{2p-2}(\mathbb{R}^2)} \left(\int_{\{|x|>R\}} u^2 dx \right)^{1/2} \le C \left(\int_{\{|x|>R\}} u^2 dx \right)^{1/2}$$

for some $C > 0$, and

$$\int_{\{|x|>R\}} u^2 dx \le \frac{1}{\ln(1+R)} \int_{\{|x|>R\}} \ln(1+|x|)u^2 dx \le \frac{C}{\ln(1+R)}$$

for some $C > 0$. Finally, working as in [21, Theorem 9.16] we conclude.

As for the estimate in dimension N, let $u \in \mathcal{X}_r \bigcap C_c^\infty(\mathbb{R}^N)$ and $r > 0$. We have

$$\begin{aligned}\left(\sqrt{\ln(1+r)}u^2 r^{N-1}\right)' &= \frac{1}{2\sqrt{\ln(1+r)}} \frac{u^2 r^{N-1}}{1+r} + 2uu' r^{N-1}\sqrt{\ln(1+r)} + (N-1)r^{N-2}u^2\sqrt{\ln(1+r)} \\ &\ge 2uu' r^{N-1}\sqrt{\ln(1+r)}.\end{aligned}$$

Integrating from r to ∞ we obtain

$$\begin{aligned}\sqrt{\ln(1+r)}u^2 r^{N-1} &\le -\int_{B_r^C} 2uu'\rho^{N-1}\sqrt{\ln(1+\rho)}d\rho = -\int_{B_r^C} 2uu'\sqrt{\ln(1+|x|)}dx \\ &\le C\left(\int_{B_r^C} |u|^2 \ln(1+|x|)\right)^{\frac{1}{2}} \left(\int_{B_r^C} |Du|^2\right)^{\frac{1}{2}} \le C\|u\|_{\mathcal{X}_r}.\end{aligned}$$

Hence

$$|u(r)| \le C \frac{\|u\|_{\mathcal{X}_r}}{x^{\frac{N-1}{2}} \sqrt[4]{\ln(1+x)}}.$$

The conclusion follows by density. □

We finally introduce the "positive" and "negative" part of the reaction term, namely

$$f_\pm(x,t) = f(x,t^\pm) \text{ for a.e. } x \in \mathbb{R}^2 \text{ and for all } t \in \mathbb{R},$$

which are Carathéodory functions if f is, and $F_\pm(x,t) = \int_0^t f_\pm(x,s)ds$; moreover, we set

$$I_\pm(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|Du|^2 + a|u|^2\right) dx - \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}|u^\pm(x)|^2|u^{\pm(y)}|^2 dxdy - \int_{\mathbb{R}^2} F_\pm(x,u)dx$$

for all $u \in X$.

3 The Existence Theorem

We assume the following hypotheses on the reaction term f:

H(i) let $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $f(x,0) = 0$ and $f(x,\cdot) = f(|x|,\cdot)$ for a.e. $x \in \mathbb{R}^2$. Moreover, there exist $c \in L^p(\mathbb{R}^2)$ for $1 < p < 2$, $d > 0$ and $q \in (2,\infty)$ such that $|f(x,t)| \le c(x) + d|t|^{q-1}$, for a.e. $x \in \mathbb{R}^2$ and for all $t \in \mathbb{R}$;

H(ii) $f(x,t) = o(|t|)$ as $t \to 0$ uniformly for a.e. $x \in \mathbb{R}^2$;

H(iii) $\lim_{|t|\to+\infty} \frac{F(x,t)}{t^2} = +\infty$ uniformly for a.e. $x \in \mathbb{R}^2$;

H(iv) if $\sigma(x,s) = f(x,s)s - 2F(x,s)$, then there exists $\mathcal{M}^\star \in L^1_+(\mathbb{R}^2)$ such that $\sigma(x,s) \le \sigma(x,t) + \mathcal{M}^\star(x)$ for a.e. $x \in \mathbb{R}^2$ and for all $0 \le s \le t$ or $t \le s \le 0$;

H(v) there exists $\tilde{u} \in X$ such that

$$\lim_{y\to+\infty} \frac{\int_{\mathbb{R}^2} F(x, y^2\tilde{u}^\pm(yx))dx}{y^4 \ln y} = +\infty.$$

Remark 3.1.

1. Condition **H(iv)** was introduced in [22] to overcome the necessity of using the Ambrosetti-Rabinowitz condition.

2. Condition **H(v)** is trivially satisfied if $f(x,t) = |t|^{p-2}t$ or if $F(x,t) \ge c|t|^{\tilde{q}} - \zeta(x)$, where $\zeta \in L^1_+(\mathbb{R}^2)$ and $\tilde{q} > 4$. The very last condition is generally a consequence of the usual Ambrosetti-Rabinowitz condition, which here should be assumed *a priori*, see [23].

