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On a Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection

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Abstract. In the present paper, some results on a Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection have been studied.

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1 Introduction

If a differentiable manifold M has a Lorentzian metric g, i.e., a symmetric non-degenerated (0,2) tensor field of index 1, then it is called a Lorentzian manifold. Generally, a differentiable manifold has a Lorentzian metric if and only if it has a 1-dimensional distribution. Hence an odd dimensional manifold is able to have a Lorentzian metric. It is very natural and interesting to define both a Sasakian structure and a Lorentzian metric on an odd dimensional manifold. In fact, odd dimensional de Sitter space and Gödel universe, that are important examples on relativity theory, have Sasakian structure with Lorentzian metric ([6], [12], [20]).
The idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [7]. Further, H. A. Hayden [11], introduced the idea of metric connection with torsion on a Riemannian manifold. K. Yano [23] have studied some curvature conditions for

semi-symmetric connections in Riemannian manifolds. S. Golab [8] defined and studied quarter-symmetric connection in a differentiable manifold with affine connection. Quarter-symmetric metric connection in a Riemannian manifold have been studied by several authors such as A. Haseeb [10], A. K. Mondal and U. C. De [15], K. T. P. Kumar, Venkatesha and C. S. Bagewadi [16], R. N. Singh and S. K. Pandey [19], R. S. Mishra and S. N. Pandey [14], S. C. Rastogi ([17], [18]), U. C. De and J. Sengupta [5], and many others.

In a Riemannian manifold M a linear connection $\bar{\nabla}$ is called a quarter-symmetric connection, if the torsion tensor T of the connection $\bar{\nabla}$ [8]

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{1.1}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a 1-form and ϕ is a $(1,1)$-tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0,$$

where $X, Y, Z \in \chi(M)$, $\chi(M)$ being the set of all differentiable vector fields on M. Then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection. If we change ϕX by X, then the quarter-symmetric metric connection reduces to a semi-symmetric metric connection.

A relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in an n-dimensional Lorentzian Sasakian manifold M is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{1.3}$$

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is said to be a concircular transformation ([22], [23]). A concircular transformation is always a conformal transformation [13]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry is generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor with respect to the Levi-Civita connection. It is defined by [22]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.4}$$

where $X, Y, Z \in \chi(M)$, R is the curvature tensor and r is the scalar curvature with respect to the Levi-Civita connection. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. Recently, concircular curvature tensor have been studied by various authors such as A. Barman [2], A. Barman and G. Ghosh [3], A. Haseeb [9], A. Taleshian and N. Asghari [21] and many others. Recently, U. C. De and P. Majhi studied ϕ-Weyl semisymmetric and ϕ-projectively semisymmetric generalized Sasakian space-forms [4].

The paper is produced in the following manner: In Section 2, we give a brief introduction of Lorentzian Sasakian manifolds. In Section 3, we deduce the relation between the curvature tensor of Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection and the Levi-Civita connection. Section 4, 5, 6 and 7 are devoted to study ξ-concircularly flat, quasi-concircularly flat, pseudoconcircularly flat and ϕ-concircularly flat Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection, respectively. In the last Section 8, we study ϕ-concircularly semisymmetric Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection.

2 Preliminaries

A differentiable manifold M of dimension n is called a Lorentzian Sasakian manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy [1]

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(X,\xi) = -\eta(X), \quad g(\xi,\xi) = -1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y) \tag{2.3}$$

for all vector fields X, Y on M.
Also Lorentzian Sasakian manifolds satisfy

$$\nabla_X \xi = -\phi X, \tag{2.4}$$

$$\Phi(X,Y) = (\nabla_X \eta)Y = g(\phi X, Y), \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.
Further, on a Lorentzian Sasakian manifold M, the following relations hold:

$$g(R(X,Y)Z,\xi) = -\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.6}$$

$$R(\xi, X)Y = -g(X, Y)\xi - \eta(Y)X, \tag{2.7}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.8}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi, \tag{2.9}$$

$$S(X, \xi) = (n-1)\eta(X), \quad S(\xi, \xi) = (n-1), \tag{2.10}$$

$$Q\xi = -(n-1)\xi, \tag{2.11}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X, Y) - (n-1)\eta(X)\eta(Y) \tag{2.13}$$

for all $X, Y, Z \in \chi(M)$, where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator.

