On a method to construct magic rectangles of even order

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Abstract

Magic rectangles are well-known for their very interesting and entertaining combinatorics. In a magic rectangle, the integers 1 to mn are arranged in an array of m rows and n columns so that each row adds to the same total M and each column to the same total N . In the present paper we provide a simple and systematic method for constructing any even by even magic rectangle.

Keywords : magic rectangles; magic constants.

1. Introduction

Magic rectangles are well-known for their very interesting and entertaining combinatorics. A magic rectangle is an arrangement of the integers 1 to mn in an array of m rows and n columns so that each row adds to the same total M and each column to the same total N . The totals M and N are termed the magic constants. Since the average value of the integers is $A = (mn + 1)/2$, we must have $M = nA$ and $N = mA$. The total of all the integers in the array is $mnA = mM = nN$. If mn is even $mn + 1$ is odd and so for $M = n(mn + 1)/2$ and $N = m(mn + 1)/2$ to be integers n and m must both be even. On the other hand, since either m or

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n being even would result in the product mn to be even, therefore if mn is odd then m and n must both be odd. In this case also M and N are integers since $mn+1$ is even. Therefore, an odd by even magic rectangle is impossible. Also, by actual construction one can see that a 2×2 magic rectangle is impossible.

For an update on available literature on magic rectangles we refer to Hagedorn (1999) and Bier and Kleinschmidt (1997). Such magic rectangles have been used in designing experiments. For example, Phillips (1964, 1968a, 1968b) illustrated the use of these magic figures for the elimination of trend effects in certain classes of one-way, factorial, latin-square, and graeco-latin-square designs. As highly balanced structures, magic rectangles can be potential tools for use in situations yet unexplored.

In the present paper we provide a method for constructing any even by even magic rectangle. The construction involves some simple steps. The method has been shaped in form of an algorithm that is very convenient for writing a computer program for constructing such rectangles.

In Section 2 we construct magic rectangle of sides $m = 2p$ and $n = 2q$. The proofs related to the construction are given in the appendix. In Section 3 we illustrate our construction method through some examples of magic rectangles.

2. The construction

We construct magic rectangle of sides $m = 2p$ and $n = 2q$ for given positive integers p and q . We consider separately the cases (i) at least one of p , q is even, and (ii) both p and q are odd.

Case A: At least one of p , q is even. Without loss of generality, let p be even.

Step A1. Write the mn consecutive integers from 1 to mn with first column having integers $1, 2, \ldots, m$; second column having integers $2m, 2m-1, \ldots, m+1$; third column having integers $2m+1, 2m+2, \ldots, 3m$; fourth column having integers $4m, 4m-1, \ldots, 3m+1$; and so on, $(n-1)$ -th column having integers

$(n-2)m+1, (n-2)m+2, \ldots, (n-1)m$; n -th column having integers $nm, nm-1, \ldots, (n-1)m+1$. We would call this a serpentine format for writing the mn consecutive integers in n columns.

Step A2. Reverse the middle p rows.

Case B: Both p and q are odd. Without loss of generality, if either p or q equals 1, let $q=1$. Note that the case $p=q=1$ does not arise since it amounts to a 2×2 magic rectangle which is impossible.

Step B1. Same as Step A1.

Step B2. Reverse the first $q-1$ columns.

Step B3. Reverse the first p rows except for the middle two elements in each row.

Step B4. Interchange the middle $p-3$ elements of q -th column with the corresponding middle $p-3$ elements in the $(q+1)$ -th column. Also, (1) interchange the element in the $\{1+(p-3)/2, q\}$ -th position with the element in the $\{1+(p-3)/2, q+1\}$ -th position and, (2) interchange the element in the $\{3+(p-3)/2, q\}$ -th position with the element in the $\{3+(p-3)/2, q+1\}$ -th position.

The proofs related to the construction are given in Appendices A and B.

3. Some illustrative examples

In this section we provide some examples of magic rectangles of orders 10×8 and 10×14 .