We start proving

Proposition 3.1. *If* **H(i)** *holds, then the functional $I : X \to \mathbb{R}$ is well-defined and of class C^1 on X. The same is true for $I_\pm$.*

Proof. We do the proof for I, the ones for $I_\pm$ being completely analogous. From hypothesis **H(i)** we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} F(x,u)dx\right| &\le \int_{\mathbb{R}^2} c(x)|u(x)|dx + \frac{d}{q}\int_{\mathbb{R}^2}|u(x)|^q dx \\ &\le \|c\|_{L^p(\mathbb{R}^2)}\|u\|_{L^{p'}(\mathbb{R}^2)} + \frac{d}{q}\|u\|^q_{L^q(\mathbb{R}^2)}.\end{aligned} \tag{3.1}$$

From (3.1), (2.6) and (2.7) we have

$$|I(u)| \leq \frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^2)} + \frac{1}{4\pi}\int_{\mathbb{R}^2} \ln(1+|x|)u^2(x)\int_{\mathbb{R}^2} u^2(y)dxdy$$
$$+ C_2\|u\|^4_{L^{\frac{8}{3}}(\mathbb{R}^2)} + \|c\|_{L^p(\mathbb{R}^2)}\|u\|_{L^{p'}(\mathbb{R}^2)} + \frac{d}{q}\|u\|^q_{L^q(\mathbb{R}^2)} < \infty,$$

for some constant $C_2 > 0$, so the associated functional is well-defined.

Now we observe that the Gâteaux derivative of $B(u^2, u^2)$ is

$$B(u^2, u\varphi) = 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}|u(x)|^2 u(y)\varphi(y)dxdy \tag{3.2}$$

for all $\varphi \in X$, so the functional I is the sum of C^1 terms and we have the desired regularity follows. □

Our purpose is to prove that both I_+ and I_- satisfy the assumptions of the mountain pass theorem. While the geometric structure is somehow standard and is obtained exploiting **H(i)** and **H(v)**, the compactness condition is the delicate part: the lack of the Ambrosetti-Rabinowitz condition makes the bound on Cerami sequences more complicated, and, indeed, by using **H(iv)** we obtain only a bound in $H^1(\mathbb{R}^2)$. Thus we move to radial functions and use Strauss' Lemma to exploit the compact embedding in $L^q(\mathbb{R}^2)$: thanks to the principle of symmetric criticality, a critical point for the functional constrained on the subset of radial functions is a free critical point, see [24]. This permits to recover the desired bound of Cerami sequences in X and finally prove that the Cerami condition holds.

Hence, from now on, we consider $I : X_r \to \mathbb{R}$, where

$$X_r = \{u \in X : u(x) = u(|x|)\}$$

and we look for critical point for $I|_{X_r}$. For the sake of simplicity we will continue to denote by I the functional $I|_{X_r}$.

Now we are ready to prove that the $(C)_d$-condition holds. In order to do that, we first give the following definition

Definition 2. We say that a sequence $(u_n)_n \subset X_r$ is a $(C)_d$**-sequence** if

$$I(u_n) \to d \quad \text{and} \quad \|I'(u_n)\|_{X_r^\star}\left(1 + \|u_n\|_{X_r}\right) \to 0 \quad \text{as } n \to \infty.$$

We say that I satisfies the $(C)_d$-condition if any $(C)_d$-sequence admits a convergent subsequence.

We prove that, under suitable hypotheses, a $(C)_d$-sequence in X_r is bounded in $H^1_r(\mathbb{R}^2)$.

Lemma 3.1. *Suppose hypotheses **H(i)**, **H(iii)** and **H(iv)** hold and let $(u_n)_n \subset X_r$ be a $(C)_d$-sequence for I_+ (I_- respectively). Then $(u_n)_n$ is bounded in $H^1_r(\mathbb{R}^2)$.*

Proof. We do the proof for the I_+, for I_- being analogous.

Let $(u_n)_n \subset X_r$ be a $(C)_d$-sequence. In particular,

$$|I_+(u_n)| \leq M_1 \text{ for some } M_1 > 0 \text{ and every } n \geq 1, \tag{3.3}$$

$$\left(1 + \|u_n\|_{X_r}\right) I'_+(u_n) \to 0 \text{ in } X_r^\star \text{ as } n \to \infty. \tag{3.4}$$

We recall that for any $v \in X_r$ we have

$$I'_+(u_n)(v) = \int_{\mathbb{R}^2} (Du_n \cdot Dv + au_n v)\,dx - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}|u_n^+(x)|^2 u_n^+(y)v(y)dxdy - \int_{\mathbb{R}^2} f_+(x,u_n)v\,dx.$$