Definition 2.1. *A Lorentzian Sasakian manifold M is said to be an η-Einstein manifold if its Ricci tensor S is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a and b are scalar functions on M. If $a = 0$, then M reduces to a special type of η-Einstein manifold.

Example. We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z > 0\}$, where (x, y, z) are the standard coordinates in R^3. Let

$$\eta = \frac{1}{2}(dz - ydx), \quad \xi = 2\frac{\partial}{\partial z}$$

and

$$\phi(X_1\frac{\partial}{\partial x} + Y_1\frac{\partial}{\partial y} + Z_1\frac{\partial}{\partial z}) = X_1\frac{\partial}{\partial y} - Y_1\frac{\partial}{\partial x} - Y_1 y\frac{\partial}{\partial z}.$$

We choose the vector fields

$$e_1 = 2\frac{\partial}{\partial y}, \quad e_2 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad e_3 = 2\frac{\partial}{\partial z} = \xi$$

which are linearly independent at each point of M and hence from a basis of T_pM. Now we define

$$g = -\eta \otimes \eta + \frac{1}{4}(dx \otimes dx + dy \otimes dy).$$

Let η be the 1-form on M defined by $\eta(X) = -g(X, e_3) = -g(X, \xi)$ for all $X \in \chi(M)$ and let ϕ be the $(1, 1)$-tensor field on M defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Let g be the Lorentzian metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=1,\ \ g(e_3,e_3)=-1,\ \ g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi)=1,\quad g(\xi,\xi)=-1,\quad \phi^2=-X+\eta(X)\xi,\quad \eta(\phi X)=0,$$
$$g(X,\xi)=-\eta(X),\quad g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y).$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1,e_2]=2e_3,\quad [e_1,e_3]=0,\quad [e_2,e_3]=0.$$

The Levi-Civita connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned}2g(\nabla_X Y,Z)&=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])\\&\quad+g(Y,[Z,X])+g(Z,[X,Y]),\end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\begin{array}{lll}\nabla_{e_1}e_1=0, & \nabla_{e_1}e_2=0, & \nabla_{e_1}e_3=-e_1,\\ \nabla_{e_2}e_1=0, & \nabla_{e_2}e_2=-e_3, & \nabla_{e_2}e_3=-e_2,\\ \nabla_{e_3}e_1=0, & \nabla_{e_3}e_2=0, & \nabla_{e_3}e_3=0.\end{array} \tag{2.14}$$

Now Let

$$X=\sum_{i=1}^{3}X^ie_i=X^1e_1+X^2e_2+X^3e_3,\qquad Y=\sum_{j=1}^{3}Y^ie_j=Y^1e_1+Y^2e_2+Y^3e_3,$$

and

$$Z=\sum_{k=1}^{3}Z^ke_k=Z^1e_1+Z^2e_2+Z^3e_3$$

for all $X,Y,Z\in\chi(M)$. It is known that

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z. \tag{2.15}$$

From the equations (2.14) and (2.15), it can be easily verified that

$$\begin{array}{lll}R(e_1,e_2)e_1=-e_2, & R(e_1,e_3)e_1=-e_3, & R(e_2,e_3)e_1=0,\\ R(e_1,e_2)e_2=e_1, & R(e_1,e_3)e_2=0, & R(e_2,e_3)e_2=-e_3,\\ R(e_1,e_2)e_3=0, & R(e_1,e_3)e_3=-e_1, & R(e_2,e_3)e_3=-e_2.\end{array} \tag{2.16}$$

From (2.16), it follows that

$$R(X,Y)Z=g(Y,Z)X-g(X,Z)Y.$$

Thus for $e_3=\xi$, the manifold M satisfies (2.4) and (2.12). Hence $M(\phi,\xi,\eta,g)$ is a Lorentzian Sasakian metric manifold of constant curvature 1 and is locally isometric to the unit sphere $S^3(1)$.