Magic rectangle of order 10×8 is the transpose of magic rectangle of order 8×10 . So, we construct a magic rectangle of order 8×10 . Here $p=4$, $q=5$. Therefore,

Step A1 gives

$$\begin{pmatrix}
1 & 16 & 17 & 32 & 33 & 48 & 49 & 64 & 65 & 80 \\
2 & 15 & 18 & 31 & 34 & 47 & 50 & 63 & 66 & 79 \\
3 & 14 & 19 & 30 & 35 & 46 & 51 & 62 & 67 & 78 \\
4 & 13 & 20 & 29 & 36 & 45 & 52 & 61 & 68 & 77 \\
5 & 12 & 21 & 28 & 37 & 44 & 53 & 60 & 69 & 76 \\
6 & 11 & 22 & 27 & 38 & 43 & 54 & 59 & 70 & 75 \\
7 & 10 & 23 & 26 & 39 & 42 & 55 & 58 & 71 & 74 \\
8 & 9 & 24 & 25 & 40 & 41 & 56 & 57 & 72 & 73
\end{pmatrix},$$

and the desired magic rectangle, as given by Step A2, is

$$\begin{pmatrix}
1 & 16 & 17 & 32 & 33 & 48 & 49 & 64 & 65 & 80 \\
2 & 15 & 18 & 31 & 34 & 47 & 50 & 63 & 66 & 79 \\
78 & 67 & 62 & 51 & 46 & 35 & 30 & 19 & 14 & 3 \\
77 & 68 & 61 & 52 & 45 & 36 & 29 & 20 & 13 & 4 \\
76 & 69 & 60 & 53 & 44 & 37 & 28 & 21 & 12 & 5 \\
75 & 70 & 59 & 54 & 43 & 38 & 27 & 22 & 11 & 6 \\
7 & 10 & 23 & 26 & 39 & 42 & 55 & 58 & 71 & 74 \\
8 & 9 & 24 & 25 & 40 & 41 & 56 & 57 & 72 & 73
\end{pmatrix}.$$

Magic rectangle of order 10×14 has $p = 5$, $q = 7$. Therefore,

Step B1 gives

$$\begin{pmatrix}
1 & 20 & 21 & 40 & 41 & 60 & 61 & 80 & 81 & 100 & 101 & 120 & 121 & 140 \\
2 & 19 & 22 & 39 & 42 & 59 & 62 & 79 & 82 & 99 & 102 & 119 & 122 & 139 \\
3 & 18 & 23 & 38 & 43 & 58 & 63 & 78 & 83 & 98 & 103 & 118 & 123 & 138 \\
4 & 17 & 24 & 37 & 44 & 57 & 64 & 77 & 84 & 97 & 104 & 117 & 124 & 137 \\
5 & 16 & 25 & 36 & 45 & 56 & 65 & 76 & 85 & 96 & 105 & 116 & 125 & 136 \\
6 & 15 & 26 & 35 & 46 & 55 & 66 & 75 & 86 & 95 & 106 & 115 & 126 & 135 \\
7 & 14 & 27 & 34 & 47 & 54 & 67 & 74 & 87 & 94 & 107 & 114 & 127 & 134 \\
8 & 13 & 28 & 33 & 48 & 53 & 68 & 73 & 88 & 93 & 108 & 113 & 128 & 133 \\
9 & 12 & 29 & 32 & 49 & 52 & 69 & 72 & 89 & 92 & 109 & 112 & 129 & 132 \\
10 & 11 & 30 & 31 & 50 & 51 & 70 & 71 & 90 & 91 & 110 & 111 & 130 & 131
\end{pmatrix},$$