From (3.4) we have

$$|I'_+(u_n)(h)| \leq \frac{\varepsilon_n \|h\|_{X_r}}{1 + \|h\|_{X_r}} \tag{3.5}$$

for all $h \in X_r(\mathbb{R}^2)$, where $\varepsilon_n \to 0$ as $n \to \infty$. We choose $h = -u_n^- \in X_r$ and we obtain

$$|I'_+(u_n)(-u_n^-)| \leq \frac{\varepsilon_n \|u_n^-\|_{X_r}}{1 + \|u_n^-\|_{X_r}} \leq \varepsilon_n,$$

so that

$$\int_{\mathbb{R}^2} \left(|Du_n^-|) + a|u_n^-|^2\right) dx = \|u_n^-\|^2_{H^1_r(\mathbb{R}^2)} \leq \varepsilon_n,$$

which means that

$$u_n^- \to 0 \text{ in } H^1_r(\mathbb{R}^2) \text{ as } n \to \infty. \tag{3.6}$$

From (3.3) we have

$$\left|\frac{1}{2}\|u_n\|^2_{H^1_r(\mathbb{R}^2)} - \frac{1}{4}B\left((u_n^+)^2, (u_n^+)^2\right) - \int_{\mathbb{R}^2} F_+(x, u_n)dx\right| \leq M_1 \tag{3.7}$$

so that

$$\|u_n\|^2_{H^1_r(\mathbb{R}^2)} - \frac{1}{2}B\left((u_n^+)^2, (u_n^+)^2\right) - 2\int_{\mathbb{R}^2} F_+(x, u_n)dx \leq M_2 \text{ for some } M_2 > 0$$

and using (3.6) we obtain

$$\|u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - \frac{1}{2}B\left((u_n^+)^2, (u_n^+)^2\right) - 2\int_{\mathbb{R}^2} F(x, u_n^+)dx \leq M_2. \tag{3.8}$$

We assume by contradiction that $(u_n)_n$ is unbounded in $H^1_r(\mathbb{R}^2)$, then by passing to a subsequence, if necessary, we assume that $\|u_n^+\|_{H^1_r(\mathbb{R}^2)} \to \infty$ as $n \to \infty$.

We set $v_n = \dfrac{u_n^+}{\|u_n^+\|_{H^1_r(\mathbb{R}^2)}}$, $n \geq 1$, so we may assume that, by Strauss' Theorem,

$$v_n \rightharpoonup v \text{ in } H^1_r(\mathbb{R}^2) \text{ and } v_n \to v \text{ in } L^s(\mathbb{R}^2),\ s \in (2, \infty),\ v \geq 0. \tag{3.9}$$

To reach our goal we show that both $v \neq 0$ and $v = 0$ lead to a contradiction. We start with the case $v \neq 0$.

We define the set $Z(v) = \left\{x \in \mathbb{R}^2 : v(x) = 0\right\}$; then $meas\left(\mathbb{R}^2 \setminus Z(v)\right) > 0$ and $u_n^+(x) \to \infty$ as $n \to \infty$ for a.e. $x \in \mathbb{R}^2 \setminus Z(v)$. By **H(iii)** we have

$$\frac{F(x, u_n^+)}{\|u_n^+\|^2_{H^1_r(\mathbb{R}^2)}} = \frac{F(x, u_n^+)}{|u_n^+|^2} v_n^2 \to \infty \text{ for a.e. } x \in \mathbb{R}^2 \setminus Z(v)$$

and by Fatou's Lemma we obtain

$$\int_{\mathbb{R}^2} \frac{F(x, u_n^+)}{\|u_n^+\|^2_{H^1_r(\mathbb{R}^2)}} dx \to \infty \text{ as } n \to \infty. \tag{3.10}$$

But from (3.7) we have

$$-\frac{1}{2} + \frac{1}{4}\frac{B\left((u_n^+)^2, (u_n^+)^2\right)}{\|u_n^+\|^2_{H^1_r(\mathbb{R}^2)}} + \int_{\mathbb{R}^2} \frac{F(x, u_n^+)}{\|u_n^+\|^2_{H^1_r(\mathbb{R}^2)}} dx \leq \frac{M_1}{\|u_n^+\|^2_{H^1_r(\mathbb{R}^2)}}$$

so that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2} \frac{F(x, u_n^+)}{\|u_n^+\|^2_{H^1_r(\mathbb{R}^2)}} dx \leq M_3 \text{ for some } M_3 > 0. \tag{3.11}$$

Comparing (3.10) and (3.11) we reach a contradiction.