3 Curvature tensor of Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection

The curvature tensor $\bar{R}$ of a Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.1}$$

Using the equations (1.3), (2.1), (2.5), (2.12) in (3.1), we find

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z + \{g(X,\phi Y) - g(Y,\phi X)\}\phi Z \\ + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi + \eta(Z)\{\eta(Y)X - \eta(X)Y\},\end{aligned} \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the Riemannian curvature tensor of the connection ∇.

Let $\{e_1, e_2, e_3, \ldots, e_n\}$ be an orthonormal basis. Then the Ricci tensor $\bar{S}$ and the scalar curvature $\bar{r}$ of the manifold M endowed with a quarter-symmetric metric connection $\bar{\nabla}$ are defined by

$$\bar{S}(X,Y) = \sum_{i=1}^{n} \epsilon_i g(\bar{R}(e_i, X)Y, e_i),$$

and

$$\bar{r} = \sum_{i=1}^{n} \epsilon_i \bar{S}(e_i, e_i),$$

where $X, Y \in \chi(M)$ and $\epsilon_i = g(e_i, e_i)$. Now, contracting X in (3.2), we obtain

$$\bar{S}(Y,Z) = S(Y,Z) + g(Y,Z) + n\eta(Y)\eta(Z), \tag{3.3}$$

where $\bar{S}$ and S are the Ricci tensors with respect to the connections $\bar{\nabla}$ and ∇, respectively on M.

Contracting again Y and Z in (3.3), we get

$$\bar{r} = r, \tag{3.4}$$

where $\bar{r}$ and r are the scalar curvatures with respect to the connections $\bar{\nabla}$ and ∇, respectively on M.

Lemma 3.1. *Let M be an n-dimensional Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection. Then we have*

$$\bar{R}(X,Y)\xi = 2\{\eta(Y)X - \eta(X)Y\}, \tag{3.5}$$

$$\bar{R}(\xi,X)Y = -\bar{R}(X,\xi)Y = -2\{g(X,Y)\xi + \eta(Y)X\}, \tag{3.6}$$

$$\bar{R}(\xi,X)\xi = 2\{-X + \eta(X)\xi\}, \tag{3.7}$$

$$\bar{S}(X,\xi) = 2(n-1)\eta(X), \tag{3.8}$$

$$\bar{Q}\xi = -2(n-1)\xi \tag{3.9}$$

for all $X, Y \in \chi(M)$,

4 ξ-concircularly flat Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection

Analogous to the equation (1.4), the concircular curvature tensor $\bar{C}$ endowed with a quarter-symmetric metric connection is defined by

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{4.1}$$

where $\bar{R}$ and $\bar{r}$ are the Riemannian curvature tensor and the scalar curvature with respect to the connection $\bar{\nabla}$, respectively on M.

Definition 4.1. *A Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection is said to be ξ-concircularly flat if*

$$\bar{C}(X,Y)\xi = 0 \tag{4.2}$$

for all X, Y on M.

Taking $Z = \xi$ in (4.1) and using (2.2), (3.4) and (3.5), we have

$$\bar{C}(X,Y)\xi = 2\{\eta(Y)X - \eta(X)Y\} - \frac{r}{n(n-1)}\{-\eta(Y)X + \eta(X)Y\}. \tag{4.3}$$