following which, Step B2 gives

$$
\left(\begin{array}{cccccccccccccc}
10 & 11 & 30 & 31 & 50 & 51 & 61 & 80 & 81 & 100 & 101 & 120 & 121 & 140 \\
9 & 12 & 29 & 32 & 49 & 52 & 62 & 79 & 82 & 99 & 102 & 119 & 122 & 139 \\
8 & 13 & 28 & 33 & 48 & 53 & 63 & 78 & 83 & 98 & 103 & 118 & 123 & 138 \\
7 & 14 & 27 & 34 & 47 & 54 & 64 & 77 & 84 & 97 & 104 & 117 & 124 & 137 \\
6 & 15 & 26 & 35 & 46 & 55 & 65 & 76 & 85 & 96 & 105 & 116 & 125 & 136 \\
5 & 16 & 25 & 36 & 45 & 56 & 66 & 75 & 86 & 95 & 106 & 115 & 126 & 135 \\
4 & 17 & 24 & 37 & 44 & 57 & 67 & 74 & 87 & 94 & 107 & 114 & 127 & 134 \\
3 & 18 & 23 & 38 & 43 & 58 & 68 & 73 & 88 & 93 & 108 & 113 & 128 & 133 \\
2 & 19 & 22 & 39 & 42 & 59 & 69 & 72 & 89 & 92 & 109 & 112 & 129 & 132 \\
1 & 20 & 21 & 40 & 41 & 60 & 70 & 71 & 90 & 91 & 110 & 111 & 130 & 131
\end{array}\right),
$$

and Step B3 gives

$$
\left(\begin{array}{cccccccccccccc}
140 & 121 & 120 & 101 & 100 & 81 & 61 & 80 & 51 & 50 & 31 & 30 & 11 & 10 \\
139 & 122 & 119 & 102 & 99 & 82 & 62 & 79 & 52 & 49 & 32 & 29 & 12 & 9 \\
138 & 123 & 118 & 103 & 98 & 83 & 63 & 78 & 53 & 48 & 33 & 28 & 13 & 8 \\
137 & 124 & 117 & 104 & 97 & 84 & 64 & 77 & 54 & 47 & 34 & 27 & 14 & 7 \\
136 & 125 & 116 & 105 & 96 & 85 & 65 & 76 & 55 & 46 & 35 & 26 & 15 & 6 \\
5 & 16 & 25 & 36 & 45 & 56 & 66 & 75 & 86 & 95 & 106 & 115 & 126 & 135 \\
4 & 17 & 24 & 37 & 44 & 57 & 67 & 74 & 87 & 94 & 107 & 114 & 127 & 134 \\
3 & 18 & 23 & 38 & 43 & 58 & 68 & 73 & 88 & 93 & 108 & 113 & 128 & 133 \\
2 & 19 & 22 & 39 & 42 & 59 & 69 & 72 & 89 & 92 & 109 & 112 & 129 & 132 \\
1 & 20 & 21 & 40 & 41 & 60 & 70 & 71 & 90 & 91 & 110 & 111 & 130 & 131
\end{array}\right),
$$

and the desired magic rectangle, as given by Step B4, is

$$
\left(\begin{array}{cccccccccccccc}
140 & 121 & 120 & 101 & 100 & 81 & 61 & 80 & 51 & 50 & 31 & 30 & 11 & 10 \\
139 & 122 & 119 & 102 & 99 & 82 & 79 & 62 & 52 & 49 & 32 & 29 & 12 & 9 \\
138 & 123 & 118 & 103 & 98 & 83 & 63 & 78 & 53 & 48 & 33 & 28 & 13 & 8 \\
137 & 124 & 117 & 104 & 97 & 84 & 77 & 64 & 54 & 47 & 34 & 27 & 14 & 7 \\
136 & 125 & 116 & 105 & 96 & 85 & 76 & 65 & 55 & 46 & 35 & 26 & 15 & 6 \\
5 & 16 & 25 & 36 & 45 & 56 & 75 & 66 & 86 & 95 & 106 & 115 & 126 & 135 \\
4 & 17 & 24 & 37 & 44 & 57 & 67 & 74 & 87 & 94 & 107 & 114 & 127 & 134 \\
3 & 18 & 23 & 38 & 43 & 58 & 68 & 73 & 88 & 93 & 108 & 113 & 128 & 133 \\
2 & 19 & 22 & 39 & 42 & 59 & 69 & 72 & 89 & 92 & 109 & 112 & 129 & 132 \\
1 & 20 & 21 & 40 & 41 & 60 & 70 & 71 & 90 & 91 & 110 & 111 & 130 & 131
\end{array}\right).
$$