Now we consider the case $v = 0$. For every $n \in \mathbb{N}$ we define the continuous function $y_n : [0,1] \to \mathbb{R}$ as

$$y_n(t) = I(tu_n^+) \text{ for all } n \geq 1 \text{ and all } t \in [0,1],$$

and let $t_n \in [0,1]$ be such that

$$y_n(t_n) = \max\{y_n(t) : t \in [0,1]\}. \tag{3.12}$$

For $\lambda > 0$, let $w_n = (2\lambda)^{\frac{1}{2}} v_n \in H_r^1(\mathbb{R}^2)$. Then $w_n \to 0$ in $L^p(\mathbb{R}^2)$ by (3.9). By **H(i)** and the Krasnoselskii's Theorem (see [25, Theorem 2.75]), we have

$$\int_{\mathbb{R}^2} F(x, w_n)dx \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Since $\|u_n^+\|_{H_r^1(\mathbb{R}^2)} \to \infty$ as $n \to \infty$, we can find $n_0 \geq 1$ such that $\frac{(2\lambda)^{\frac{1}{2}}}{\|u_n^+\|_{H_r^1(\mathbb{R}^2)}} \in (0,1)$ for all $n \geq n_0$. Then, by (3.12),

$$y(t_n) \geq y\left(\frac{(2\lambda)^{\frac{1}{2}}}{\|u_n^+\|_{H_r^1(\mathbb{R}^2)}}\right) \text{ for all } n \geq n_0.$$

Hence, by (2.2) and (2.6) we get

$$\begin{aligned}
I(t_n u_n^+) \geq I((2\lambda)^{\frac{1}{2}} v_n) &= \lambda - \frac{\lambda^2}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}|v_n(x)|^2|v_n(y)|^2 dxdy - \int_{\mathbb{R}^2} F(x,w_n)dx \\
&= \lambda + \frac{\lambda^2}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\,|v_n(x)|^2|v_n(y)|^2 dxdy \\
&\quad - \frac{\lambda^2}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|v_n(x)|^2|v_n(y)|^2 dxdy - \int_{\mathbb{R}^2} F(x,w_n)dx \\
&\geq \lambda - \frac{\lambda^2}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|v_n(x)|^2|v_n(y)|^2 dxdy - \int_{\mathbb{R}^2} F(x,w_n)dx \\
&\geq \lambda - C\|v_n\|^4_{L^{\frac{8}{3}}(\mathbb{R}^2)} - \int_{\mathbb{R}^2} F(x,w_n)dx.
\end{aligned}$$

Now we observe that $\|v_n\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \to 0$ as $n \to \infty$ by Strauss' Theorem, and by (3.13) we have

$$I(t_n u_n^+) \geq \lambda + o(1) \geq \frac{\lambda}{2}.$$

Being $\lambda > 0$ arbitrary, we finally find

$$I(t_n u_n^+) \to \infty \text{ as } n \to \infty. \tag{3.14}$$

Since $0 \leq t_n u_n^+ \leq u_n^+$ for all $n \geq 1$, from **H(iv)** we have

$$\int_{\mathbb{R}^2} \sigma(x, t_n u_n^+)dx \leq \int_{\mathbb{R}^2} \sigma(x, u_n^+)dx + \|\mathcal{M}^\star\|_{L^1(\mathbb{R}^2)} \text{ for all } n \geq 1. \tag{3.15}$$

Moreover, by (3.3) and (3.6) there exists $M_4 > 0$ such that

$$I(u_n) = I_+(u_n) + o(1) \leq M_4 \text{ for all } n \geq 1. \tag{3.16}$$

Thus, (3.14) and (3.16) imply that $t_n \in (0,1)$ for all $n \geq n_1 \geq 1$. Hence, by (3.12) we obtain that

$$\begin{aligned} 0 = t_n \frac{d}{dt} I(tu_n^+)\Big|_{t=t_n} &= I'(t_n u_n^+)(t_n u_n^+) = \int_{\mathbb{R}^2} \left(|D(t_n u_n^+)|^2 + a|t_n u_n^+|^2 \right) dx \\ &- \frac{1}{2\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} |t_n u_n^+(x)|^2 |t_n u_n^+(y)|^2 dxdy - \int_{\mathbb{R}^2} f(x, t_n u_n^+)(t_n u_n^+) dx \\ &= \|t_n u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - B\left((t_n u_n^+)^2, (t_n u_n^+)^2 \right) - \int_{\mathbb{R}^2} f(x, t_n u_n^+)(t_n u_n^+) dx \end{aligned} \tag{3.17}$$

for all $n \geq 1$, that is

$$\int_{\mathbb{R}^2} f(x, t_n u_n^+)(t_n u_n^+) dx = \|t_n u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - B\left((t_n u_n^+)^2, (t_n u_n^+)^2 \right) \tag{3.18}$$

for all $n \geq n_1$. Replacing (3.18) in (3.15), we obtain

$$\|t_n u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - B\left((t_n u_n^+)^2, (t_n u_n^+)^2 \right) - 2\int_{\mathbb{R}^2} F(x, t_n u_n^+) dx \leq \int_{\mathbb{R}^2} \sigma(x, u_n^+) dx + \|\mathcal{M}^\star\|_{L^1(\mathbb{R}^2)}$$

for all $n \geq n_1$.