Combining the equations (4.2) and (4.3), we find

$$\{2 + \frac{r}{n(n-1)}\}\{\eta(Y)X - \eta(X)Y\} = 0. \tag{4.4}$$

Now taking $Y = \xi$ in (4.4) and using (2.2), we obtain

$$\{2 + \frac{r}{n(n-1)}\}\{X - \eta(X)\xi\} = 0$$

which by taking inner product with W and using (2.2) takes the form

$$\{2 + \frac{r}{n(n-1)}\}\{g(X, W) + \eta(X)\eta(W)\} = 0. \tag{4.5}$$

By replacing X by QX in (4.5) and using the fact that $S(X, W) = g(QX, W)$, we obtain

$$\{2 + \frac{r}{n(n-1)}\}\{S(X, W) - (n-1)\eta(X)\eta(W)\} = 0.$$

This implies that either $r = -2n(n-1)$ or

$$S(X, W) = (n-1)\eta(X)\eta(W).$$

Hence, we can state the following theorem:

Theorem 4.1. *For a ξ-concircularly flat Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection, either the scalar curvature with respect to the Levi-Civita connection is a negative constant or the manifold is a special type of η-Einstein manifold with respect to the Levi-Civita connection.*

5 Quasi-concircularly flat Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection

Definition 5.1. *A Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection is said to be quasi-concircularly flat if*

$$g(\bar{C}(X, Y)Z, \phi W) = 0 \tag{5.1}$$

for all X, Y, Z, W on M.

Let M be an n-dimensional quasi-concircularly flat Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection. Therefore from (4.1) and (5.1), it follows that

$$g(\bar{R}(X, Y)Z, \phi W) = \frac{\bar{r}}{n(n-1)}\{g(Y, Z)g(X, \phi W) - g(X, Z)g(Y, \phi W)\}. \tag{5.2}$$

By using (3.2) and (3.4) in (5.2), we have

$$\begin{aligned} g(R(X,Y)Z,\phi W) &= -\{g(X,\phi Y)-g(Y,\phi X)\}g(\phi Z,\phi W) \\ &\quad - \{g(X,\phi W)\eta(Y)-g(Y,\phi W)\eta(X)\}\eta(Z) \\ &\quad + \frac{r}{n(n-1)}\{g(Y,Z)g(X,\phi W)-g(X,Z)g(Y,\phi W)\}. \end{aligned} \tag{5.3}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis in M. If we put $X = \phi e_i$ and $W = e_i$ in (5.3) and sum up with respect to i. Then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i,Y)Z,\phi e_i) &= -\sum_{i=1}^{n-1}\{g(\phi e_i,\phi Y)-g(Y,\phi^2 e_i)\}g(\phi Z,\phi e_i) \\ &\quad - \sum_{i=1}^{n-1}\{g(\phi e_i,\phi e_i)\eta(Y)-g(Y,\phi e_i)\eta(\phi e_i)\}\eta(Z) \\ &\quad + \frac{r}{n(n-1)}\sum_{i=1}^{n-1}\{g(Y,Z)g(\phi e_i,\phi e_i)-g(\phi e_i,Z)g(Y,\phi e_i)\}. \end{aligned} \tag{5.4}$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g(R(\phi e_i,Y)Z,\phi e_i) = S(Y,Z)+g(Y,Z)+\eta(Y)\eta(Z), \tag{5.5}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i) = g(Y,Z)+\eta(Y)\eta(Z), \tag{5.6}$$

$$\sum_{i=1}^{n-1} \eta(\phi e_i)g(Y,\phi e_i) = 0, \tag{5.7}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i) = n-1, \tag{5.8}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,Z)g(Y,\phi e_i) = g(Y,Z)+\eta(Y)\eta(Z). \tag{5.9}$$

By virtue of (5.5)-(5.9), the equation (5.4) takes the form

$$S(Y,Z) = \{-3+\frac{(n-2)r}{n(n-1)}\}g(Y,Z) - \{n+2+\frac{r}{n(n-1)}\}\eta(Y)\eta(Z)$$

from which it follows that

$$r = -2n(n-1).$$

Hence, we can state the following theorem:

Theorem 5.1. *An n-dimensional quasi-concircularly flat Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection is an η-Einstein manifold with respect to the Levi-Civita connection and the scalar curvature with respect to the Levi-Civita connection is a negative constant.*

6 Pseudoconcircularly flat Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection

Definition 6.1. *A Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection is said to be pseudoconcircularly flat if*

$$g(\bar{C}(\phi X, Y)Z, \phi W) = 0 \tag{6.1}$$

for all X, Y, Z, W on M.