Appendix A

Proof for the case when at least one of p , q is even.

i) Without loss of generality, let p be even (since a magic rectangle of order $n \times m$ is the transpose of magic rectangle of order $m \times n$ and visa verse). The serpentine format of generating the columns ensures that for $1 \le i \le m$, the elements in the i -th row are $\{i, 4p+1-i, 4p+i, 8p+1-i, 8p+i, 12p+1-i, \ldots, 4p(q-1)+i, 4pq+1-i\}$. Thus, the sum of the elements in the i -th row is $(4p+1)+(12p+1)+(20p+1)+\cdots+(4p(2q-1)+1) = q+4p(1+3+5+\cdots+(2q-1)) = q+4pq^2 = q(1+4pq) = M$, the magic constant.

ii) After Step A1, for $1 \le j \le n$, the j -th column sum is $m^2 j - m(m-1)/2$ and the sum of the middle p elements in the j column is $m^2 j/2 - m(m-1)/4$ (being half of the column sum since the elements in the sum are in arithmetic progression). Note that there always exists p middle rows since p is even and there are in all $m = 2p$ rows. Also, for $1 \le j \le q$, the j -th column sum is less than the magic constant $N = m(mn+1)/2$ by a quantity $(n+1-2j)m^2/2$ and for $q+1 \le j \le n$, the j -th column sum is greater than the magic constant N by a quantity $(2j-n-1)m^2/2$. Thus, in Step A2, for $1 \le j \le q$, the interchange of the middle p elements between j and $(n+1-j)$ -th columns increases and decreases the respective column totals by the desired amounts and thus reduces it to a magic rectangle.

Appendix B

Proof for the case when both p and q are odd.

i) Up to Step B3, the proof follows on lines similar to Case A.

ii) After Step B3, certain elements between the two middle columns are interchanged. Note that the elements in the q -th column are $\{(q-1)m+1,(q-1)m+2,(q-1)m+3,\ldots,qm\}$ and those in the $q+1$ -th column are $\{(q+1)m,(q+1)m-1,(q+1)m-2,\ldots,qm+1\}$. Since $n=2q$, the q -th column sum is less than the magic constant N by a quantity $m^2/2$. Similarly, the $(q+1)$ -th column sum is greater than the magic constant N by a quantity $m^2/2$. Now, the sum of the middle $p-3$ elements in the q -th column is $\Sigma_{i=(p+5)/2}^{3(p-1)/2}\{(q-1)m+i\}=(p-3)(4pq-2p+1)/2$ and the sum of the middle $p-3$ elements in the $q+1$ -th column is $\Sigma_{i=(p+5)/2}^{3(p-1)/2}\{(q+1)m+1-i\}=(p-3)(4pq+2p+1)/2$. Thus, the interchange of the middle $p-3$ elements of q -th column with the corresponding middle $p-3$ elements in the $(q+1)$ -th column increases the q -th column sum by $m(p-3)$ and decreases the $q+1$ -th column sum by $m(p-3)$. Now, noting that (a) the $\{1+(p-3)/2,q\}$ -th position has the element $(q-1)m+1+(p-3)/2$, (b) the $\{1+(p-3)/2,q+1\}$ -th position has the element $(q+1)m+1-1-(p-3)/2$, (c) the $\{3+(p-3)/2,q\}$ -th position has the element $(q-1)m+3+(p-3)/2$ and (d) the $\{3+(p-3)/2,q+1\}$ -th position has the element $(q+1)m+1-3-(p-3)/2$, the final two interchanges carried out in Step B4 increases the q -th column sum and decreases the $q+1$ -th column sum by $4m-2(p-3)-6$. Thus, the overall interchanges in Step B4 lead to the increase and decrease of the respective column totals by $m(p-3)+4m-2(p-3)-6=mp=m^2/2$, thereby ensuring a magic rectangle.

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