Again by **H(iv)**

$$f(x, t_n u_n^+)(t_n u_n^+) - 2F(x, t_n u_n^+) \leq f(x, u_n^+)(u_n^+) - 2F(x, u_n^+) + \mathcal{M}^\star,$$

so that

$$\begin{aligned} -2\int_{\mathbb{R}^2} F(x, t_n u_n^+) dx \leq &\int_{\mathbb{R}^2} \left(f(x, u_n^+)(u_n^+) - f(x, t_n u_n^+)(t_n u_n^+) \right) dx \\ &- 2\int_{\mathbb{R}^2} F(x, u_n^+) dx + \int_{\mathbb{R}^2} \mathcal{M}^\star dx. \end{aligned}$$

Using (3.17) the previous inequality reads as

$$\begin{aligned} -2\int_{\mathbb{R}^2} F(x, t_n u_n^+) dx \leq -I'(u_n^+)(u_n^+) &+ \int_{\mathbb{R}^2} \left(|Du_n^+|^2 + a|u_n^+|^2 \right) dx - B\left((u_n^+)^2, (u_n^+)^2 \right) \\ &- \int_{\mathbb{R}^2} \left(|D(t_n u_n^+)|^2 + a|t_n u_n^+|^2 \right) dx + B\left((t_n u_n^+)^2, (t_n u_n^+)^2 \right) \\ &- 2\int_{\mathbb{R}^2} F(x, u_n^+) dx + \int_{\mathbb{R}^2} \mathcal{M}^\star dx \end{aligned}$$

and from (3.4)

$$\begin{aligned} -2\int_{\mathbb{R}^2} F(x, t_n u_n^+) dx \leq \|u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - B\left((u_n^+)^2, (u_n^+)^2 \right) - \|t_n u_n^+\|^2_{H^1_r(\mathbb{R}^2)} + B\left((t_n u_n^+)^2, (t_n u_n^+)^2 \right) - 2\int_{\mathbb{R}^2} F(x, u_n^+) dx \\ + \int_{\mathbb{R}^2} \mathcal{M}^\star dx + o(1). \end{aligned} \tag{3.19}$$

Now

$$2I(t_n u_n^+) = \|t_n u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - \frac{1}{2} B\left((t_n u_n^+)^2, (t_n u_n^+)^2 \right) - 2\int_{\mathbb{R}^2} F(x, t_n u_n^+) dx. \tag{3.20}$$

Thus, replacing (3.19) in (3.20), since $B\left((t_n u_n^+)^2,(t_n u_n^+)^2\right) \le B\left((u_n^+)^2,(u_n^+)^2\right)$ being $t_n < 1$, we have

$$\begin{aligned}2I(t_n u_n^+) &\le \|t_n u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - \frac{1}{2}B\left((t_n u_n^+)^2,(t_n u_n^+)^2\right) + \|u_n^+\|^2_{H^1_r(\mathbb{R}^2)} \\ &\quad - B\left((u_n^+)^2,(u_n^+)^2\right) - \|t_n u_n^+\|^2_{H^1_r(\mathbb{R}^2)} + B\left((t_n u_n^+)^2,(t_n u_n^+)^2\right) \\ &\quad - 2\int_{\mathbb{R}^2} F(x,u_n^+)dx + \int_{\mathbb{R}^2} \mathcal{M}^\star dx + o(1) \\ &\le \|u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - \frac{1}{2}B\left((u_n^+)^2,(u_n^+)^2\right) - 2\int_{\mathbb{R}^2} F(x,u_n^+)dx + \int_{\mathbb{R}^2} \mathcal{M}^\star dx + o(1).\end{aligned}$$

This last formula, together with (3.14), tells us that

$$\|u_n^+\|^2_{H^1_r(\mathbb{R}^2)} - \frac{1}{2}B\left((u_n^+)^2,(u_n^+)^2\right) - 2\int_{\mathbb{R}^2} F(x,u_n^+)dx \to \infty \text{ as } n\to\infty. \tag{3.21}$$

Comparing (3.8) and (3.21) we reach a contradiction.

So $(u_n^+)_n$ is bounded in $H^1_r(\mathbb{R}^2)$. □

We use this result to finally prove the Cerami compactness condition.

Proposition 3.2. *Let $(u_n)_n \subset X_r$ be a $(C)_d$-sequence for I_+ (I_- respectively), with $d > 0$. Then, up to a subsequence,*

$$u_n \to u \text{ in } X_r \text{ as } n \to \infty$$

for some nonzero critical point $u \in X_r$ of I_+ (I_- respectively). In particular, the $(C)_d$-condition holds.