Let M be an n-dimensional pseudoconcircularly flat Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection. Therefore from (4.1) and (6.1), it follows that

$$g(\bar{R}(\phi X, Y)Z, \phi W) = \frac{\bar{r}}{n(n-1)}\{g(Y,Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)\}. \tag{6.2}$$

By using (3.2) and (3.4) in (6.2), we have

$$\begin{aligned} g(R(\phi X, Y)Z, \phi W) = & -\{g(\phi X, \phi Y) - g(Y, \phi^2 X)\}g(\phi Z, \phi W) \\ & - g(\phi X, \phi W)\eta(Y)\eta(Z) \\ & + \frac{r}{n(n-1)}\{g(Y,Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)\}. \end{aligned} \tag{6.3}$$

In view of (2.3), (6.3) becomes

$$\begin{aligned} g(R(\phi X, Y)Z, \phi W) = & -2g(X,Y)g(Z,W) - 2g(X,Y)\eta(Z)\eta(W) \\ & - 2g(Z,W)\eta(X)\eta(Y) - 3\eta(X)\eta(Y)\eta(Z)\eta(W) \\ & - g(X,W)\eta(Y)\eta(Z) + \frac{r}{n(n-1)}\{g(Y,Z)g(X,W) \\ & + g(Y,Z)\eta(X)\eta(W) - g(\phi X, Z)g(Y, \phi W)\}. \end{aligned} \tag{6.4}$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (6.4) and sum up with respect to i. Then we have

$$\begin{aligned}\sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) &= -2\sum_{i=1}^{n-1} g(e_i, Y)g(Z, e_i) - 2\sum_{i=1}^{n-1} g(e_i, Y)\eta(Z)\eta(e_i) \\ &- 2\sum_{i=1}^{n-1} g(Z, e_i)\eta(e_i)\eta(Y) - 3\sum_{i=1}^{n-1} \eta(e_i)\eta(Y)\eta(Z)\eta(e_i) \\ &- \sum_{i=1}^{n-1} g(e_i, e_i)\eta(Y)\eta(Z) + \frac{r}{n(n-1)}\{\sum_{i=1}^{n-1} g(Y, Z)g(e_i, e_i) \\ &+ \sum_{i=1}^{n-1} g(Y, Z)\eta(e_i)\eta(e_i) - \sum_{i=1}^{n-1} g(\phi e_i, Z)g(Y, \phi e_i)\}.\end{aligned} \tag{6.5}$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g(e_i, Z)g(Y, e_i) = g(Y, Z) + \eta(Y)\eta(Z), \tag{6.6}$$

$$\sum_{i=1}^{n-1} g(e_i, e_i) = (n-1). \tag{6.7}$$

By virtue of (5.5), (5.9), (6.6) and (6.7), the equation (6.5) takes the form

$$S(Y, Z) = \{-3 + \frac{(n-2)r}{n(n-1)}\}g(Y, Z) - \{n + 2 + \frac{r}{n(n-1)}\}\eta(Y)\eta(Z)$$

from which it follows that

$$r = -2n(n-1).$$

Hence, we can state the following theorem:

Theorem 6.1. *An n-dimensional pseudoconcircularly flat Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection is an η-Einstein manifold with respect to the Levi-Civita connection and the scalar curvature with respect to the Levi-Civita connection is a negative constant.*

7 ϕ-concircularly flat Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection

Definition 7.1. *A Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection is said to be ϕ-concircularly flat if*

$$\phi^2\bar{C}(\phi X, \phi Y)\phi Z = 0 \tag{7.1}$$

for all X, Y, Z on M.