Proof. From Lemma 3.1, we know that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } H^1_r(\mathbb{R}^2) \text{ as } n\to\infty.$$

Now, we show that

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}^2} \int_{B_r(x)} u_n^2(y)dy > 0 \tag{3.22}$$

for every $r > 0$. We argue by contradiction, so we suppose that (3.22) is false. Since $(u_n)_n$ is bounded in $H^1_r(\mathbb{R}^2)$, by [26, Lemma I.1], we have that $u_n \to 0$ as $n\to\infty$ in $L^s(\mathbb{R}^2)$ for every $s \in (2,\infty)$.

By our assumptions

$$\begin{aligned}I_+'(u_n)u_n &= \|u_n\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)|u_n^+(x)|^2|u_n^+(y)|^2dxdy \\ &\quad - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|u_n^+(x)|^2|u_n^+(y)|^2dxdy \\ &\quad - \int_{\mathbb{R}^2} f_+(x,u_n)u_n dx,\end{aligned}$$

and so

$$\begin{aligned}&\|u_n\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)|u_n^+(x)|^2|u_n^+(y)|^2dxdy \\ &= I_+'(u_n)u_n + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|u_n^+(x)|^2|u_n^+(y)|^2dxdy + \int_{\mathbb{R}^2} f_+(x,u_n)u_n dx.\end{aligned}$$

By **H(i)** and (2.6) we have

$$\begin{aligned}
&\|u_n\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)|u_n^+(x)|^2|u_n^+(y)|^2\,dxdy \\
&\le I'_+(u_n)u_n + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|u_n^+(x)|^2|u_n^+(y)|^2\,dxdy \\
&+ \|c\|_{L^p(\mathbb{R}^2)}\|u_n\|_{L^{p'}(\mathbb{R}^2)} + d\|u_n^+\|^q_{L^q(\mathbb{R}^2)} \\
&\le I'_+(u_n)u_n + C\|u_n^+\|^4_{L^{\frac{8}{3}}(\mathbb{R}^2)} + \|c\|_{L^p(\mathbb{R}^2)}\|u_n\|_{L^{p'}(\mathbb{R}^2)} + d\|u_n^+\|^q_{L^q(\mathbb{R}^2)}.
\end{aligned} \tag{3.23}$$

Since $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s \in (2,\infty)$, and $p' > 2$, we have

$$\|u_n\|_{H^1_r(\mathbb{R}^2)} \to 0,$$

and then

$$\|u_n^\pm\|_{H^1_r(\mathbb{R}^2)} \to 0,$$

and also

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)|u_n^+(x)|^2|u_n^+(y)|^2\,dxdy \to 0.$$

Hence,

$$\begin{aligned}
I_+(u_n) &= \frac{1}{2}\|u_n\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)|u_n^+(x)|^2|u_n^+(y)|^2\,dxdy \\
&- \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|u_n^+(x)|^2|u_n^+(y)|^2\,dxdy - \int_{\mathbb{R}^2} F_+(x,u_n)\,dx \\
&\le \frac{1}{2}\|u_n\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)|u_n^+(x)|^2|u_n^+(y)|^2\,dxdy \\
&+ \|c\|_{L^p(\mathbb{R}^2)}\|u_n\|_{L^{p'}(\mathbb{R}^2)} + d\|u_n^+\|^q_{L^q(\mathbb{R}^2)} \to 0,
\end{aligned}$$

but $I_+(u_n) \to d > 0$, so we reach a contradiction. Thus (3.22) holds.

This means that *vanishing* (see [27]) cannot occur. Moreover, since we use radial functions, dichotomy cannot take place, either. Hence, we can conclude that $u \neq 0$. By [17, Lemma 2.1] we can conclude that

$$(u_n)_n \text{ is bounded in } X_r.$$

Then we can assume that

$$u_n \rightharpoonup u \text{ in } X_r,$$

with $u \neq 0$, and by Proposition 2.1 we also have that $u_n \to u$ in $L^s(\mathbb{R}^2)$ for every $s \in [2,\infty)$.

Finally, we claim that $u_n \to u$ in X_r. In (3.5), we take $h = u_n - u$ and, using (2.2), we have

$$\begin{aligned}
I'_+(u_n)(u_n-u) &= \int_{\mathbb{R}^2} \left(|Du_n|^2 + a|u_n|^2\right)dx - \int_{\mathbb{R}^2} (Du_n\cdot Du + au_nu)\,dx \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|u_n^+(x)|^2u_n^+(y)(u_n-u)(y)\,dxdy \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\,|u_n^+(x)|^2u_n^+(y)(u_n-u)(y)\,dxdy \\
&- \int_{\mathbb{R}^2} f_+(x,u_n)(u_n-u)\,dx.
\end{aligned}$$

Hence,

$$\begin{aligned}\|u_n\|^2_{H^1_r(\mathbb{R}^2)} - (u_n|u) &= I'_+(u_n)(u_n-u)\\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|u_n^+(x)|^2 u_n^+(y)(u_n-u)(y)\,dxdy\\ &- \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\,|u_n^+(x)|^2 u_n^+(y)(u_n-u)(y)\,dxdy\\ &+ \int_{\mathbb{R}^2} f_+(x,u_n)(u_n-u)\,dx. \end{aligned} \tag{3.24}$$

By Theorem 2.1 and the Hölder inequality, **H(i)** and (2.3) we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|u_n^+(x)|^2 |u_n^+(y)(u_n-u)(y)|\,dxdy\\ &\le C\|u_n^+\|^3_{L^s(\mathbb{R}^2)}\|u_n-u\|_{L^s(\mathbb{R}^2)} \to 0\end{aligned}$$

with $s \in (2,\infty)$,

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\,|u_n^+(x)|^2 |u_n^+(y)(u_n-u)(y)|\,dxdy\\ &= \int_{\mathbb{R}^2} \ln(1+|x-y|)\,|u_n^+(x)|^2 \int_{\mathbb{R}^2} |u_n(y)(u_n-u)(y)|\,dxdy \to 0\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} f_+(x,u_n)(u_n-u)\,dx\right|\\ &\le \|c\|_{L^p(\mathbb{R}^2)}\|u_n^+\|_{L^p(\mathbb{R}^2)}\|u_n-u\|_{L^{p'}(\mathbb{R}^2)} + d\|u_n^+\|^{q-1}_{L^q(\mathbb{R}^2)}\|u_n\|_{L^{q'}(\mathbb{R}^2)} \to 0\end{aligned}$$

Hence, from (3.24)

$$u_n \to u \text{ in } X_r$$

and so the $(C)_d$-condition hold. □

Now we are ready to produce two nontrivial solutions of (P) using the Mountain Pass Theorem.

Theorem 3.1. *Under hypotheses* ***H(i)*** *-* ***H(v)****, problem* (P) *has two nontrivial constant sign solutions.*

Proof. We do the proof for the functional I_+; for I_- it is analogous. First, $I_+(0) = 0$. By Proposition 3.1 we have the regularity of I_+ and by Proposition 3.2 the $(C)_d$-condition is verified.

Now, take $\tilde{u}$ as in **H(v)**, $t > 0$ and, following [10], we set $u_t(x) = t^2\tilde{u}(tx)$. Then

$$\begin{aligned}I_+(u_t) = &\frac{t^4}{2}\int_{\mathbb{R}^2}\left(|D\tilde{u}|^2 + a\tilde{u}^2\right)dx\\ &+ \frac{t^4}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)|\tilde{u}^+(x)|^2|\tilde{u}^+(y)|^2\,dxdy\\ &- \frac{t^4\ln t}{8\pi}\left(\int_{\mathbb{R}^2}|\tilde{u}^+(x)|^2dx\right)^2 - \int_{\mathbb{R}^2} F_+(x, t^2\tilde{u}(tx))\,dx,\end{aligned}$$

and by **H(v)**,

$$\lim_{t\to+\infty} I_+(u_t) = -\infty.$$

In order to complete the proof it only remains to show that $I_+(u) \geq \alpha \geq 0$ with $\|u\| = r$, for some $r > 0$.

By **H(i)** we have $|f(x,t)| \leq c(x) + d|t|^{q-1}$ and then

$$|F(x,t)| \leq c(x)|t| + \frac{d}{q}|t|^q \text{ a.e. } x \in \mathbb{R}^2 \text{ and for all } t \in \mathbb{R}. \tag{3.25}$$

Hypothesis **H(ii)** says that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for $|t| < \delta$ we have

$$\frac{|f(x,t)|}{|t|} \leq \varepsilon \text{ a.e. } x \in \mathbb{R}^2,$$

thus

$$|F(x,t)| \leq \frac{\varepsilon}{2}t^2 \text{ a.e. } x \in \mathbb{R}^2 \text{ and } |t| \leq \delta. \tag{3.26}$$

On the other hand, when $|t| \geq \delta$

$$|F(x,t)| \leq \frac{c(x)|t|\delta^{q-1}}{\delta^{q-1}} + \frac{d}{q}|t|^q \leq \left(\frac{c(x)}{\delta^{q-1}} + \frac{d}{q}\right)|t|^q.$$