Let M be an n-dimensional ϕ-concircularly flat Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection. Therefore from (7.1), it follows that

$$g(\bar{C}(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{7.2}$$

From the equations (4.1) and (7.2), we have

$$\begin{aligned} g(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{\bar{r}}{n(n-1)}\{&g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &- g(\phi X, \phi Z)g(\phi Y, \phi W)\} \end{aligned}$$

which in view of (3.2) and (3.4) takes the form

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z,\phi W) &= -2g(\phi X, Y)g(Z, \phi W) \\ &+ \frac{r}{n(n-1)}\{g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\}. \end{aligned} \tag{7.3}$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis in M. If we put $X = W = e_i$ in (7.3) and sum up with respect to i. Then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z, \phi e_i) &= -2\sum_{i=1}^{n-1} g(\phi e_i, Y)g(Z, \phi e_i) \\ &+ \frac{r}{n(n-1)}\sum_{i=1}^{n-1}\{g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}. \end{aligned} \tag{7.4}$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z). \tag{7.5}$$

By virtue of (5.6), (5.8), (5.9) and (7.5), the equation (7.4) takes the form

$$S(\phi Y, \phi Z) = \left\{ \frac{(n-2)r}{n(n-1)} - 3 \right\} g(\phi Y, \phi Z)$$

which by using (2.3) and (2.13) yields

$$S(Y, Z) = \left\{ \frac{(n-2)r}{n(n-1)} - 3 \right\} g(Y, Z) + (n-2) \left\{ \frac{r}{n(n-1)} - 1 \right\} \eta(Y)\eta(Z)$$

from which it follows that

$$r = -2n(n-1).$$

Hence, we can state the following theorem:

Theorem 7.1. *An n-dimensional ϕ-concircularly flat Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection is an η-Einstein manifold with respect to the Levi-Civita connection and the scalar curvature with respect to the Levi-Civita connection is a negative constant.*

8 ϕ-concircularly semisymmetric Lorentzian Sasakian manifolds endowed with a quarter-symmetric metric connection

Definition 8.1. *A Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection is said to be ϕ-concircularly semisymmetric if*

$$\bar{C}(X, Y).\phi = 0 \tag{8.1}$$

for all X, Y on M.

Let M be an n-dimensional ϕ-concircularly semisymmetric Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection. Therefore (8.1) turns into

$$(\bar{C}(X, Y).\phi)Z = \bar{C}(X, Y)\phi Z - \phi\bar{C}(X, Y)Z = 0. \tag{8.2}$$

Taking $X = \xi$ in (8.2), we have

$$(\bar{C}(\xi, Y).\phi)Z = \bar{C}(\xi, Y)\phi Z - \phi\bar{C}(\xi, Y)Z = 0. \tag{8.3}$$

Now by taking $X = \xi$ in (4.1) and using (2.2), we have

$$\bar{C}(\xi, Y)Z = -\left\{2 + \frac{r}{n(n-1)}\right\}(g(Y, Z)\xi + \eta(Z)Y) \tag{8.4}$$

from which, we get

$$\phi\bar{C}(\xi, Y)Z = -\left\{2 + \frac{r}{n(n-1)}\right\}\eta(Z)\phi Y, \tag{8.5}$$

and

$$\bar{C}(\xi, Y)\phi Z = -\left\{2 + \frac{r}{n(n-1)}\right\}g(Y, \phi Z)\xi. \tag{8.6}$$

Therefore from (8.3), (8.5) and (8.6), we have

$$-\left\{2 + \frac{r}{n(n-1)}\right\}g(Y, \phi Z)\xi + \left\{2 + \frac{r}{n(n-1)}\right\}\eta(Z)\phi Y = 0$$

which by taking inner product with ξ and using (2.2) reduces to

$$\left\{2 + \frac{r}{n(n-1)}\right\}g(Y, \phi Z) = 0. \tag{8.7}$$

Since $g(Y, \phi Z) \neq 0$, therefore we get

$$r = -2n(n-1).$$

Hence, we can state the following theorem:

Theorem 8.1. *For an n-dimensional ϕ-concircularly semisymmetric Lorentzian Sasakian manifold endowed with a quarter-symmetric metric connection, the scalar curvature with respect to the Levi-Civita connection is a negative constant.*

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