Combining the inequality above with (3.26) we get that for a.e. $x \in \mathbb{R}^2$ and for all $t \in \mathbb{R}$

$$|F(x,t)| \leq \left(\frac{c(x)}{\delta^{q-1}} + \frac{d}{q}\right)|t|^q + \frac{\varepsilon}{2}t^2.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^2} F(x,u)dx &\leq \int_{\mathbb{R}^2} \frac{c(x)}{\delta}|u|^q + \frac{d}{q}\int_{\mathbb{R}^2}|u|^q + \frac{\varepsilon}{2}\int_{\mathbb{R}^2}|u|^2 \\ &\leq \frac{1}{\delta}\|c\|_{L^p(\mathbb{R}^2)}\|u\|^q_{L^{qp'}(\mathbb{R}^2)} + \frac{d}{q}\|u\|^q_{L^q(\mathbb{R}^2)} + \frac{\varepsilon}{2}\|u\|^2_{L^2(\mathbb{R}^2)}\end{aligned}$$

that is

$$\int_{\mathbb{R}^2} F(x,u)dx \leq \varepsilon C_1\|t\|^2_{H^1_r(\mathbb{R}^2)} + C_\delta C_2\|t\|^q_{H^1_r(\mathbb{R}^2)}.$$

We use this estimates on functional I_+:

$$\begin{aligned}I_+(u) &= \frac{1}{2}\|u\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\,|u^+(x)|^2|u^+(y)|^2dxdy \\ &\quad - \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)|u^+(x)|^2|u^+(y)|^2dxdy - \int_{\mathbb{R}^2} F_+(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{4}B_1\left((u^+)^2,(u^+)^2\right) - \frac{1}{4}B_2\left((u^+)^2,(u^+)^2\right) \\ &\quad - \varepsilon\|u^+\|^2_{L^2(\mathbb{R}^2)} - B_\delta\|u^+\|^q_{L^q(\mathbb{R}^2)}\end{aligned}$$

and by the Hardy-Littlewood-Sobolev inequality

$$\begin{aligned}I_+(u) &\geq \frac{1}{2}\|u\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{4}B_1\left((u^+)^2,(u^+)^2\right) - C_3\|u^+\|^4_{H^1_r(\mathbb{R}^2)} - \varepsilon C_1\|u^+\|^2_{H^1_r(\mathbb{R}^2)} \\ &\quad - B_\delta C_2\|u^+\|^q_{H^1_r(\mathbb{R}^2)} \\ &= \frac{1}{2}\|u\|^2_{H^1_r(\mathbb{R}^2)} + \frac{1}{4}B_1\left((u^+)^2,(u^+)^2\right) - C_3\|u^+\|^4_{H^1_r(\mathbb{R}^2)} \\ &\quad - \varepsilon C_1\|u^+\|^2_{H^1_r(\mathbb{R}^2)} - B_\delta C_2\|u^+\|^q_{H^1_r(\mathbb{R}^2)}.\end{aligned}$$

Since $\|u^+\|_{H^1_r(\mathbb{R}^2)} \le \|u\|_{H^1_r(\mathbb{R}^2)}$ we get

$$I_+(u) \ge \left(\frac{1}{2} - \varepsilon C_1\right) \|u\|^2_{H^1_r(\mathbb{R}^2)} - \left(C_3 + B_\delta C_2 \|u\|^{q-4}_{H^1_r(\mathbb{R}^2)}\right) \|u\|^4_{H^1_r(\mathbb{R}^2)}.$$

Choosing $\varepsilon \in \left(0, \frac{1}{2C_1}\right)$ and $\|u\|_{H^1_r(\mathbb{R}^2)} = r$, we have

$$\begin{aligned} I_+(u) &\ge C_4 \|u\|^2_{H^1_r(\mathbb{R}^2)} - \left[C_3 + B_\delta C_2 \|u\|^{q-4}\right] \|u\|^4_{H^1_r(\mathbb{R}^2)} \\ &= \|u\|^2_{H^1_r(\mathbb{R}^2)} \left(C_4 - \left[C_3 + B_\delta C_2 r^{q-4}\right] r^2\right), \end{aligned}$$

for some $C_4 > 0$. We take r such that $C_4 - \left(C_3 + B_\delta C_2 r^{q-4}\right) > 0$ and so

$$I_+(u) \ge \alpha \ge 0,$$

thus we have the Mountain Pass geometry, and we can apply [25, Theorem 5.40].

Hence u^+ satisfies $-\Delta u + au - \frac{1}{2\pi}\left[\ln \frac{1}{|x|} \star |u^+|^2\right] u^+ = f(x, u^+)$. Now, multipliyng by u^-, we get

$$-\int_{\mathbb{R}^2} \left(|(Du^-(x)|^2 + a|u^-(x)|^2\right) dx = 0,$$

thus $u^- \equiv 0$, then $u \ge 0$ and it is a nontrivial solution of problem (P). Working with I_- we find another nontrivial solution of (P) which is non positive in $\mathbb{R}^2$. □

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